The Translational Hulls of Inverse wpp Semigroups

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Abstract

It is proved that the translational hull of an inverse wpp semigroup is still an inverse wpp semigroup.

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1 Introduction and Main Result

Let \( S \) be a semigroup. A mapping \( \lambda \) of \( S \) into itself is called a \textit{left translation} of \( S \) if \( \lambda(ab) = (\lambda a)b \) for all \( a, b \in S \); a mapping \( \rho \) of \( S \) into itself is called a \textit{right translation} of \( S \) if \( (ab)\rho = a(bp) \) for all \( a, b \in S \). A left translation \( \lambda \) and a right translation \( \rho \) of \( S \) are \textit{linked} if \( a(\lambda b) = (a\rho)b \) for all \( a, b \in S \). In this case, the linked pair \( (\lambda, \rho) \) is called a \textit{bitranslation} of \( S \). Denote by \( \Omega(S) \) the set of all linked pairs of \( S \). It is not difficult to check that \( \Omega(S) \) forms a semigroup under the multiplication \( (\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_1\lambda_2, \rho_1\rho_2) \). We call the semigroup \( \Omega(S) \) the \textit{translational hull} of \( S \).

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Translational hull is an important concept in the theory of semigroups. For the role played by the translational hull in general theory of semigroup, the reader may consult Petrich [1]. There are many authors having been studying the translational hulls of semigroups (see, [2], [3]-[6] and [7]).

Tang [8] introduced Green’s $\ast\ast$-relations $L^{\ast\ast}$ and $R^{\ast\ast}$, which are common generalizations of usual Green’s relations $L, R$ and Green’s $\ast$-relations $L^*, R^*$. A semigroup $S$ is called wpp if for any $a \in S$, we have

- The $L^{\ast\ast}$-class $L^{\ast\ast}_a$ of $S$ containing $a$ has an idempotent $e_a$ such that $ae_a = a$.
- The $R^{\ast\ast}$-class $R^{\ast\ast}_a$ of $S$ containing $a$ has an idempotent $f_a$ such that $f_aha = a$.

Following [9], a wpp semigroup is called inverse wpp semigroup if the idempotents commute.

In this paper we are concerned with analogues in the theory of inverse wpp semigroups of Ponizovski’s theorem which states that the translational hull of an inverse semigroup is itself inverse. In [3], Fountain and Lawson proved that the translational hull of adequate semigroups are still adequate semigroups. Because inverse wpp semigroups are common generalizations of inverse semigroups and adequate semigroups, these raises a natural question: whether is the translational hull of an inverse wpp semigroup still an inverse wrpp semigroup? We shall prove the following theorem.

**Theorem 1.1** The translational hull of an inverse wpp semigroup is still an inverse wpp semigroup.

**2 Preliminaries**

We begin by listing some basic results and notations form [10] and [8] which we shall use without further comments. It is easy to see that in an inverse wpp semigroup, each $L^{\ast\ast}$-class and each $R^{\ast\ast}$-class contains precisely one idempotent. For convenience, we shall denote by $a^s$ the idempotent in the $L^{\ast\ast}$-class of $S$ containing $a$ and by $a^1$ one in the $R^{\ast\ast}$-class of $S$ containing $a$.

For a semigroup $S$, we define relations on $S$ given by: for all $a, b \in S$,

- $aL^{\ast\ast}b$ if and only if $(\forall x, y \in S^1)(ax, ay) \in R \iff (bx, by) \in R$

and

- $aR^{\ast\ast}b$ if and only if $(\forall x, y \in S^1)(xa, ya) \in L \iff (xb, yb) \in L$.

**Lemma 2.1** [8] (1) $L^{\ast\ast}$ is a right congruence and $L \subseteq L^* \subseteq L^{\ast\ast}$.

(2) $R^{\ast\ast}$ is a left congruence and $R \subseteq R^* \subseteq R^{\ast\ast}$.
Lemma 2.2 Let $S$ be a wpp semigroup. If $a$ and $b$ are regular elements of $S$, then

1. $aL^*b$ if and only if $aLb$.
2. $aR^*b$ if and only if $aRb$.

Proof. We only prove (1) and (2) is similarly proved. For (1), it suffices to verify that if $aL^*b$, then $aLb$. Since $a$ and $b$ are regular, there exist $a', b' \in S^1$ such that $a = aa'a, b = bb'b$. Obviously, $a'aLa$ and $b'bLb$. By Lemma 2.1, these can show that $aL^*a'a$ and $b'bL^*b$. Thus $a'aL^*b'b$ and so $a'a = (a'a)(b'b)$ and $b'b = (b'b)(a'a)$, that is, $a'aLb'b$. Therefore $aLb$. \hfill \square

Lemma 2.3 [9] Let $S$ be an inverse wpp semigroup and $a, b \in S$. Then

1. $aL^*b$ if and only if $a^* = b^*$; $aR^*b$ if and only if $a^\dagger = b^\dagger$.
2. $(ab)^* = (a^*b)^*$ and $(ab)^\dagger = (ab^\dagger)^\dagger$.
3. $aa^* = a = a^\dagger a$.

3 Proof of Theorem 1.1

To begin with, we prove

Lemma 3.1 Let $S$ be a wpp semigroup. If $\lambda, \lambda'$ ($\rho, \rho'$) are left (right) translations of $S$ whose restrictions to the set of idempotents of $S$ are equal. Then $\lambda = \lambda'$ ($\rho = \rho'$).

Proof. Let $a$ be an element of $S$ and $e$ be an idempotent in the $R^*$-class of $a$. Then $ea = a$ and so

$$\lambda a = \lambda(ea) = \lambda(e)a = (\lambda e)a = \lambda'(ea) = \lambda'a,$$

thereby $\lambda = \lambda'$. Similarly, we have $\rho = \rho'$. \hfill \square

Lemma 3.2 Let $S$ be an inverse wpp semigroup and $(\lambda_i, \rho_i) \in \Omega(S)$ with $i = 1, 2$. Then the following statements are equivalent:

1. $(\lambda_1, \rho_1) = (\lambda_2, \rho_2)$.
2. $\rho_1 = \rho_2$.
3. $\lambda_1 = \lambda_2$.

Proof. Note that (1) $\Leftrightarrow$ (2) is dual to (1) $\Leftrightarrow$ (3), it suffices to verify (1) $\Leftrightarrow$ (2).

But (1) $\Rightarrow$ (2) is clear, so we need only to show (2) $\Rightarrow$ (1). For this, we let $\rho_1 = \rho_2$. For all $e \in E(S)$, we have $e\rho_1 = e\rho_2$, and so $\lambda_1 e = (\lambda_1 e)^\dagger(\lambda_1 e) = [(\lambda_1 e)^\dagger\rho_1]e = [(\lambda_1 e)^\dagger\rho_2]e = (\lambda_1 e)^\dagger(\lambda_2 e) = (\lambda_1 e)^\dagger(\lambda_2 e)^\dagger(\lambda_2 e)$. On the other hand, since $R^*$ is a left congruence and by $\lambda_2 eR^*(\lambda_2 e)^\dagger$, we have

$$(\lambda_1 e)^\dagger R^*\lambda_1 e = (\lambda_1 e)^\dagger(\lambda_2 e)R^*(\lambda_1 e)^\dagger(\lambda_2 e)^\dagger,$$
thereby \((\lambda_1 e)^\dagger = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger\) since each \(R^{**}\)-class of an inverse wpp semigroup contains exactly one idempotent; similarly, \((\lambda_2 e)^\dagger = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger\). Thus \((\lambda_1 e)^\dagger = (\lambda_2 e)^\dagger\). Consequently, \(\lambda_1 e = (\lambda_1 e)^\dagger (\lambda_2 e)^\dagger = (\lambda_2 e)^\dagger (\lambda_2 e) = \lambda_2 e\) and \(\lambda_1 = \lambda_2\). We complete the proof. \(\square\)

Now let \(S\) be an inverse wpp semigroup with semilattice \(E\) of idempotents. Let \((\lambda, \rho) \in \Omega(S)\) and define mappings \(\lambda^\dagger, \lambda^*, \rho^\dagger, \rho^*\) of \(S\) to itself as follows:

\[
\begin{align*}
\lambda^\dagger a &= (a^\dagger \rho)^\dagger a, \\
\lambda^* a &= (\lambda a^\dagger)^* a \\
\rho^\dagger &= a(a^\star \rho)^\dagger
\end{align*}
\]

and

\[a^\star \rho^* = a(\lambda a^*)^*,\]

for all \(a \in S\).

**Lemma 3.3** Let \(S\) be an inverse wpp semigroup with semilattice of idempotents \(E\). Then

1. \(\lambda^\dagger e = e \rho^\dagger\) for all \(e \in E\).
2. \(\lambda^* e = e \rho^*\) for all \(e \in E\).
3. \(\lambda^\dagger e, \lambda^* e \in E(S)\) for all \(e \in E\).
4. For all \(a, b \in S\), \((ab)^\dagger = a^\dagger (ab)^\dagger = (ab)^\dagger a^\dagger\) and \((ab)^* = (ab)^* b^* = b^* (ab)^*\).

**Proof.**

(1) Since \(S\) is an inverse wpp semigroup, we have that \(E(S)\) is a semilattice. This shows that for \(e \in E(S)\), \(\lambda^\dagger e = (e \rho)^\dagger e = e (e \rho)^\dagger = e \rho^\dagger\).

(2) It is similar to (1).

(3) It is a routine calculation.

(4) Since \(a^\dagger ab = ab\) and by \(ab R^{**} (ab)^\dagger\), we have \((ab)^\dagger R^{**} ab = a^\dagger (ab) R^{**} a^\dagger (ab)^\dagger\) and \((ab)^\dagger = a^\dagger (ab)^\dagger = (ab)^\dagger a^\dagger\) since each \(R^{**}\)-class of \(S\) contains unique idempotent. Similarly, \((ab)^* = (ab)^* b^* = b^* (ab)^*\). \(\square\)

**Lemma 3.4** The pairs \((\lambda^\dagger, \rho^\dagger)\) and \((\lambda^*, \rho^*)\) are members of the translational hull \(\Omega(S)\) of \(S\).

**Proof.** Let \(a, b\) be elements of \(S\). Compute

\[
\begin{align*}
\lambda^\dagger (ab) &= ((ab)^\dagger \rho)^\dagger ab = ((ab)^\dagger a^\dagger \rho)^\dagger ab \\
&= (ab)^\dagger (a^\dagger \rho)^\dagger ab \\
&= (a^\dagger \rho)^\dagger ab \\
&= (\lambda^\dagger a)b
\end{align*}
\]
This means that $\lambda^\dagger$ is a left translation on $S$; dually, $\rho^*$ is a right translation on $S$. And, we have

\[
\lambda^*(ab) = (\lambda(ab)^\dagger)^*ab = (\lambda(a^\dagger(ab)^\dagger))^*ab = (\lambda a^\dagger)^*(ab)^\dagger ab = (\lambda a^\dagger)^*ab = (\lambda a^\dagger)b
\]

and whence $\lambda^*$ is a left translation on $S$; dually, $\rho^\dagger$ is a right translation on $S$.

Because

\[
a(\lambda^\dagger b) = aa^* (\lambda^\dagger (b^\dagger b)) = aa^* (\lambda^\dagger b^\dagger) b
\]

\[
= a(\lambda^\dagger b^\dagger) a^* b = a(\lambda^\dagger (b^\dagger a^*)) b
\]

\[
= a(\lambda^\dagger (a^\dagger b^\dagger)) b = a(\lambda^\dagger a^\dagger) b^\dagger b
\]

\[
= a(a^* \rho^\dagger)^\dagger b^\dagger b
\]

\[
= ((aa^*)(\rho^\dagger)) b
\]

\[
= (a \rho^\dagger) b,
\]

we obtain that $(\lambda^\dagger, \rho^\dagger)$ is a linked pair and so is in $\Omega(S)$. Similarly, $(\lambda^*, \rho^*) \in \Omega(S)$. \hfill \Box

**Lemma 3.5** $E(\Omega(S)) = \{(\lambda, \rho) \in \Omega(S) | \lambda E \cup E \rho \subseteq E(S)\}.$

**Proof.** Suppose $(\lambda, \rho) \in \Omega(S)$. If $\lambda E \cup E \rho \subseteq E$, then for all $e \in E(S)$,

\[
e \rho^2 = (e \rho) \rho = (e(e \rho)) \rho = (e \rho)(e \rho) = e \rho,
\]

and by Lemma 3.1, $\rho^2 = \rho$, thus by Lemma 3.2, $(\lambda^2, \rho^2) = (\lambda, \rho)$, that is, $(\lambda, \rho)^2 = (\lambda, \rho)$. Conversely, if $\lambda^2 = \lambda$ and $\rho^2 = \rho$, then for all $e \in E(S)$

\[
\lambda e = \lambda^2 e = \lambda(\lambda e) = (\lambda e)^2 \subseteq E(S);
\]

and similarly, $e \rho \subseteq E(S)$. This completes the proof. \hfill \Box

By Lemmas 3.3, 3.4 and 3.5, the following corollary is immediate.

**Corollary 3.6** For any member $(\lambda, \rho) \in \Omega(S)$, the elements $(\lambda^\dagger, \rho^\dagger)$ and $(\lambda^*, \rho^*)$ are idempotents.

**Lemma 3.7** The elements of $E(\Omega(S))$ commute with each other.

**Proof.** If $(\lambda_1, \rho_1), (\lambda_2, \rho_2) \in E(\Omega(S))$, then for any $e \in E(S)$,

\[
\lambda_1 \lambda_2 (e) = \lambda_1 (\lambda_2 e) = \lambda_1 ((\lambda_2 e)e) = (\lambda_1 e)(\lambda_2 e) = (\lambda_2 e)(\lambda_1 e) = \lambda_2 (\lambda_1 e)
\]

and by Lemma 3.1, $\lambda_1 \lambda_2 = \lambda_2 \lambda_1$, thus by Lemma 3.2, $(\lambda_1 \lambda_2, \rho_1 \rho_2) = (\lambda_2 \lambda_1, \rho_2 \rho_1)$, that is, $(\lambda_1, \rho_1)(\lambda_2, \rho_2) = (\lambda_2, \rho_2)(\lambda_1, \rho_1)$. We complete the proof. \hfill \Box
Lemma 3.8 Let \((\lambda, \rho) \in \Omega(S)\). Then \((\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho) = (\lambda^\dagger, \rho^\dagger)(\lambda, \rho)\).

Proof. For all \(e \in E(S)\), since

\[
\lambda\lambda^*(e) = \lambda(\lambda^*e) = (\lambda e)(\lambda^*e) = (\lambda e)(\lambda^*e) = (\lambda e)(\lambda^*e) = \lambda e,
\]

we have \(\lambda\lambda^* = \lambda\), and by Lemma 3.2, \(\rho = \rho\rho^*\). Therefore \((\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho)\); and similarly, \((\lambda^\dagger, \rho^\dagger)(\lambda, \rho) = (\lambda, \rho)\).

Lemma 3.9 Let \((\lambda, \rho) \in \Omega(S)\). Then \((\lambda^*, \rho^*)\mathcal{L}^{**}(\lambda, \rho)\mathcal{R}^{**}(\lambda^\dagger, \rho^\dagger)\).

Proof. We only show \((\lambda, \rho)\mathcal{L}^{**}(\lambda^*, \rho^*)\) since \((\lambda, \rho)\mathcal{R}^{**}(\lambda^\dagger, \rho^\dagger)\) can be similarly obtained. For this, it suffices to show that

\[
(*) \quad (\lambda, \rho)(\lambda_1, \rho_1)\mathcal{R}(\lambda, \rho)(\lambda_2, \rho_2) \Leftrightarrow (\lambda^*, \rho^*)(\lambda_1, \rho_1)\mathcal{R}(\lambda^*, \rho^*)(\lambda_2, \rho_2)
\]

for all \((\lambda_i, \rho_i) \in \Omega(S)\) with \(i = 1, 2\). Note that \(\mathcal{R}\) is a left congruence and by Lemma 3.8, we observe that if \((\lambda^*, \rho^*)(\lambda_1, \rho_1)\mathcal{R}(\lambda^*, \rho^*)(\lambda_2, \rho_2)\), then

\[
(\lambda, \rho)(\lambda_1, \rho_1) = (\lambda, \rho)[(\lambda^*, \rho^*)(\lambda_1, \rho_1)]\mathcal{R}(\lambda, \rho)[(\lambda^*, \rho^*)(\lambda_2, \rho_2)] = (\lambda, \rho)(\lambda_2, \rho_2).
\]

Now, to verify the implication \((*)\), we need only to prove that

\[
(**) \quad (\lambda, \rho)(\lambda_1, \rho_1)\mathcal{R}(\lambda, \rho)(\lambda_2, \rho_2) \Rightarrow (\lambda^*, \rho^*)(\lambda_1, \rho_1)\mathcal{R}(\lambda^*, \rho^*)(\lambda_2, \rho_2).
\]

Finally, we prove the implication \((***)\). Now let \((\lambda, \rho)(\lambda_1, \rho_1)\mathcal{R}(\lambda, \rho)(\lambda_2, \rho_2)\) for \((\lambda_1, \rho_1), (\lambda_2, \rho_2) \in \Omega(S)\). Then there exist \((\lambda_3, \rho_3), (\lambda_4, \rho_4) \in \Omega(S)\) such that \((\lambda, \rho)(\lambda_1, \rho_1) = (\lambda, \rho)(\lambda_2, \rho_2)(\lambda_3, \rho_3)\) and \((\lambda, \rho)(\lambda_2, \rho_2) = (\lambda, \rho)(\lambda_1, \rho_1)(\lambda_4, \rho_4)\). Comparing components of these two equalities, we have \(\rho\rho_1 = \rho\rho_2\rho_3\) and \(\rho_2 = \rho\rho_1\rho_4\). By the prior equality, \(g\rho\rho_1 = g\rho\rho_2\rho_3\) for all \(g \in E(S)\), which implies that \((g\rho)(g\rho)^*\rho_1 = g\rho\rho_1 = g\rho\rho_2\rho_3 = (g\rho)(g\rho)^*\rho_2\rho_3\), thereby \((g\rho)(g\rho)^*\rho_1\mathcal{R}(g\rho)(g\rho)^*\rho_2\rho_3\) since \((g\rho)(g\rho)^*\mathcal{L}^{**}g\rho\). This means that

\[
(***) \quad (\lambda e)^\dagger\rho_1 = ((\lambda e)^\dagger\rho)^*\rho_2\rho_3x
\]

for some \(x \in S^1, e \in E(S)\). On the other hand, we have \(((\lambda e)^\dagger\rho)e = (\lambda e)^\dagger(\lambda e) = \lambda e\), and by Lemma 2.3 (2),

\[
(\dagger) \quad (\lambda e)^* = ((\lambda e)^\dagger\rho)^*e.
\]
Now, we have

$$e^*\rho_1 = e(\lambda^*)e = (\lambda^*)e$$

(by $\dagger$)

$$= (e((\lambda^*)^e)\rho_1 = e(((\lambda^*)^e)\rho_1)$$

(by $\ast\ast\ast$)

$$= (\lambda^*)e\rho_2\rho_3\rho_x$$

(by $\dagger$)

$$= (\lambda^*)e\rho_2\rho_3\rho_x$$

(by Lemma 3.3)

where $\rho_x$ is the inner right translation on $S$ determined by $x$, and by Lemma 3.1, $\rho_1 = \rho^*\rho_2\rho_3\rho_x$; similarly, by $\rho_2 = \rho\rho_4$, we have $\rho^*\rho_2 = \rho^*\rho_1\rho_4\rho_y$ for some $y \in S^1$. Therefore by Lemma 3.2, $(\lambda^*\lambda_1, \rho^*\rho_1) = (\lambda^*\lambda_2\lambda_3\lambda_x, \rho^*\rho_2\rho_3\rho_x)$ and $(\lambda^*\lambda_2, \rho^*\rho_2) = (\lambda^*\lambda_1\lambda_4\lambda_x, \rho^*\rho_1\rho_4\rho_y)$. In other words,

$$(\lambda^*, \rho^*)(\lambda_1, \rho_1) = (\lambda^*, \rho^*)(\lambda_2, \rho_2)[(\lambda_3, \rho_3)(\lambda_x, \rho_x)]$$

and

$$(\lambda^*, \rho^*)(\lambda_2, \rho_2) = (\lambda^*, \rho^*)(\lambda_1, \rho_1)[(\lambda_4, \rho_4)(\lambda_y, \rho_y)].$$

Consequently, $(\lambda^*, \rho^*)(\lambda_1, \rho_1)R(\lambda^*, \rho^*)(\lambda_2, \rho_2)$. We have proved the implication ($\ast\ast$), as required.

**Proof of Theorem 1.1:** Let $S$ be an inverse wpp semigroup. Then by Lemma 3.6 and 3.9, $\Omega(S)$ is wpp, and by Lemma 3.7, $\Omega(S)$ is an inverse wpp semigroup. This completes the proof.

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