State of the Art of
Compactness and Circularity Measures

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Abstract

Compactness is an intrinsic property of objects. This feature is often associated with the old and classical ratio \( \text{perimeter}^2/\text{area} \). This ratio is used in many scientific fields as shape descriptor in shape analysis tasks. However, when we use this ratio for measuring shape compactness in the digital domain some problems arise. Several approaches have been proposed for measuring shape compactness of discrete regions. This paper presents a review of several methods for measuring shape circularity and compactness and surveys some of them when scaling transformation is applied to discrete regions. Also, we present a comparison of the studied methods using the same set of examples of binary shapes under different conditions: shapes with holes and with noisy perimeters.

Mathematics Subject Classification: 68T45

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1 Introduction

The representation, analysis, and description of the shapes of objects are important topics in pattern recognition and computer vision. Even though several approaches for describing the shape of objects have been developed in past decades [33, 29, 42], one simple shape descriptor has been widely used in pattern recognition tasks: shape compactness. Shape compactness is an intrinsic characteristic of the shape of objects. The importance of shape compactness and its utility as shape descriptor has been demonstrated both in computer vision [22, 28, 34, 16] and psychological studies [1, 43]. The shape compactness is frequently included as shape descriptor in diverse shape analysis processes [35, 30, 15, 5] and it has been used as the main property to evaluate morphological change in biological entities [31, 7].

Shape compactness is usually associated with the old ratio:

\[ C = \frac{P^2}{A} \]  

Where \( C \) is the value of shape compactness, \( A \) is the shape area and \( P \) is the shape perimeter. This way to measure shape compactness is taken from the isoperimetric inequality.

\[ P^2 \geq 4\pi A \]  

The isoperimetric inequality has been used since Babylonian time and this inequality establishes the isoperimetric problem: What is the domain of a prescribed perimeter enclosing the largest area?. The expression (1) is called shape compactness [2, 33, 18, 8, 40, 39, 26], circularity [10, 28, 11, 38], dispersion [1, 19] or roundness measure [23], and sometimes it is expressed as shape factor [12, 25] or \( P^2A \). The Greeks used and studied the isoperimetric inequality and knew that a circle is the geometric shape that, for a prescribed perimeter, encloses the largest area [3]. Therefore, a logical consequence is that the circle is the most compact shape in the continuous plane. Furthermore, if (1) is used as compactness measure, then we are comparing an object shape with a circle that has an equal perimeter. Although the \( P^2A \) is the most widely used compactness measure [13], there are more than 20 proposals to evaluate shape compactness in the Euclidean plane [32, 21]. In the case of the continuous space, the concept of shape compactness is commonly associated with the volumetric version of the isoperimetric inequality

\[ A^3 \geq 36\pi V^2 \]  

Where \( A \) is the enclosing surface of a 3-Dimensional object and \( V \) is its volume.
Both isoperimetric radii, when they are used as measures of shape compactness, are invariant under translation, rotation, and scaling transformations, dimensionless and minimized by a circle and sphere, respectively. Furthermore, the process of normalized shape compactness measure to be unity for a circle is simple and is given by

\[ C = \frac{P^2}{4\pi A} \]  

This idea of measuring circularity or compactness with the isoperimetric ratios is correct in the continuous plane and space. However, these ratios present aberrant values of compactness and circularity when are tested on roughness shapes or fractals. In Figure 1, examples of P2A behavior are shown. Fig. 1(a) present a circle, Fig. 1(b) a circle with an irregular border and Fig. 1 (c) a square. All figures show their area, perimeter, and compactness values. Notice that shape areas are very similar among these shapes. However, their compactness values are very different. Indeed, the square is more compact than the roughness circle. These figures show the enormous influence of the perimeter on P2A.

Shape compactness and circularity in the continuous plane and 3D shape compactness and sphericity in the Euclidean spaces are the same concepts. However, shape compactness, circularity and sphericity have different meaning in the digital world [36, 20, 8]. Rosenfeld showed that the most compact shape in the digital domain is not a circle [36]. Hence, the measure of shape compactness and circularity are different concepts.

Furthermore, when P2A is applied on a digital region, which is formed by a set of discrete element or cells (called pixels), the evaluation of the length of a border of a digital region is the main problem. According to the specialized literature, the perimeter of a digital region may be measured by three methods [36, 20, 40, 18]: the sum of the length of edge of the cell on border of a digital region; the sum of border cells; and the sum of all distances between the centers of two adjacent boundary pixels. Each of this methods yield a different border length. Therefore, P2A has three different values of a digital region, depending
upon how the perimeter of a digital region is measured [36, 20]. In consequence, the value of compactness of the shape is not unique.

Several approaches have been designed to measure digital shape compactness and circularity. These measures try to overcome $P2A$ drawbacks without losing their properties that have in Euclidean plane and space. In the specialized literature of computer vision, we can find eleven proposals to measure digital shape compactness and circularity. We may group ten of these eleven proposals in three main classes: inner shape distances [20, 12, 41, 14], reference shape [24, 6, 34] and geometric pixel [8, 4]. In the content of this work, the state-of-the-art of compactness and circularity of digital regions is reviewed, and the eleven digital compactness and circularity measures are described. Moreover, a comparative study of invariant scaling digital measures is included.

The outline of this work can be summarized as follows: In Section 2, the different measuring compactness and circularity approaches are described. At the end of Section 2, we show two measures of compactness for 3D-shapes. In Section 3, two comparative study of compactness and circularity are presented. Finally, some conclusions are described in Section 4.

2 State-of-the-art Review

The drawbacks of the classical ratio $P2A$, both on the Euclidean and digital plane and space, have motivated the design of different digital shape compactness and circularity measures. These measures can be grouped in three approaches: inner shape distance, reference shape, and geometric pixel properties.

In this section, we describe the eleven circularity and compactness measures divided into the inner distances, reference shape, and geometric pixel properties. Within each approach, the different measures are described in chronological order. Some of those measures are tested using a digital region set, shown in Fig. 2, and their circularity or compactness values being given in Table 1.

2.1 Inner Distance Approach

The approach based on the measurement of distances of the interior of a digital region has been used to calculate either shape digital compactness or shape circularity. We found two proposals to measure both digital compactness and shape circularity. In chronological order, the first measure based on circularity was proposed by Haralick [20]. That author calculated a shape centroid, and measured all the Euclidean distances from the centroid to each boundary pixel. With this set of distances, the media ($\mu$) and the standard deviation ($\sigma$) were
Fig 2: Set of test digital region
calculated. These statistical parameters are used on a ratio that calculates the circularity, $C$, of a shape. This circularity measure is given by:

$$C = \mu_R / \sigma_R.$$  

(5)

Where $\mu_R$ and $\sigma_R$ are the mean and standard deviation, respectively, of the distances between the centroid of a shape and its boundary pixels. The circularity value of $C$ grows whether the digital shape is more similar to a circle. We used the algorithm proposed in [38] to test this circularity measure on the set of digital shapes illustrated in Fig. 2 and the table 1 shows the obtained results. The Fig. 2(d), a digitized circle has the highest value of this measure. The least circular shapes are figures 2(h) and (i). However, the Fig. 2(f) a digitized rectangle is more compact than the Fig. 2(c), a circle with one hole. Finally, the squares are more circular than the digitized character C.

Although this measure does not include explicitly the shape perimeter, this parameter has a strong influence on its value if the digital resolution changes. Therefore, this circularity measure will be sensitive to shape resolution.

Four years later, another $P2A$ substitute measure appeared, the shape factor $G$ [12]. This shape descriptor was defined in such a way as to obtain a quantitative compactness value of a shape. The shape factor $G$ uses the average distance between each picture element and its nearest border, $\overline{d}$. This distance is defined by:

$$\overline{d} = \sum_{i=0}^{N} \frac{x_i}{N}.$$  

(6)

Where $x_i$ is the value generated by the well-known discrete distance transform, and $N$ is the number of region pixels. To calculate the shape factor, $G$, a ratio is defined between the average distance, $\overline{d}$, and the area of the studied digital region which is given by:

$$G = A / 9\pi(\overline{d})^2.$$  

(7)

In the Euclidean plane, if the shape is a circle, then the minimum value of the shape factor, $G$, is equal to one. This shape descriptor is dimensionless, and much more robust to irregular contours than $P2A$. However, when the shape factor, $G$, was applied to the set of digital tested regions presented in Fig. 2, the digital crosses yielded the minimum value of $G$ and turned out to be less than one. The digital squares proved more compact than the digitalized circle. The resolution of digital regions showed itself to be the main factor in modifying the value of shape descriptors as the $G$ measure. This effect is demonstrated by the digital squares (figures 2(a) and (g)), where the $G$ value has an important variation. Nevertheless, one advantage of this shape descriptor is that it can be applied on digital regions with holes.
In 1983, another digital compactness measure was presented [41]. In that work, the approach of inner distances is used to define a different way to measure distances into a shape. The proposal involved the measurement of the length of a set of line segments with an angle $\phi$, with respect to a horizontal axis which connects opposite border points of a digital region. With continuous variations in the angle $\phi$, a new set of distances was obtained. Each set of distances was used to set up a map of distances. In Fig. 3, a digital region and all its distance border-to-border maps are shown (figures 3(a) to (d)).

The set of distance maps is used to set up two new distance maps, called the maximum and minimum distance border-to-border maps, presented in Fig. 3(e) and (f). These maps are used to calculate the mean of the distance values on each distance map, called $\overline{d}_{\text{min}}$ and $\overline{d}_{\text{max}}$, respectively. These new parameters are used to obtain two quantitative values of shape compactness: $f_1$ and $f_2$, which are defined by:

$$f_1 = c_1(A/(\overline{d}_{\text{min}})^2)$$ (8)
\[ f_2 = c_2 \left( A / (\bar{d}_{\text{max}})^2 \right). \]  

Where \( A \) is the area of the digital shape, and \( c_1 \) and \( c_2 \) are constants equal to 0.6122 and 1.2388, respectively.

Finally, Di Ruperto and Dempster proposed a set of circularity measures in [14]. They used a mathematical morphology approach to design four ratios as digital circularity measures. The first circularity measure was denominated \( M \), and is given by:

\[ M = \sum_{i=1}^{L} \left( \text{var} \{ d_{i,j} \} / \text{max} \{ d_{i,j} \} \right). \]  

To calculate this measure, a digital region is eroded with a disk as structure element, \( nB \), where \( n \) is the diameter of disk \( B \). After the last erosion, a maximal region, or a set of maximal regions, is found. Once the maximal region or regions is found, a set of distances \( d_{i,j} \), is calculated from the maximal region \( i \), in several directions \( j \), 8 or 16 cardinal directions, to the border of the digital region. The variance of the distances \( d_{i,j} \) is \( \text{var} \{ d_{i,j} \} \) and \( \text{max} \{ d_{i,j} \} \) is the maximal distance found for each maximal region.

The second proposed circularity measure was \( V \). This measure is calculated as the sum of each distance transform values, divided by the cube of the maximum distance transform value, \( h \). This measure is calculated by:

\[ V = \sum_{p \in X} \frac{\text{dist}_X(p)}{h^3}. \]  

The third measure to calculate shape circularity is the \( T \) measure, which uses the area, \( A \), of a digital region. \( A \) is divided by the square of the maximum distance transform value, \( h \), and it is defined by:

\[ T = A / h^2. \]  

Finally, the circularity measure, \( E \), is the ratio between the maximum disk that can be contained in a digital region and the root of its area, \( A \), which is given by:

\[ E = n / \sqrt{A}. \]  

According to Di Ruperto and Dempster, the best results were produced by the measure, \( M \), which is invariant under geometric transformations. This measure evaluates the deformity of a shape by the variance of \( d_{i,j} \) distances, where the minimum value of \( M \) is zero if a shape is a circle. The measure, \( M \), was implemented in this paper using the digital distance transform, and eight cardinal directions were evaluated to obtain the variance value of each maximal region. In Table 1, the circularity values of the digital region set
State of the art of compactness and circularity measures

<table>
<thead>
<tr>
<th>Region</th>
<th>Area</th>
<th>Perimeter</th>
<th>P2A</th>
<th>( P_{2A} )</th>
<th>( C_{DN} )</th>
<th>Haralick's C</th>
<th>G</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>225</td>
<td>60</td>
<td>16</td>
<td>1</td>
<td>8.9335</td>
<td>0.8712</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>199</td>
<td>112</td>
<td>63.035</td>
<td>0.8382</td>
<td>6.957</td>
<td>1.1849</td>
<td>0.134</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>84</td>
<td>56</td>
<td>37.333</td>
<td>0.8549</td>
<td>3.4182</td>
<td>1.6711</td>
<td>20.54</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>89</td>
<td>44</td>
<td>21.752</td>
<td>0.9559</td>
<td>11.831</td>
<td>0.9270</td>
<td>0.166</td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>63</td>
<td>52</td>
<td>42.92</td>
<td>0.7896</td>
<td>3.1298</td>
<td>1.3818</td>
<td>1.27</td>
<td></td>
</tr>
<tr>
<td>(f)</td>
<td>50</td>
<td>30</td>
<td>18</td>
<td>0.9767</td>
<td>3.4719</td>
<td>0.6907</td>
<td>1.59</td>
<td></td>
</tr>
<tr>
<td>(g)</td>
<td>25</td>
<td>20</td>
<td>28.8</td>
<td>0</td>
<td>7.5950</td>
<td>0.4511</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(h)</td>
<td>5</td>
<td>12</td>
<td>28.8</td>
<td>0</td>
<td>7.5950</td>
<td>0.4511</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>(i)</td>
<td>9</td>
<td>20</td>
<td>44.44</td>
<td>0</td>
<td>7.5950</td>
<td>0.4511</td>
<td>0.3183</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Table 1: Compactness and circularity values from the digital regions shown in Fig. 1

presented in Fig. 1 are shown. The results show that the digitalized circle (Fig. 2(d)) produces a different value than zero at this resolution. Moreover, the digitized squares shown in figures 2(a) and 1(g) obtained the minimum \( M \) value, which is zero. Generally speaking, if the shape is a square that has odd number of pixels in its sides, then the \( M \) measure value evaluated in 8 cardinal direction will be zero. This measure is a part of the set of digital measures that are tested in Section 3.

### 2.2 Reference Shape Approach

Comparing a shape with a circle is a compactness and circularity measuring approach often used in the Euclidean plane [32, 21]. However, there are only four measures of shape compactness in \( \mathbb{Z}^2 \) that have been developed using this approach [24, 34, 6, 27].

The first measure in this approach was proposed in [24]. This measure compares the area, \( A_p \), of a discrete convex region, \( p \), with the area of the smallest digitalized circle, \( A_{p_0} \), that contains the region, \( p \). This measure is denominated Digital Compactness Measure, \( m_p \), and is defined by:

\[
m_p = \frac{A_p}{A_{p_0}}. \tag{14}
\]

This approach is illustrated in Fig. 4. Fig. 4(a) and (b) show the continuous and digital versions of a square and their respective circumscribed circles. As it is expressed by (14), the ratio between shape area and its circumscribed circle area is the value of compactness of a digital region. The maximum value of compactness of a digital region is equal to one if the digital shape is a circle. In the case of Fig 4 (b), the ratio is 49/89 area units and \( m_p \) is equal to 0.55. On the other hand, circularity value \( m_p \) for the continuous shapes in Fig 4(a) is 0.63. The digital shape area depends on the cell number of a digital region.
Therefore, from each resolution, we will have a different value of circularity. Another drawback of this measure is that it cannot be applied on digital shapes with holes.

A second measure using this approach is Bottema’s circularity measure, $C$ [6]. This measure intersects two sets of pixels, a digital region, $S$, and a disk, $D$, that can be formed with the same number of pixels from, $S$. Otherwise, the circularity measure, $C$, is zero if the digital shape is a disk. Originally, this manner of calculating the circularity of a shape was proposed in [17], but Bottema formulated an algorithm to implement this measure in $\mathbb{Z}^2$. The circularity measure, $C$, is given by:

$$C = 1 - \frac{|S \cap D|}{S}.$$  \hspace{1cm} (15)

This shape compactness is another compactness measure based on the area of two regions. Then, in the same way as Kim measure, scaling invariant is not supported by the Bottema’s proposal. The localization of the circle, which will be intersected with the digital region, is essential in this compactness measure. As it is illustrated in Fig 5(a), the circle position calculated and the digital shape does not have an intersection area. The Fig 5(b) is another example where the shape compactness can yield erroneous results. In this figure, the intersection area between a digital shape and a digital disk is partial.

A third proposal appeared in [34]. This compactness measure is a ratio between two perimeters. The perimeter of a digital shape with area, $A$, is compared with the perimeter of a circle with the same area in the continuous plane, $A$. This measure is calculated by:
State of the art of compactness and circularity measures

\[ \text{comp} = \frac{P_{\text{circle}}}{P_{\text{shape}}}. \] (16)

The Fig 6 shows a simple example of this circularity measure. These disks have the same area; however, their perimeters are very different. Therefore, the ratio to measure shape circularity cannot yield one as circularity value in this case. Indeed, there does not exit a digital disk that yields one as shape circularity value. In the same way as two measures above mentioned, shape resolution plays an important role for calculating shape circularity value. Additionally, shapes with holes contribute to bad evaluation of circularity.

The last measure in this approach appeared in [27]. This measure was called Digital Compactness, \(C_d\). The measure \(C_d\) is a consequence of a selection process among all possible compactness values of the \(P2A\) that can be calculated with different perimeter definitions (using the border of each boundary pixel, counting the number of boundary pixels and measuring the segments between two adjacent pixels). Thus, the value most similar to the Euclidean value of \(P2A\) is used as a compactness measure.

Using reference shape approach in \(Z^2\) has some drawbacks. First, it is necessary to know the real object, in order to generate a digitalized disk to compare with or calculate the disk according to shape resolution. Second, the generation of digitalized disks adds more complexity with respect to computing shape compactness. This approach is not included in the study outlined in Section 3.
2.3 Geometric pixel property approach

A digital shape can be viewed as a set of cells that form a region on a grid. These cells can be square, hexagonal, or triangular depending on the kind of grid in which the image object is sampled. The first work that used this approach to measuring digital compactness was presented in [8]. This work proposed the Normalized Discrete Compactness, $C_{DN}$.

The $C_{DN}$ is based on a simple concept: counting the number of cell sides that are shared between region cells. This quantitative feature is known as Contact Perimeter or Discrete Compactness, $C_D$. The $C_{DN}$ is given by:

$$C_{DN} = \frac{C_D - C_{D_{\text{min}}}}{C_{D_{\text{max}}}}$$  \hspace{1cm} (17)

where $C_D$ is the perimeter of contact of a digital region. If the pixel cell is a square, then $C_D$ is given by:

$$C_D = \frac{4n - P}{2}$$  \hspace{1cm} (18)

where $n$ is the number of region pixels, and $P$ is the perimeter of the digital region. $C_{D_{\text{min}}}$ and $C_{D_{\text{max}}}$ are the bottom and the top limits of the Contact Perimeter of a shape composed of $n$ number of region cells, respectively. These parameters are calculated by:

$$C_{D_{\text{min}}} = n - 1$$  \hspace{1cm} (19)

and

$$C_{D_{\text{max}}} = \frac{4n - 4\sqrt{n}}{2}$$  \hspace{1cm} (20)
The Normalized Discrete Compactness is the unique measure of the set of measures described in this paper that has been implemented to 3D shapes [9]. The 3D version of $C_{DN}$ is called Discrete Compactness and it is given by:

$$C_D = \frac{A_C - A_{C_{\text{min}}}}{A_{C_{\text{max}}} - A_{C_{\text{min}}}}. \quad (21)$$

where $A_C$ is the contact surface area. If the solid is composed of voxels, then $A_C$ is given by:

$$A_C = \frac{6an - A}{2}. \quad (22)$$

where $a$ is the area of a voxel face, and $n$ is the number of volume voxels. $A_{C_{\text{min}}}$ and $A_{C_{\text{max}}}$ are the minimum and maximum contact surface areas, respectively. They are calculated by:

$$A_{C_{\text{min}}} = a(n - 1) \quad (23)$$

and

$$A_{C_{\text{max}}} = 3(n - (\sqrt[3]{n})^2) \quad (24)$$

Table 1 shows the results of the set of digital regions in Fig 2 for $C_{DN}$ measure, where the rough square is less compact than the circle, the rectangle, or the circle with a hole. Therefore, the $C_{DN}$ is sensitive to rough contours to small resolutions. On the other hand, the digital regions of figures 2(h) and 2(i) obtained the minimum value of $C_{DN}$ without being the most disperse digital shapes. This measure is included into the set of measures tested in Section 3.

In the case of 3D shapes, the figures 7 show $C_{DN}$ and $C$ classical shape compactness behavior. Fig 7 illustrate how compact are two objects. Fig 7(a) shows a bull and its shape compactness values while Fig 7(b) shows human shape.

Finally, four ratios were proposed in [4]. These measures compare the area, $A(a)$, of a digital region, $a$, or its perimeter, $P(a)$, with the area or perimeter of reference regions formed with the same $n$ number of region cells. These measures are defined by:

$$\phi_1 = \frac{A_{\text{min}}(p)}{A(a)} \quad (25)$$

$$\phi_2 = \frac{A(a)}{A_{\text{max}}(p)} \quad (26)$$

$$\phi_3 = \frac{P(a)}{P_{\text{max}}(a)} \quad (27)$$

$$\phi_4 = \frac{P_{\text{min}}(a)}{P(a)} \quad (28)$$

$P_{\text{min}}(a)$ is the perimeter of the most compact shape that can be formed with the same $n$ number of cells as the original shape. This parameter may
Fig 7: Two different 3D shapes and their Discrete and Classical Compactness measures

be calculated in three different ways. First, if there is an integer number \( x_1 \), where \( x_1^2 = A(a) \), then

\[
P_{\min}(a) = 4\sqrt{n}
\]  
(29)

If \( A(a) \leq x_1^2 + x_1 \) then

\[
P_{\min}(a) = 4x_1 + 2
\]  
(30)

Finally, if \( A(a) > x_1^2 + x_1 \) then

\[
P_{\min}(a) = 4(x_1 + 1)
\]  
(31)

\( P_{\max}(a) \) is the perimeter of the most disperse region. This parameter is calculated according to the connectivity used (4 or 8 - connected). If the four - connected is used, then the value of \( P_{\max}(a) \) is defined by:

\[
P_{\max}(a) = 2(A(a) + 1)
\]  
(32)

In the case of eight connected, \( P_{\max}(a) \) is given by

\[
P_{\max}(a) = 4(A(a))
\]  
(33)

For \( A_{\max}(a) \), on shapes with minimum perimeter, \( P_{\min}(a) \), is calculated. If \( A(p) = x_1^2 \) then this parameter is calculate by

\[
A_{\max}(p) = (P(a)/4)^2
\]  
(34)
and

$$A_{\text{max}}(p) = 1/4((P(a)^2/4) - 1)$$  (35)

otherwise.

The $A_{\text{min}}(a)$ is based on the connectivity used. At four-connectivity, this parameter is given by:

$$A_{\text{min}}(p) = (P(a) - 2)/2$$  (36)

And if the eight-connectivity is used, and the shape contact perimeter, $C_D$, is zero, the minimum area is calculated by:

$$A_{\text{min}}(p) = P(a)/4$$  (37)

This set of ratios is very sensitive to the resolution of digital shapes, therefore, they are not invariant under scaling. The best measure of this set is $\phi_4$, however, the authors of this work suggested to use a combination of this measure and the ratio $P2A$ to produce a good measurement of shape compactness.

### 2.4 Sankar’s compactness measure

There is a measure that cannot be classified using the above-mentioned inner shape distances, reference shape and geometric pixel property approaches: the non-compactness measure designed by Sankar [37]. Sankar saw a digital region as a lattice, and used his approach to propose a unique manner to measure the perimeter of a digital region without holes. Using this approach, the area of a digital region, $S$, is defined by:

$$A_p(S) = 1/2(b + i - 1).$$  (38)

Where $b$ is the number of border points, and $i$ is the number of interior points of $S$. The perimeter of the digital region is given by:

$$P_p(S) = 1/2 \sum_j p(X_j).$$  (39)

where $p(X_j)$ is the perimeter value in the point $X_j \in S$. $p(X_j)$, which is calculated by:

$$p(X_j) = \sum_{\Delta \in 0,2,4,6} ((X_j \land X_{j,\Delta})$$

$$\quad - (X_{j,\Delta} \land (X_{j,\Delta-1} \lor X_{j,\Delta-2}) \land (X_{j,\Delta+1} \lor X_{j,\Delta+2})))$$

$$\quad + \sum_{\Delta \in 1,3,5,7} (2)^{\frac{1}{2}} ((X_j \land X_{j,\Delta})$$

$$\quad - (X_{j,\Delta-1} \land X_{j,\Delta} \land X_{j,\Delta+1})).$$  (40)
Where $\land$ and $\lor$ are the Boolean AND and OR operators, respectively. The value of each $X_{j,\Delta}$ is one if $X_{j,\Delta} \in S$ and zero otherwise. Each neighbor of $X_j$ is determined by the relation of neighborhood described below:

$\begin{align*}
X_{j,3} & \quad X_{j,2} \quad X_{j,1} \\
X_{j,4} & \quad X_j \quad X_{j,0} \\
X_{j,5} & \quad X_{j,6} \quad X_{j,7}
\end{align*}$ (41)

The mentioned parameters, area and perimeter, are used in the classical ratio $P2A$. This modification of the classical ratio is described by:

$$\Pi = \frac{P_p(S)^2}{A_p(S)}$$ (42)

This measure cannot be used in regions with holes, thus, this measure does not form part of the set of digital measures tested in Section 3, where these kinds of regions will be used.

3 Comparative Study

Eleven digital proposals to measure the shape circularity or compactness have been described, and some results have been reported. However, it is important to know how effective these measures are in classifying a digital shape based only on the circularity or compactness property, and whether the scale change on digital shape affects these measures. Unfortunately, only four among the eleven measures were designed to be scaling invariant. Because of this, we only tested two digital scaling invariant circularity and two compactness measures.

These digital measures are tested on two sets of digital binary shapes, presented in figures 8 and 9. The first set of shapes has different roundness, while the second group has a different degree of compactness. Both groups of shapes contain elements with holes. The results of this shape analysis are shown in Tables 2 and 3, and their classifications according with circularity and compactness are shown in figures 10 and 11.

If the measure was designed to measure the shape circularity or compactness, then we hope that the most circular or compact digital shape would be classified first. In Section 3.2, these measures were tested on three different digital shapes which have different resolutions. The measures tested in this section were: Danielsson’s Shape Factor $G$; Bribiesca’s Normalized Discrete Compactness; Haralick’s Circularity Measure $C$; and Di Ruperto and Dempster’s $M$ Circularty.
<table>
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<th>Region (Set 1)</th>
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<th>Region (Set 2)</th>
<th>$P2A$</th>
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<td>(n)</td>
<td>16</td>
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<td>(g)</td>
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<td>(a)</td>
<td>31.93</td>
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<td>(c)</td>
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<td>(k)</td>
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<td>(e)</td>
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<td>(l)</td>
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<td>(d)</td>
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<td>(h)</td>
<td>372.1</td>
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Table 2: $P2A$ ratio values from the digital regions of set 1 and 2 presented in figures 3 and 4.

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<th>$M$</th>
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<td>(d)</td>
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<td>(f)</td>
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<td>(g)</td>
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<td>(b)</td>
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<td>(h)</td>
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<td>(k)</td>
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<td>(j)</td>
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<td>(i)</td>
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<td>(m)</td>
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<td>(l)</td>
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<td>(a)</td>
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<td>(d)</td>
<td>9.91</td>
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<td>(d)</td>
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<td>(k)</td>
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Table 3: $M$ and Haralick's $C$ circularity values from the digital regions of set 1 and 2 illustrated in figures 3 and 4.
3.1 Classification

The tested compactness or circularity measures have different value ranges. The most circular shape has a value equal to zero if the $M$ circularity measure is used. Instead, zero is the value of the less circular shape using the Haralick’s measure $C$. However, shape factors, $G$ and $C_{DN}$, have been designed to return a value of one whether the shape is the most compact shape. All classifications of these measures are compared with the shape $P2A$ classification.

One of the main problems associated with the $P2A$ measure is its high sensitivity to contours produced by the measurement of the shape perimeter. This fact is evident, first, in the set 1 classification, presented in Fig. 10, where some circular shapes are classified at the end of this set. The influence of the number of holes generates an important growth in shape perimeter. Therefore, the shape without holes was classified first. In the case of set 2, the shape without holes was again classified first, while the shape with more perimeter, or with holes, was classified as a less compact shape. Hence, if the criterion is that a shape without holes is more compact than a shape with holes, then the $P2A$ is a better measure shape compactness.

In the first set of binary shapes, there are six shapes evidently more circular (figures 8(a) to (e) and (o)). We expected that these shapes would be classified in the first five places by the Haralick’s and $M$ measures. The classification results showed that these measures could classify three and four of these shapes in the first six places (Fig. 10). Nevertheless, one circular shapes, in both
measures, was classified at the end of the circularity classification. Again, the shape holes affect significantly the performance of these circularity measures. For the $M$ circularity measure, we used the distance transform to find the region with maximal distance value which yields the maximum number of points with maximal distance value. These points increase computational work to measure shape circularity on digital regions. Therefore, the selection of the size of the disk as structure element is an important factor in the performance of the $M$ measure.

Using the criterion of shape compactness already mentioned, the compactness classification was better evaluated by $G$ and $C_{DN}$ measures. These measures could classify the shapes without holes in the first places and the most disperse shapes or shapes with holes, in the last places both set 1 and 2, figures 10 and 11. We can also see that both measures could separate similar shapes, one with holes and the other without holes. Thus, these measures can give additional information about the structure of the shape. On the other hand, using these two measures, the most circular shapes were ordered according to shape dispersion and number of holes.

### 3.2 Changes of spacial resolution

Many shape descriptors are affected when spacial resolution of a digital region is changed [40]. However, these four measures were designed to be invariant
Fig 10: Circularity shape classification.
<table>
<thead>
<tr>
<th>$P2A$</th>
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<th>$M$</th>
<th>$G$</th>
<th>$C_{an}$</th>
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</table>

Fig 11: Compactness shape classification.
under this kind of transformation. In this test, we assessed whether these measures are affected under this transformation. We chose three kinds of shapes, two without holes and one with two holes. The shapes, a digitalized circle, a shape of a scissor and a shape of a lion are illustrated in Fig. 12. A total of 123 scale changes were tested with respect to these shapes. Out of the 123 shapes, 60 are variations of the lion shape, 30 of the digitized circle, and 33 of the scissors shape. The shape descriptor results are plotted in figures 13-15.

The $P2A$ measure is affected by the magnitude of the shape perimeter. This fact is illustrated by the plotted results in Fig. 13, where different versions of a digital disk shows how the $P2A$ value becomes greater as shape perimeter grows. Haralick’s $C$ circularity measure produced similar results. The approach of the Haralick’s measure is based on the number of sides of a polygon, with each border pixel being taken as a side of a polygon. Thus, if the number of border pixels is increased, the evaluation of circularity of a shape is enhanced. The plotted results in figures 13-15 illustrate this fact, where as more border pixels are added to the digitalized circle, the value of this measure increases.

The $M$ measure was especially affected by scaling transformation. The erratic behavior of this measure (shown by the plots in figures 13-15) is, in part, yielded by the algorithm used, and by the number of regions with the maximal value. Hence, it is vitally important to know the correct size of the structure

<table>
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<tr>
<th>Set 1</th>
<th>$C_{DN}$</th>
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<th>$C_{DN}$</th>
<th>Set 1</th>
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<td>(o)</td>
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<td>(o)</td>
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<td>(f)</td>
<td>30.97</td>
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Table 4: $C_{DN}$ and $G$ compactness values from the digital regions of set 1 and 2 shown in Fig. 3 and 4
Fig 12: Set of shapes tested under scaling transformation; (a) Shape of a lion, (b) Digital Disk and, (c) Shape of a scissor.

element, in order to obtain an accurate measurement of the shape circularity. Finally, compactness measures, Shape Factor $G$ and Normalized Discrete Compactness, were more stables in their values. They produced similar results with regard to the three tested shapes and were less affected by the changes of resolution of digital regions. These tests showed that both shape circularity and compactness measures were affected by scaling transformation (changes in resolution). However the circularity measures were those most affected under this transformation.

4 Conclusions

In this paper, shape descriptors based on shape circularity and compactness property have been reviewed, and some of them have been tested. With exception of Sankar’s non-compactness measure, these shape descriptors can be grouped in three main approaches: inner shape distance, reference shape, and geometric pixel properties. Among these approaches, the reference shape approach exhibits more drawbacks with respect to being implemented as a circularity or compactness measure, because the reference shape, the circle, has digitalized problems of its own. The inner shape distance and geometric pixel properties are better suited to being implemented as compactness or circularity measures. Moreover, these approaches can obtain all of their parameters from the digitalized shape.

The study in Section 3 shows that it is more difficult to measure shape circu-
Fig 13: Values of compactness and circularity of a digital disk with several space resolutions.
Fig 14: Values of compactness and circularity of different representations of a shape of a lion.
Fig 15: Values of compactness and circularity of a shape of a scissor.
larity than shape compactness. The two circularity measures exhibited important variations in their values under changes of resolution, $M$ and Haralick’s $C$, and had more problems with respect to classification of shapes by their circularity property. The holes increased the difficulty when using these measures. In contrast, the two compactness measures, Shape Factor $G$ and $C_{DN}$, performed better with respect to evaluating the set of digitized shapes based on compactness characteristic. Furthermore, the compactness measures tested could tell us more about the structure of a shape, since they are able to yield a different value depending on whether similar shapes have holes or not. Although all the shape descriptors were affected by scale changes, the compactness measures were less affected in our tests. Finally, if the measure produces important variations in its circularity or compactness value, then this would have to be considered a further disadvantage, additional to those already mentioned for these kind of measures. Therefore, the universe of shapes can be mixed more easily and produce aberrant classifications.

1 This work is a section of the Doctoral Dissertation of Raul Santiago Montero presented to Universidad Nacional Autónoma de Mexico, under the direction of Dr. Ernesto Bribiesca.

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References


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