Sharper Inequalities for Powers of the Numerical Radii of Hilbert Space Operators

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Abstract
We give several sharp inequalities involving powers for the numerical radii of Hilbert space operators; which generalize earlier inequalities.

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1 Introduction

Let $B(\mathcal{H})$ denote the $C^*$- algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $A \in B(\mathcal{H})$, let $w(A)$ and $\|A\|$ denote the numerical radius and the operator norm of $A$, respectively. It is well-known that $w(\cdot)$ defines a norm on $B(\mathcal{H})$, and that for every $A \in B(\mathcal{H})$,

$$w(A) \leq \frac{1}{2} \left(\|A\| + \|A^2\|^{\frac{1}{2}}\right) \leq \|A\|. \quad (1)$$
For basic properties of the numerical radius, we refer to [4] and [7]. It has been recently shown in [5], that if $A^2$ does not converge to the zero operator in $B(H)$, then

$$\text{w}(A) \leq \|A^2\|^{\frac{1}{2}},$$

and it has been shown in [2] that if $A \in B(H)$, then

$$\frac{1}{4} \|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2} \|A^*A + AA^*\|.$$  \hfill (3)

Recently, Al-Badawi and shebrawi [3] generalized some inequalities for powers of the numerical radii. It has been shown that if $A \in B(H)$, and $f$ and $g$ be nonnegative on $[0, \infty)$ which are continuous and satisfying $f(t)g(t) = t$, for all $t \in [0, \infty)$, then for $r > 1$, we have

$$w^r(A) \leq \frac{1}{2} \|f^{2r}(|A|) + g^{2r}(|A^*|)\|.$$ \hfill (4)

Also it has been proved in [1]; that if $A, B \in B(H)$ for $0 < \alpha < 1$, then for $r > 2$ we have

$$w^r(A + B) \leq 2^{r-1} \|\alpha (|A|^r + |B|^r) + (1 - \alpha) (|A^*|^r + |B^*|^r)\|.$$ \hfill (5)

In the next section of this paper, we generalize inequalities (2), (4), and (5) using some classical inequalities for nonnegative real numbers and some operator inequalities.

## 2 Numerical radius inequalities

In this section, we prove useful numerical radius inequalities for Hilbert space operators. To prove our generalized numerical radius inequalities, we need several well-known lemmas. The first lemma follows from spectral theorem for positive operators and Jensen’s inequality [1].

**Lemma 2.1.** Let $A \in B(H)$ be positive operator and let $x \in H$ be any unit vector. Then

$$\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle \text{ for all } r \geq 1,$$ \hfill (6)

$$\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r \text{ for all } 0 \leq r < 1.$$ \hfill (7)
The second lemma is a generalization from of the mixed Schwarz inequality which has been proved by Kittaneh [1].

**Lemma 2.2.** Let $T \in B(\mathcal{H})$ be an operator and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$, for all $t \in [0, \infty)$. Then

$$\langle Tx, y \rangle \leq \|f(|T|x)\| \|g(|T^*|y)\| \text{ for all } x, y \in \mathcal{H}. \quad (8)$$

The fourth lemma is known as the generalized mixed Schwarz inequality [1].

**Lemma 2.4.** Let $A \in B(\mathcal{H})$, and $0 < \alpha < 1$, then

$$|\langle Ax, y \rangle|^2 \leq \langle |A|^{2\alpha} x, x \rangle \langle |A^*|^{2(1-\alpha)} y, y \rangle, \text{ for all } x, y \in \mathcal{H}. \quad (9)$$

Our first result is a generalization of inequality (2).

**Theorem 2.1.** Let $A \in B(\mathcal{H})$ be any operator and let $f$ and $g$ be as in Lemma 2.2. Then for $r \geq 1$, we have

$$w^r(A) \leq \frac{1}{2} \|f^{2r}(|A|) g^{2r}(|A^*|)\|^{\frac{1}{2}}. \quad (10)$$

where $f^{2r}(|A|) g^{2r}(|A^*|)$ does not converge to the zero operator in $B(\mathcal{H})$.

**Proof:** For every unit vector $x \in \mathcal{H}$, we have

$$|\langle Ax, x \rangle|^{2r} \leq \langle f^2(|A|) x, x \rangle^r \langle g^2(|A^*|) x, x \rangle^r$$

$$\leq \langle f^{2r}(|A|) x, x \rangle \langle g^{2r}(|A^*|) x, x \rangle$$

$$\leq \langle f^{2r}(|A|) g^{2r}(|A^*|) x, x \rangle.$$

Thus

$$\sup \{|\langle Ax, x \rangle|^{2r} : x \in \mathcal{H}\} \leq \{\sup \langle f^{2r}(|A|) g^{2r}(|A^*|) x, x \rangle : x \in \mathcal{H}\},$$

and so

$$w^{2r}(A) \leq \|f^{2r}(|A|) g^{2r}(|A^*|)\|.$$ 

Hence,

$$w^r(A) \leq \|f^{2r}(|A|) g^{2r}(|A^*|)\|^{\frac{1}{2}}. \quad \blacksquare$$

The inequality (10) better than inequality (4) because
Our second result generalizes the inequality (5), which gives a numerical radius inequality for sum of two operators.

**Theorem 2.2.** Let $A, B \in B(\mathcal{H})$ be operator two operators, and let $\alpha$ be any positive real number such that $0 < \alpha < 1$. Then for $r \geq 1$,

$$w^r(A + B) \leq 2^{r-1} \left\| \left( |A|^{\alpha r} |A^*|^{(1-\alpha)r} \right) + \left( |B|^{\alpha r} |B^*|^{(1-\alpha)r} \right) \right\|,$$

(11)

where $\left( |A|^{\alpha r} |A^*|^{(1-\alpha)r} \right) + \left( |B|^{\alpha r} |B^*|^{(1-\alpha)r} \right)$ does not converge to the zero operator in $B(\mathcal{H})$.

**Proof.** For every unite vector $x \in \mathcal{H}$, we have

$$|\langle (A + B) x, x \rangle|^r \leq (|\langle Ax, x \rangle| + |\langle Bx, x \rangle|)^r \leq 2^{r-1} (|\langle Ax, x \rangle|^r + |\langle Bx, x \rangle|^r) \leq 2^{r-1} \left( |\langle A|^{2\alpha} x, x \rangle| \frac{2}{r} (|A^*|^{2(1-\alpha)} x, x)^\frac{2}{r} + |\langle B|^{2\alpha} x, x \rangle| \frac{2}{r} (|B^*|^{2(1-\alpha)} x, x)^\frac{2}{r} \right) \leq \left( |\langle A|^{\alpha r} x, x \rangle| \right) + \left( |\langle B|^{\alpha r} x, x \rangle| \right) = 2^{r-1} \left( \left| \langle A|^{\alpha r} x, x \rangle \right| + \left| \langle B|^{\alpha r} x, x \rangle \right| \right). \text{■}$$

**Corollary 2.1.** Letting $r = 2$, $\alpha = \frac{1}{2}$ in (11) we obtain that

$$w^2(A + B) \leq 2 \left( \left| \langle A| A^* \rangle \right| + \left| \langle B| B^* \rangle \right| \right),$$

(12)

where $\left| \langle A| A^* \rangle \right| + \left| \langle B| B^* \rangle \right|$ does not converge to the zero operator in $B(\mathcal{H})$.

**Corollary 2.2.** If $A = B$, then the inequality (12) reduces to the inequality (2).

**Corollary 2.3.** If $A = B$, then the inequality (11) reduces to

$$w^r(A) \leq 2^{r-1} \left| \langle A|^{\alpha r} x, x \rangle \right|,$$

(13)

where $|A|^{\alpha r}$ does not converge to the zero operator in $B(\mathcal{H})$. 

(see [8]).
If we let $\alpha = \frac{1}{2}$ and $r \geq 1$ then we obtain,

$$w^r(A) \leq 2^{r-1} \left\| |A|^\frac{r}{2} |A^*|^\frac{r}{2} \right\| \leq 2^{r-1} \|A\|^r. \quad (14)$$

References


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