The Schur Multiplicative Convexity of the Weighted Generalized Logarithmic Mean in $n$ Variables

Weifeng Xia
School of Educational Science and Technology
Huzhou Teachers College, Huzhou 313000, P.R. China

Yuming Chu
Department of Mathematics
Huzhou Teachers College, Huzhou 313000, P.R. China

Abstract

In this paper, we investigate the Schur multiplicative convexity of the weighted generalized logarithmic mean in $n$ variables, and prove that $L_r(x)$ is Schur multiplicatively convex if $r \geq 2$ or $r \leq 0$ and $F_r(x)$ is Schur multiplicatively convex if $r \geq \frac{1}{2}$ or $r = 0$.

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1. Introduction

Throughout the paper, For $n \geq 2$, $n \in \mathbb{N}$, $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in R_+^n$, and $\alpha \in R$, we denotes by $\log x = (\log x_1, \ldots, \log x_n)$, $x^\alpha = (x_1^\alpha, \ldots, x_n^\alpha)$, $xy = (x_1y_1, \ldots, x_ny_n)$ and $e^x = (e^{x_1}, \ldots, e^{x_n})$. Let $E_{n-1} \subset R^{n-1}$ be the simplex $E_{n-1} = \{(u_1, \ldots, u_{n-1}) : u_i > 0 (1 \leq i \leq n - 1), \sum_{i=1}^{n-1} u_i \leq 1\}$, and let $du = du_1 \cdots du_n$ be the differential of the volume in $E_{n-1}$.

The weighted arithmetic mean $A(x, u)$ and the power mean $M_r(x, u)$ of order $r$ with respect to $x_1, \ldots, x_n$ and the positive weights $u_1, \ldots, u_n$ with $\sum_{i=1}^{n} u_i = 1$ are defined, respectively, as $A(x, u) = \sum_{i=1}^{n} u_i x_i$, $M_r(x, u) = \left(\sum_{i=1}^{n} u_i x_i^r\right)^{\frac{1}{r}}$ for $r \neq 0$, and $M_0 = \prod_{i=1}^{n} u_i^{-u_i}$.

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Corresponding author. E-mail address : chuyuming@hutc.zj.cn
For $x_1, x_2 > 0$, the well-known logarithmic mean $L(x_1, x_2)$ of $x_1, x_2$ is defined as

$$L(x_1, x_2) = \begin{cases} \frac{x_1 - x_2}{\log x_1 - \log x_2}, & x_1 \neq x_2, \\ x_1, & x_1 = x_2. \end{cases} \quad (1.1)$$

The logarithmic mean has numerous applications in physics. Many properties and inequalities are obtained by many mathematicians (see [4,10,13,18,20,21]).

As generalization of $L(x_1, x_2)$, Pittenger [17] and Pearce et al. [16] defined the weighted generalized logarithmic mean $L_r(x)$ and $F_r(x)$, respectively, as follows

$$L_r(x) = \begin{cases} ((n - 1)! \int_{E_{n-1}} (A(x,u))^r du)^{\frac{1}{r}}, & r \neq 0, \\ \exp((n - 1)! \int_{E_{n-1}} \log A(x,u) du), & r = 0. \end{cases} \quad (1.2)$$

$$F_r(x) = (n - 1)! \int_{E_{n-1}} M_r(x,u) du, \quad (1.3)$$

where $u_n = 1 - \sum_{i=1}^{n-1} u_i$. It is obviously that $L_r(x)$ and $F_r(x)$ are symmetric with respect to $x_1, \cdots, x_n$ and continuous with respect to $r$ for all $x \in R^n$.

In the case $n = 2$, Stolarsky [19] and Alzer [1,2] investigate $L_r(x_1, x_2)$ and $F_r(x_1, x_2)$. In [17], Pittenger proved that $L_r(x)$ was increasing with respect to $r$, and Pearce et al. obtained several properties of $F_r(x)$ in [16]. In [22], Zheng et al. studied the Schur convexities of $L_r(x)$ and $F_r(x)$. In this paper, we investigate the Schur multiplicative convexities of $L_r(x)$ and $F_r(x)$.

For convenience of readers, we recall some definitions as follows.

**Definition 1.1.** A set $E_1 \subseteq R^n$ is called a convex set if $\frac{x+y}{2} \in E_1$ whenever $x, y \in E_1$. A set $E_2 \subseteq R^n_+$ is called a multiplicatively convex set if $x^\frac{1}{2}y^\frac{1}{2} \in E_2$ whenever $x, y \in E_2$.

It is easy to see that $E \subseteq R^n_+$ is a multiplicatively convex set if and only if $\log E = \{ \log x : x \in E \}$ is a convex set, and $F \subseteq R^n$ is a convex set if and only if $e^F = \{ e^x : x \in F \}$ is a multiplicatively convex set.

**Definition 1.2.** Let $E \subseteq R^n$ be a convex set. A function $f : E \rightarrow R$ is said to be a convex function on $E$ if $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in E$. Moreover, $f$ is called a concave function if $-f$ is a convex function.

**Definition 1.3.** Let $E \subseteq R^n_+$ be a multiplicatively convex set. A function $f : E \rightarrow (0, \infty)$ is called a multiplicatively convex function on $E$ if $f(x^\frac{1}{2}y^\frac{1}{2}) \leq f^\frac{1}{2}(x)f^\frac{1}{2}(y)$ for all $x, y \in E$. We say $f$ is a multiplicatively concave function if $\frac{1}{f}$ is a multiplicatively convex function.
Definition 1.2 and 1.3 have the following consequences.

Remark 1.1. If \( E_1 \subseteq R^n_+ \) is a multiplicatively convex set and \( f : E_1 \to (0, \infty) \) is a multiplicatively convex function, then \( F(x) = \log f(e^x) : \log E_1 \to R \) is a convex function. Conversely, if \( E_2 \) is a convex set and \( F : E_2 \to R \) is a convex function, then \( f(x) = e^{F(\log x)} : e^{E_2} \to (0, \infty) \) is a multiplicatively convex function.

Definition 1.4. Let \( E \subseteq R^n_+ \) be a set. A function \( F : E \to R \) is called a Schur convex function on \( E \) if \( F(x) \leq F(y) \) for each pair of \( n \)-tuples \( x = (x_1, \cdots, x_n) \) and \( y = (y_1, \cdots, y_n) \) in \( E \), such that \( x \prec y \), i.e. \( \sum_{i=1}^k x[i] \leq \sum_{i=1}^k y[i] \) and \( \sum_{i=1}^n x[i] = \sum_{i=1}^n y[i] \), where \( 1 \leq k \leq n-1 \) and \( x[i] \) denotes the \( i \)th largest component in \( x \). A function \( F \) is called a Schur concave function if \( -F \) is a Schur convex function.

Definition 1.5. Let \( E \subseteq R^n_+ \) be a set. A function \( F : E \to (0, \infty) \) is called a Schur multiplicatively convex function on \( E \) if \( F(x) \leq F(y) \) for each pair \( x = (x_1, \cdots, x_n) \) and \( y = (y_1, \cdots, y_n) \) in \( E \), such that \( \log x \prec \log y \). And \( F \) is called Schur multiplicatively concave if \( \log F \) is Schur multiplicatively convex.

Remark 1.2. Let \( E \subseteq R^n_+ \) be a set, and \( H = \log E = \{ \log x : x \in E \} \). Then \( f : E \to (0, \infty) \) is a Schur multiplicatively convex(or Schur multiplicatively concave, respectively) function on \( E \) if and only if \( \log f(e^x) \) is a Schur convex(or Schur concave, respectively) function on \( H \).

The theory of Schur convex functions in the sense of arithmetic mean is one of the most important theory in the fields of modern analysis and geometry. It can be used in global Riemannian geometry [7,8], operator inequalities [3], nonlinear PDE of elliptic type [11], combinatorial optimization [9], graphs and metrics [5], inequalities and extremal problems [6] and other related fields.

The notation of multiplicative convexity was first introduced by P.Montel [14], in a beautiful paper where the analogues of the notion of convex function in \( n \) variables are discussed. Once upon a time, the theory of multiplicative convexity seemed to be hidden, which is a pity because of its richness. In 2000, C.P.Niculescu [15] discussed an attractive class of inequalities, which arise from the notion of multiplicatively convex functions.

Our aim in what follows is to prove the following two results.

**Theorem 1.1.** For \( r \in R \), if \( r \geq 2 \) or \( r \leq 0 \), then \( L_r(x) \) is Schur multiplicatively convex.

**Theorem 1.2.** For \( r \in R \), if \( r \geq \frac{1}{2} \) or \( r = 0 \), then \( F_r(x) \) is Schur multiplicatively convex.

2. Lemmas and Proof of Theorem
In this section we first introduce and establish two lemmas, which are used in the proof of Theorem 1.1 and 1.2.

**Lemma 2.1.** (see [12]) Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior $\text{int}E$ and $\varphi : E \to \mathbb{R}$ be a continuous symmetric function on $E$. If $\varphi$ is differentiable on $\text{int}E$, then $\varphi$ is Schur convex (or Schur concave respectively) on $E$ if and only if

$$(x_i - x_j)(\frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_j}) \geq 0 \text{ (or } \leq 0, \text{ respectively)}$$

for $i, j = 1, 2, \ldots, n$ and all $x = (x_1, \ldots, x_n) \in \text{int}E$.

**Lemma 2.2.** Let $E \subseteq \mathbb{R}^n_+$ be symmetric multiplicatively convex set with nonempty interior $\text{int}E$ and $\varphi : E \to (0, \infty)$ be a continuous symmetric function on $E$. If $\varphi$ is differentiable on $\text{int}E$, then $\varphi$ is Schur multiplicatively convex (or Schur multiplicatively concave, respectively) on $E$ if and only if

$$(\log x_i - \log x_j)(x_i \frac{\partial \varphi}{\partial x_i} - x_j \frac{\partial \varphi}{\partial x_j}) \geq 0 \text{ (or } \leq 0, \text{ respectively)}$$

for $i, j = 1, 2, \ldots, n$ and all $x = (x_1, \ldots, x_n) \in \text{int}E$.

**Proof.** Lemma 2.2 follows from Lemma 2.1 and Remark 1.2 together with the elementary computation.

**Proof of Theorem 1.1.** We use Lemma 2.2 to discuss the nonegativity and nonpositivity of $(\log x_i - \log x_j)(x_i \frac{\partial L_r(x)}{\partial x_i} - x_j \frac{\partial L_r(x)}{\partial x_j})$ for all $x \in \mathbb{R}^n_+$ and for $r \in \mathbb{R}$. Since $L_r(x)$ is symmetric with respect to $x_1, \ldots, x_n$, without loss of generality we only need to discuss the nonegativity and nonpositivity of $(\log x_1 - \log x_2)(x_1 \frac{\partial L_r(x)}{\partial x_1} - x_2 \frac{\partial L_r(x)}{\partial x_2})$ for all $x \in \mathbb{R}^n_+$ and for $r \in \mathbb{R}$. We divide our proof into two cases.

**Case 1.** $r \neq 0$. Let $u = (u_1, u_2, \ldots, u_n)$ and $\tilde{u} = (u_2, u_1, \ldots, u_n)$, then by the symmetry of $L_r(x)$ with respect to $x_1, \ldots, x_n$, we have

$$g_r(x) \triangleq \int_{E_{n-1}} (A(x, u))^r du = \int_{E_{n-1}} (A(x, \tilde{u}))^r du. \quad (2.1)$$

Therefore,

$$x_1 \frac{\partial g_r(x)}{\partial x_1} - x_2 \frac{\partial g_r(x)}{\partial x_2} = r \int_{E_{n-1}} u_1 [x_1 (A(x, u))^{r-1} - x_2 (A(x, \tilde{u}))^{r-1}] du. \quad (2.2)$$
By Lagrang’s mean value theorem and simple computation yields
\[
x_1(A(x, u))^{r-1} - x_2(A(x, \tilde{u}))^{r-1}
= \left( \frac{A(x, u)}{x_1^{r-1}} \right)^{r-1} - \left( \frac{A(x, \tilde{u})}{x_2^{r-1}} \right)^{r-1}
= (r - 1) \left( u_1 x_1 x_2^{1-r} + u_2 x_2^2 - x_2^r \sum_{i=3}^{n} u_i x_i - u_1 x_1^{1-r} x_2 - u_2 x_2^{2-r} \right)
\]
\[
- x_1^{1-r} \sum_{i=3}^{n} u_i x_i \times \left( x_1^{r-1} - x_2^{r-1} \theta_1^{-2} \right)
\]
\[
= (r - 1) \left[ u_1 x_1 x_2 (x_2^{1-r} - x_1^{1-r}) + u_2 (x_2^{2-r} - x_1^{2-r}) + \left( x_2^{1-r} - x_1^{1-r} \right) \sum_{i=3}^{n} u_i x_i \right]
\times \left( x_1^{r-1} - x_2^{r-1} \theta_1^{-2} \right)
\]
\[
= (r - 1) \left( x_1 - x_2 \right) \left[ ru_1 x_1 x_2 \theta_2^{r-1} + (r - 2) u_2 \theta_3^{1-r} + \left( r - 1 \right) \sum_{i=3}^{n} u_i x_i \theta_4^{-r} \right]
\times \left( x_1^{r-1} - x_2^{r-1} \theta_1^{-2} \right),
\] (2.3)

where \( \theta_1 \) is between \( \frac{A(x,u)}{x_1^{1-r}} \) and \( \frac{A(x,u)}{x_2^{1-r}} \), \( \theta_2, \theta_3, \theta_4 \) are between \( x_1 \) and \( x_2 \).

Making use of (2.1)-(2.3) and (1.2) we can get
\[
(\log x_1 - \log x_2) \left( x_1 \frac{\partial L_r(x)}{\partial x_1} - x_2 \frac{\partial L_r(x)}{\partial x_2} \right)
= (n - 1)! \frac{1}{r} (L_r(x))^{1-r} \left( \log x_1 - \log x_2 \right) (x_1 \frac{\partial g_r(x)}{\partial x_1} - x_2 \frac{\partial g_r(x)}{\partial x_2})
\]
\[
= (n - 1)! (L_r(x))^{1-r} \left( \log x_1 - \log x_2 \right) \frac{x_1 - x_2}{x_1 x_2^{1-r}} (r - 1) \left( \sum_{i=3}^{n} u_i x_i \theta_4^{-r} \right)
\]
\[
[r u_1 x_1 x_2 \theta_2^{r-1} + (r - 2) u_2 \theta_3^{1-r} + (r - 1) \sum_{i=3}^{n} u_i x_i \theta_4^{-r}] \theta_1^{-2} du.
\] (2.4)

If \( r \geq 2 \) or \( r < 0 \), then from (2.4) and Lemma 2.2 we show that \( L_r(x) \) is Schur multiplicatively convex.

**Case 2.** \( r = 0 \). Then (1.2) lead to the following identity
\[
(\log x_1 - \log x_2) \left( x_1 \frac{\partial L_r(x)}{\partial x_1} - x_2 \frac{\partial L_r(x)}{\partial x_2} \right)
= (n - 1)! L_r(x) \left( \log x_1 - \log x_2 \right) (x_1 - x_2) \int_{E_{n-1}} u_1 u_2 (x_1 + x_2)
\]
\[
+ \sum_{i=3}^{n} u_i x_i \left( A(x,u)A(x,\tilde{u}) \right)^{-1} du.
\] (2.5)
It is obviously to see that $L_r(x)$ is Schur multiplicatively convex for $r = 0$ by lemma 2.2 and (2.5).

**Proof of Theorem 1.2.** By similar discuss as in proof of Theorem 1.1, we only need to discuss the the nonegativity and nonpositivity of $(\log x_1 - \log x_2)(x_1 \frac{\partial F_r(x)}{\partial x_1} - x_2 \frac{\partial F_r(x)}{\partial x_2})$ for all $x \in R^n$ and for $r \in R$. We also divide our proof into two cases.

**Case 1.** $r \neq 0$. Then (1.3) leads to the following identity

$$x_1 \frac{\partial F_r(x)}{\partial x_1} - x_2 \frac{\partial F_r(x)}{\partial x_2} = (n-1)! \int_{E_{n-1}} u_1 \left[ \left( \frac{M_r(x, u)}{x_1^{\frac{r}{1-r}}} \right)^{1-r} - \left( \frac{M_r(x, \bar{u})}{x_2^{\frac{r}{1-r}}} \right)^{1-r} \right] du. \quad (2.6)$$

Now by Lagrang’s mean value theorem together with simple computation lead to

$$\left( \frac{M_r(x, u)}{x_1^{\frac{r}{1-r}}} \right)^{1-r} - \left( \frac{M_r(x, \bar{u})}{x_2^{\frac{r}{1-r}}} \right)^{1-r} = (1-r) \left[ \left( \frac{u_1 x_1^r + u_2 x_2^r + \sum_{i=3}^{n} u_i x_i^r}{x_1^{\frac{r}{1-r}}} \right) - \left( \frac{\sum_{i=3}^{n} u_i x_i^r}{x_2^{\frac{r}{1-r}}} \right) \right]^{\frac{1}{r}} \eta_1^{-r}$$

$$= \frac{1-r}{r} \left( u_1 x_1^2 x_2^{r-2} + u_2 x_2^{r+2} + \sum_{i=3}^{n} u_i x_i^r - u_1 x_1^r x_2^{2} - u_2 x_1^{r+2} - \sum_{i=3}^{n} u_i x_i^r \right) \eta_1^{-r}$$

$$= (x_1 - x_2) \left[ u_1 x_1^r x_2^r \eta_5^{\frac{r}{r-2}} + u_2 (2r-1) \eta_4^{\frac{r}{r-2}} + \sum_{i=3}^{n} u_i x_i^r \eta_5^{\frac{2r-2r+1}{r-2}} \right] \eta_1^{-r} \eta_2^{-1} \eta_2^{-1}$$

Here $\eta_1$ is between $(M_r(x, u)) x_1^{\frac{r}{1-r}}$ and $(M_r(x, \bar{u})) x_2^{\frac{r}{1-r}}$, $\eta_2$ is between $(M_r(x, u))^r x_1^{\frac{r}{r-1}}$ and $(M_r(x, \bar{u}))^r x_2^{\frac{r}{r-1}}$, $\eta_3, \eta_4, \eta_5$ are between $x_1$ and $x_2$. 

From (2.6)-(2.7) we know that
\[
(\log x_1 - \log x_2)(x_1 \frac{\partial F_r(x)}{\partial x_1} - x_2 \frac{\partial F_r(x)}{\partial x_2})
= (n - 1)!(\log x_1 - \log x_2)(x_1 - x_2)\int_{E_{n-1}} u_1 \left[ u_1 x_1^r x_2^r \eta_3^{-1} + u_2(2r - 1)
\times \eta_4^r - \eta_5^{-r}\right] (x_1 x_2)^{1-r-1} \eta_2^{-1} \eta_1^{-r} du.
\]
(2.8)

If \( r \geq \frac{1}{2} \), then from (2.8) and Lemma 2.2 we know that \( F_r(x) \) is Schur multiplicatively convex.

**Case 2.** \( r = 0 \). Then (1.3) and simple computation lead to \( (\log x_1 - \log x_2)(x_1 \frac{\partial F_r(x)}{\partial x_1} - x_2 \frac{\partial F_r(x)}{\partial x_2}) = 0 \), hence \( F_r(x) \) is Schur multiplicatively convex for \( r = 0 \) by Lemma 2.2.

**References**


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