On the Involutes of the Spacelike Curve with a Timelike Binormal in Minkowski 3-Space

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Abstract

The involute of a given curve is a well-known concept in 3-dimensional Euclidean space $\mathbb{R}^3$ (see [7],[8]).

According to reference [6], $M_1$ is a timelike curve then the involute curve $M_2$ is a spacelike curve with a spacelike or timelike binormal. On the other hand, it has been investigated the involute and evolute curves of the spacelike curve $M_1$ with a spacelike binormal in Minkowski 3-space and it has been seen that the involute curve $M_2$ is timelike, [1].

In this paper, we have defined the involute curves of the spacelike curve $M_1$ with a timelike binormal in Minkowski 3-space $\mathbb{R}^3_1$. We have seen that the involute curve $M_2$ must be a spacelike curve with a spacelike or timelike binormal.

The relationship between the Frenet frames of the involute-evolute curve couple and some new characterizations with relation to the curve couple have been found.

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1. INTRODUCTION

Let Minkowski 3-space $\mathbb{IR}^3$ be the vector space $\mathbb{IR}^3$ provide with the Lorentzian inner product $g$ given by $g(X,X) = -x_1^2 + x_2^2 + x_3^2$, where $X = (x_1,x_2,x_3) \in \mathbb{IR}^3$.

A vector $X = (x_1,x_2,x_3) \in \mathbb{IR}^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathbb{IR}^3$ where $s$ is a pseudo-arclenght parameter, can locally be timelike spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively timelike, spacelike or null, for every $s \in I \subset \mathbb{IR}$. A lightlike vector $X$ is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$) and a timelike vector $X$ is said to be positive (resp. negative) if and only if $x_1 < 0$ (resp. $x_1 > 0$). The norm of a vector $X$ is defined by $\|X\|_L = \sqrt{g(X,X)}$.

The vectors $X = (x_1,x_2,x_3), Y = (y_1,y_2,y_3) \in \mathbb{IR}^3$ are orthogonal if and only if $g(X,Y) = 0$, [7].

Now let $X$ and $Y$ be two vectors in $\mathbb{IR}^3$, then the Lorentzian cross product is given by

$$X \times Y = (x_1y_2 - x_2y_1, x_2y_3 - x_3y_2, x_3y_1 - x_1y_3), \quad [5].$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha$. Then $T$, $N$ and $B$ are the tangent, the principal normal and the binormal vector of the curve $\alpha$, respectively. Depending on the causal character of the curve $\alpha$, we have the following Frenet formulae and instantaneous rotation vectors:

**i )** Let $\alpha$ be a unit speed timelike space curve with curvature $\kappa$ and torsion $\tau$. Let Frenet frames of $\alpha$ be $\{T, N, B\}$.

In this trihedron, $T$ is timelike vector, $N$ and $B$ are spacelike vectors. For this vectors, we can write

$$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

where $\times$ is the Lorentzian cross product, [5] in space $\mathbb{IR}^3$. In this situation, the Frenet formulas are given by

$$T' = \kappa N, \quad N' = \kappa T - \tau B, \quad B' = \tau N, \quad [4].$$

The Frenet instantaneous rotation vector for the timelike curve is given by

$$W = \tau T - \kappa B, \quad [2].$$

**ii )** Let $\alpha$ be a unit speed spacelike space curve with a spacelike binormal. In this trihedron, we assume that $T$ and $B$ spacelike vectors and $N$ timelike vector. In this situation,

$$T \times N = -B, \quad N \times B = -T, \quad B \times T = N,$$

The Frenet formulas are given by
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\[ T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N, \quad [4]. \]

The Frenet instantaneous rotation vector for the spacelike curve is given by
\[ W = -\tau T + \kappa B, \quad [2]. \]

\( \text{iii) Let } \alpha \text{ be a unit speed spacelike space curve. In this trihedron, we assume that } T \text{ and } N \text{ spacelike vektors and } B \text{ timelike vector. For this trihedron we write} \]
\[ T \times N = B, \quad N \times B = -T, \quad B \times T = -N, \]

The Frenet formulas are given by
\[ \tau = \frac{\alpha' \times \alpha''}{\|\alpha'\|}, \quad \kappa = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha'\times\alpha''\|^2}. \]

Lemma 1. Let X and Y be nonzero Lorentz orthogonal vektors in \( IR^3 \). If X is timelike, then Y is spacelike, [8].

Lemma 2. Let X, Y be positive (negative) timelike vectors in \( IR^3 \). Then \( g(X, Y) \leq \|X\|\|Y\| \) with equality if and only if X and Y are linearly dependent, [8].

Lemma 3. i) The Timelike Angle between Timelike Vectors
Let X and Y be positive (negative) timelike vectors in \( IR^3 \). By the Lemma 2, there is a unique nonnegative real number \( \varphi(X, Y) \) such that
\[ g(X, Y) = \|X\|\|Y\| \cosh \varphi(X, Y) \]

The Lorentzian timelike angle between X and Y is defined to be \( \varphi(X, Y) \).

ii) The Spacelike Angle between Spacelike Vectors
Let X and Y be spacelike vektors in \( IR^3 \) that span a spacelike vector subspace. Then we have \( |g(X, Y)| \leq \|X\|\|Y\| \). Hence, there is a unique real number \( \varphi(X, Y) \) between 0 and \( \pi \) such that
\[ g(X, Y) = \|X\|\|Y\| \cos \varphi(X, Y) \]

The Lorentzian spacelike angle between X and Y is defined to be \( \varphi(X, Y) \).

iii) The Timelike Angle between Spacelike Vectors
Let X and Y be spacelike vectors in \( IR^3 \) that span a timelike vector subspace. Then we have \( |g(X, Y)| > \|X\|\|Y\| \). Hence, there is a unique positive real number \( \varphi(X, Y) \) such that
\[ |g(X, Y)| = \|X\|\|Y\| \cosh \varphi(X, Y) \]

The Lorentzian timelike angle between X and Y is defined to be \( \varphi(X, Y) \).

iv) The Angle between Spacelike and Timelike Vectors
Let X be a spacelike vector and Y a positive timelike vector in \( IR^3 \). Then there is a unique nonnegative real number \( \varphi(X, Y) \) such that
\[ |g(X,Y)| = \|X\|\|Y\| \sinh \varphi(X,Y). \]

The **Lorentzian timelike angle** between \(X\) and \(Y\) is defined to be \(\varphi(X,Y)\), [8].

### 2. THE INVOLUTE OF THE SPACELIKE CURVE WITH A TIMELIKE BINORMAL IN \(IR^3\)

**Definition 1.** Let \(M_1, M_2 \subset IR^3\) be two curves which are given by \((I, \alpha)\) and \((I, \beta)\) coordinate neighbourhoods, resp. Let Frenet frames of \(M_1\) and \(M_2\) be \([T, N, B]\) and \([T^*, N^*, B^*]\), resp. \(M_2\) is called the involute of \(M_1\) \((M_1\) is called the evolute of \(M_2\)) if

\[ g(T, T^*) = 0. \] (6)

If \(M_1\) is a unit speed spacelike curve with a timelike binormal then the involute curve \(M_2\) must be a spacelike curve with a spacelike or timelike binormal. In this situation, the causal characteristic of the Frenet frames of the curves \(M_1\) and \(M_2\) must be of the form

\[ \{T \text{ spacelike}, N \text{ spacelike}, B \text{ timelike}\} \]

and

\[ \{T^* \text{ spacelike}, N^* \text{ timelike}, B^* \text{ spacelike}\} \text{ or } \{T^* \text{ spacelike}, N^* \text{ spacelike}, B^* \text{ timelike}\}. \]

So we can write

\[ g(T^*, T^*) = +1, \quad g(N^*, N^*) = \mp I = \varepsilon_\alpha, \quad g(B^*, B^*) = \mp I = \varepsilon_\alpha. \] (7)

**Definition 2.** (Unit Vectors \(C\) of Direction \(W\) for Nonnull Curves):

\(i)\) For the curve \(M_i\) with a timelike tangent, \(\theta\) being a Lorentzian timelike angle between the spacelike binormal unit vector \(-B\) and the Frenet instantaneous rotation vector \(W\),

\(a)\) If \(|\kappa| > |\tau|\), then \(W\) is a spacelike vector. In this situation, from the Lemma 3. \(iii)\) we can write

\[ \begin{aligned}
\kappa &= \|W\| \cosh \theta, \\
\tau &= \|W\| \sinh \theta,
\end{aligned} \]

\[ \|W\|^2 = g(W, W) = \kappa^2 - \tau^2 \] (8)

and

\[ C = \frac{W}{\|W\|} = \sinh \theta T - \cosh \theta B, \] (9)

where \(C\) is unit vector of direction \(W\).

\(b)\) If \(|\kappa| < |\tau|\), then \(W\) is a timelike vector. In this situation, from the Lemma 3.\(iv)\) we can write

\[ \begin{aligned}
\kappa &= \|W\| \sinh \theta, \\
\tau &= \|W\| \cosh \theta,
\end{aligned} \]

\[ \|W\|^2 = -g(W, W) = -(\kappa^2 - \tau^2) \] (10)

and

\[ C = \cosh \theta T - \sinh \theta B. \] (11)
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\textit{ii) } For the curve $M_i$ with a timelike principal normal, $\theta$ being an angle between the $B$ and the $W$, if $B$ and $W$ spacelike vectors that span a spacelike vector subspace then by the Lemma 3. \textit{ii)} we can write

\[ \kappa = \|W\| \cos \theta, \quad \tau = \|W\| \sin \theta, \quad \|W\|^2 = g(W, W) = \kappa^2 + \tau^2. \]  

(12)

and

\[ C = -\sin \theta T + \cos \theta B \]  

(13)

\textit{iii) } For the curve $M_i$ with a timelike binormal, $\theta$ being a Lorentzian timelike angle between the $B$ and the $W$,

\textbf{a)} If $|\tau| > |\kappa|$, then $W$ is a spacelike vector. From the Lemma 3.\textit{iv)} , we can write

\[ \kappa = \|W\| \sinh \theta, \quad \tau = \|W\| \cosh \theta, \quad g(W, W) = \|W\|^2 = (\tau^2 - \kappa^2) \]  

(14)

and

\[ C = \cosh \theta T - \sinh \theta B \]  

(15)

\textbf{b)} If $|\tau| < |\kappa|$, then $W$ is a timelike vector. In this situation, from the Lemma 3.\textit{i)} we have

\[ \kappa = \|W\| \cosh \theta, \quad \tau = \|W\| \sinh \theta, \quad g(W, W) = -\|W\|^2 = -(\tau^2 - \kappa^2) \]  

(16)

and

\[ C = \sinh \theta T - \cosh \theta B . \]  

(17)

\textbf{Theorem 1.} Let $(M_2, M_1)$ be the involute-evolute curve couple which are given by $(1, \alpha)$ and $(1, \beta)$ coordinate neighbourhoods, respectively. The distance between the points $\alpha(s) \in M_1$ and $\beta(s) \in M_2$ of the $(M_2, M_1)$ is given by

\[ d_{\text{II}}(\alpha(s), \beta(s)) = |c - s|, \quad c = \text{const} \quad \forall s \in I . \]

\textbf{Proof:} If $M_2$ is the involute of $M_1$, we have

$\beta(s) = \alpha(s) + \lambda T(s), \quad \lambda = c - s, \quad \alpha' = T .$

Let us derivative both side with respect to $s$:

\[ \frac{d\beta}{ds} = \frac{d\alpha}{ds} + \frac{d\lambda}{ds} T + \lambda \frac{dT}{ds} . \]

For being $\frac{dT}{ds} = T' = \kappa N ,$

\[ \frac{d\beta}{ds} = \left( I + \frac{d\lambda}{ds} \right) T + \lambda \kappa N \]

where $s$ and $s^\ast$ are arc parameter of $M_1$ and $M_2$, respectively. Thus we have

\[ T^\ast \frac{ds^\ast}{ds} = \left( I + \frac{d\lambda}{ds} \right) T + \lambda \kappa N . \]

Making inner product with $T$ this equation’s both side, we have
\[
\frac{ds^*}{ds} g(T, T^*) = \left(1 + \frac{d\lambda}{ds}\right) g(T, T) + \lambda g(N, T)
\]

From the definition of the involute-evolute curve couple, \( g(T, T^*) = 0 \). If \( M_1 \) is a spacelike then we can write \( g(T, T) = +1 \) and \( g(T, N) = 0 \). Thus we obtain
\[
1 + \frac{d\lambda}{ds} = 0 \Rightarrow \lambda = c - s, \quad c = \text{const}
\]

From the definition of the distance in the Lorentzian space, we easily find
\[
d_{i\alpha}(\alpha(s), \beta(s)) = ||\beta(s) - \alpha(s)|| = ||\alpha||\tau||, \quad \tau = |c - s|.
\]

**Theorem 2.** Let \((M_2, M_1)\) be the involute-evolute curve couple which are given by \((1, \alpha)\) and \((1, \beta)\) coordinate neighbourhoods, respectively. Let Frenet frames of \(M_1\) and \(M_2\), in the points \(\alpha(s) \in M_1\) and \(\beta(s) \in M_2\), be \(\{T, N, B\}\) and \(\{T^*, N^*, B^*\}\), respectively. For the curvature and torsion of curve \(M_2\), we have
\[
\kappa^{*2} = \frac{\epsilon_0 (\kappa^2 - \tau^2)}{(c - s)^2 \kappa^2}, \quad \tau^* = \frac{\kappa \tau^2}{\left(c - s \kappa \tau^2 - \kappa^2\right)}.
\]

**Proof:** If \(M_2\) is the involute of \(M_1\), we have
\[
\beta(s) = \alpha(s) + AT(s), \quad \lambda = c - s, \quad \alpha' = T.
\]
Let us derivative both side with respect to \(s\):
\[
\frac{d\beta}{ds} \frac{ds^*}{ds} - \frac{d\beta^*}{ds} \frac{ds^*}{ds} = T^* \frac{ds^*}{ds} = (c - s)\kappa N, \quad (19)
\]
where \(s\) and \(s^*\) are arc parameter of \(M_1\) and \(M_2\), respectively. We can find
\[
\frac{ds^*}{ds} = (c - s)\kappa \quad (20)
\]
thus we have
\[
T^* = N. \quad (21)
\]
Taking the derivative of the last equation
\[
\frac{dT^*}{ds} \frac{ds^*}{ds} = \frac{dN}{ds} \frac{ds^*}{ds} = \kappa T - \tau B, \\
\kappa^* N^* = \left(-\kappa T + \tau B\right) \frac{ds^*}{ds}, \\
\kappa^* g(N^*, N^*) = \left[\kappa^* g(T, T) + \tau^2 g(B, B)\left(\frac{ds^*}{ds}\right)^2\right].
\]
From the equations (6), (7) and (20), we have
\[
\kappa^{*2} \epsilon_0 = \left(\kappa^2 - \tau^2\right) \frac{1}{(c - s)^2 \kappa^2},
\]
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\[ \kappa'^2 = \frac{\varepsilon_0 (\kappa^2 - \tau^2)}{(c - s)^2 \kappa^2}. \]

From the equation (19) we can write

\[ \beta' = (c - s) \kappa N \]

and

\[ \beta'' = -(c - s) \kappa^2 T + ((c - s) \kappa' - \kappa) N + (c - s) \kappa \pi B. \]  \hspace{1cm} (22)

If we calculate vector \( \beta' \times \beta'' \), then we get

\[ \beta' \times \beta'' = -(c - s)^2 \kappa^2 \tau T + (c - s)^2 \kappa^4 B, \]  \hspace{1cm} (23)

\[ \| \beta' \times \beta'' \|_H = |c - s|^2 \| \kappa' \| \tau^2 - \kappa^2 |. \]  \hspace{1cm} (24)

If we take the derivative of equation (22) we have

\[ \beta'' = (2 \kappa'' - 3 \lambda \kappa \kappa') T + (- \lambda \kappa'' - 2 \kappa' + \lambda \kappa + \lambda \kappa') N + (2 \lambda \kappa' - 2 \kappa' + \lambda \kappa \tau') B \]

From the Lemma 1, we find

\[ \text{det}(\beta', \beta'', \beta''') = (c - s)^3 \tau' - (c - s)^3 \kappa^4 \kappa' \tau. \]  \hspace{1cm} (25)

For being \( \tau^* \), from the equations (24) and (25), we get

\[ \tau^* = \frac{(c - s)^3 \kappa^3 (\kappa' - \kappa') {\kappa \tau' - \kappa'}}{|c - s|^3 \| \kappa' \| \tau^2 - \kappa^2 |}. \]

Theorem 3. Let \( (M_2, M_1) \) be the involute-evolute curve couple. The Frenet vectors of the curve couple \( (M_2, M_1) \) as follow.

(1) If \( W \) is a spacelike vector \( |\mathbf{A}| < |\mathbf{r}| \), then

\[
\begin{bmatrix}
T^* \\
N^* \\
B^*
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
\sinh \theta & 0 & -\cosh \theta \\
-\cosh \theta & 0 & \sinh \theta
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}. \]  \hspace{1cm} (26)

(2) If \( W \) is a timelike vector \( |\mathbf{A}| > |\mathbf{r}| \), then

\[
\begin{bmatrix}
T^* \\
N^* \\
B^*
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
-\cosh \theta & 0 & \sinh \theta \\
-sinh \theta & 0 & \cosh \theta
\end{bmatrix}
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix}. \]  \hspace{1cm} (27)

Proof: (1) If \( M_2 \) is the involute of \( M_1 \), we have

\[ \beta(s) = \alpha(s) + \lambda T(s), \quad \lambda = c - s, \quad \alpha' = T \ (c = \text{constant}). \]

From the equation (21) we have

\[ T^* = N. \]

For being \( B^* = \frac{1}{\| \beta' \times \beta'' \|_H} (\beta' \times \beta'') \), from the equations (23) and (24), we obtain
\[ B^* = -\frac{\tau}{\sqrt{\tau^2 - \kappa^2}} T + \frac{\kappa}{\sqrt{\tau^2 - \kappa^2}} B, \]

Substituting (14) into the last equation, we obtain
\[ B^* = -\cosh \theta T + \sinh \theta B. \quad (28) \]

Since \( N^* = -(B^* \times T^*), \) then we have
\[ N^* = \sinh \theta T - \cosh \theta B. \quad (29) \]

If the equations (21), (28), (29) are written by matrix form, then the theorem is proved.

**Note:** In this situation, the causal characteristics of the Frenet frames of the curves \( M_1 \) and \( M_2 \) are
\[ \{T \text{ spacelike, } N \text{ spacelike, } B \text{ timelike}\} \]
and
\[ \{T^* \text{ spacelike, } N^* \text{ spacelike, } B^* \text{ timelike}\}. \]

**Theorem 4.** Let \( (M_2, M_1) \) be be the involute-evolute curve couple. If \( W \) and \( W^* \) are the Frenet instantaneous rotation vectors of \( M_1 \) and \( M_2 \) respectively, then

(i) \[ W^* = \frac{1}{|c - s|\kappa} (\theta' N - W), \quad (|\kappa| > |\tau|) \]

and

(ii) \[ W^* = \frac{1}{|c - s|\kappa} (\theta' N + W), \quad (|\kappa| < |\tau|) \]

**Proof:** (i) For the Frenet instantaneous rotation vector of the curve \( M_2 \), from the equation (3) we can write
\[ W^* = -\tau^* T^* + \kappa^* B^* \quad (30) \]

Using the equations (18), (26) in the equation (30), we have
\[ W^* = -\frac{\kappa \tau' - \kappa' \tau}{|c - s|\kappa |\tau^2 - \kappa^2|} N + \frac{\sqrt{|\tau^2 - \kappa^2|}}{|c - s|\kappa} (-\cosh \theta T + \sinh \theta B) \]

\[ W^* = \frac{\left(\frac{\kappa}{\tau}\right)^{\tau^2}}{|c - s|\kappa |\tau^2 - \kappa^2|} N + \frac{|W|}{|c - s|\kappa} (-\cosh \theta T + \sinh \theta B). \]

Using the equation (14) in the last equation, then we obtain
\[ W^* = \frac{1}{|c - s|\kappa} (\theta' N - (\tau T - \kappa B)); \]

and then, we get
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\[ W^* = \frac{1}{|c - s|\kappa}(\theta' N - W). \]

(ii) The proof is analogous to the proof of the statement (i).

**Theorem 5.** Let \((M_2, M_1)\) be a first type involute-evolute curve couple. If \(C\) and \(C^*\) are unit vectors of direction of \(W\) and \(W^*\), respectively, then we have

\[ (i) \quad C^* = -\frac{\theta'}{\sqrt{\theta'^2 + \kappa^2 - \tau^2}} N + \frac{\sqrt{\kappa^2 - \tau^2}}{\sqrt{\theta'^2 + \kappa^2 - \tau^2}} C, \quad (\text{for } |\kappa| > |\tau|) \]

\[ (ii) \quad C^* = \frac{\theta'}{\sqrt{\theta'^2 + \kappa^2 - \tau^2}} N + \frac{\sqrt{\tau^2 - \kappa^2}}{\sqrt{\theta'^2 + \kappa^2 - \tau^2}} C, \quad (\text{for } |\kappa| < |\tau|) \]

**Proof:** (i) Let \(\theta^*\) be the angle between \(B^*\) and \(W^*\). Then we can write

\[ C^* = -\sinh \theta^* T^* + \cosh \theta^* B^*. \quad (31) \]

For the curvatures and torsions of the curve \(M_2\), we have

\[ \begin{align*}
\kappa^* &= \|W^*\| \cosh \theta^* \\
\tau^* &= \|W^*\| \sinh \theta^*
\end{align*} \quad (32) \]

Using the equation (18) and Theorem 4. into the equation (32), we get

\[ \sinh \theta^* = \frac{\theta'}{\sqrt{\theta'^2 + \kappa^2 - \tau^2}}, \quad (33) \]

\[ \cosh \theta^* = \frac{\sqrt{\kappa^2 - \tau^2}}{\sqrt{\theta'^2 + \kappa^2 - \tau^2}}. \quad (34) \]

Substituting by the equations (33), (34) into the equation (31), the theorem is proved.

(ii) The proof is analogous to the proof of the statement (i).

**Corollary 1.** Let \((M_2, M_1)\) be the first type involute-evolute curve couple. If \(M_1\) evolute curve is helix, then

i) The vectors \(W^*\) and \(B^*\) of the involute curve \(M_2\) are linearly dependent.

ii) \(C = C^*\)

**Proof:** i) If \(M_1\) evolute curve is helix, then we have

\[ \frac{\tau}{\kappa} = \tanh \theta = \text{constant}. \]

and then we have

\[ \theta' = 0. \quad (35) \]

Substituting by the equation (35) into the equations (33), (34)

\[ \sinh \theta^* = 0 \]
\[ \cosh \theta^* = 1 \]

are found. Thus we have
\( \theta^* = 0 \).

**ii)** Substituting by the equation (35) into the Theorem 5., we have

\[ C = C^* \]

**Theorem 6.** Let \((M_2, M_1)\) be the first type involute-evolute curve couple. Let curvatures and torsions of the curve couple \((M_2, M_1)\) be \(\kappa, \tau, \kappa^*, \tau^* \) (\(\kappa \neq \tau, \kappa \neq 0\)). If \(M_1\) is a cylindrical helix, then \(M_2\) is a plane curve.

**Proof:** If \(M_1\) is a cylindrical helix, then we can write

\[ \frac{\kappa}{\tau} = \text{constant}, \]

\[ \left( \frac{\kappa}{\tau} \right)' = 0 \]

and we easily obtain

\[ \kappa'\tau - \kappa\tau' = 0. \quad (36) \]

On the other hand from the equation (18), we can write

\[ \frac{\tau^*}{\kappa^*} = \frac{\kappa\tau' - \kappa'\tau}{(c-s)\kappa(c^2 - \tau^2)}, \]

\[ \sqrt{\left[\frac{(c-s)^2}{\kappa^2}\right]} \]

substituting by the equation (36) into the last equation, we get

\[ \tau^* = 0. \]

**Example 2.1:** Let \(\alpha_1(\theta) = (\sqrt{2} \sinh \theta, \sqrt{2} \cosh \theta, \theta)\) be a unit speed time-like curve. If \(\alpha_1\) is a time-like curve then the involute curve is a space-like. In this situation, involute curve \(\beta_1\) of the curve \(\alpha_1\) can be given below.

\[ \beta_1(\theta) = (\sqrt{2} \sinh \theta + (c-\theta)\sqrt{2} \cosh \theta, \sqrt{2} \cosh \theta + (c-\theta)\sqrt{2} \sinh \theta, c) \quad c \in IR, \]

for specially \(c=2\) and \(0 \leq \theta \leq \pi\). We can draw involute-evolute curve couple \((\beta_1, \alpha_1)\) with helping the programme of Mapple 11 as follow.
Example 2.2: The involute of the unit speed space-like curve
\[
\alpha_2(\theta) = \left( \frac{\sqrt{3}}{2} \cosh \theta, \frac{\sqrt{3}}{2} \sinh \theta, \frac{\theta}{2} \right)
\]
is a space-like or time-like. The involute of the curve \( \alpha_2 \) can be given below.

\[
\beta_2(\theta) = \left( \frac{\sqrt{3}}{2} \left[ \cosh \theta + (c - \theta) \sinh \theta \right], \frac{\sqrt{3}}{2} \left[ \sinh \theta + (c - \theta) \cosh \theta \right], \frac{c}{2} \right), \quad c \in \mathbb{R}.
\]

\( \beta_2 \) is a space-like curve. For specially \( c=2 \) and \( \theta \in [0, \pi] \), once again we can draw involute-evolute curve couple \( (\beta_2, \alpha_2) \) with helping the programme of Mapple 11 as follow
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