Inclusion Theorems for Absolute Matrix Summability

Methods of Infinite Series

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Abstract

In this paper, two general theorems concerning $\phi - |T|_k$ summability factors of infinite series has been proved.

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1. Introduction

Let $(\phi_n)$ be a sequence of positive real numbers, let $\sum a_n$ be an infinite series with the sequence of partial sums $(s_n)$. Let $(u_n)$ denote the n-th (C,1) means of the sequence $(na_n)$. The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \frac{1}{n^k} |u_n| < \infty. \quad (1.1)$$

and it is said to be summable $\phi - |C,1|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} \frac{\phi_n}{n^k} |u_n| < \infty. \quad (1.2)$$

If we are taking $\phi_n = n$, $\phi - |C,1|_k$ reduces to $|C,1|_k$ summability.

Let $(p_n)$ be a sequence of positive numbers such that

$$P_n = \sum_{n=0}^{n} p_n \to \infty \quad as \quad n \to \infty \quad (p_{-1} = P_{-1} = 0).$$

The sequence-to-sequence transformation
(1.3) \[ v_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \]
defines the sequence \((v_n)\) of the Riesz mean or simply the \((\overline{N}, p_n)\) mean of the sequence \((s_n)\) generated by the sequence of coefficients \((p_n)\) (see [2]). The series \(\sum a_n\) is said to be summable \(|R, p_n|_k\), \(k \geq 1\) if \(n\)

\[
\sum_{n=1}^{\infty} n^{k-1} |v_n - v_{n-1}|^k < \infty.
\]

In the special case when \(p_n = 1\) for all \(n\), then \(|R, p_n|_k\) summability is the same as \(|C, I|_k\) summability. The series \(\sum a_n\) is summable \(\varphi - |N, p_n|_k\), \(k \geq 1\), if \(n\)

\[
\sum_{n=1}^{\infty} \varphi^{k-1} n^{k-1} |v_n - v_{n-1}|^k < \infty.
\]

For \(\varphi_n = n\), \(\varphi - |R, p_n|_k\) summability is the same as \(|R, p_n|_k\) summability.

For arbitrary lower triangular matrix \(T = (t_{nv})\), the series \(\sum a_n\) is summable \(|T|_k\), \(k \geq 1\), if \(n\)

\[
\sum_{n=1}^{\infty} n^{k-1} |\Delta t_{n-1}|^k < \infty,
\]

where \(n\)

\[ t_n = \sum_{k=0}^{n} t_{nk}s_k. \]

The series \(\sum a_n\) is summable \(\varphi - |T|_k\), \(k \geq 1\), if \(n\)

\[
\sum_{n=1}^{\infty} \varphi^{k-1} n^{k-1} |\Delta t_{n-1}|^k < \infty.
\]

Two matrices \(\overline{T} = (\overline{t}_{nv})\) and \(\hat{T} = (\hat{t}_{nv})\) can be associated with \(T\) as follows:

The entries \(\overline{t}_{nv}\) and \(\hat{t}_{nv}\) are defined by \(n\)

\[
\overline{t}_{nk} = \sum_{v=0}^{n} t_{nv}, \quad \hat{t}_{nv} = \overline{t}_{nv} - \overline{t}_{n-1,v}, \quad \text{obviously} \quad \hat{t}_{nn} = t_{nn}. \]

By above, we have \(n\)

\[
t_n = \sum_{k=0}^{n} t_{nk}s_k = \sum_{k=0}^{n} t_{nk} \sum_{v=0}^{n} a_v = \sum_{v=0}^{n} a_v \sum_{k=0}^{n} t_{nv} = \sum_{v=0}^{n} \overline{t}_{nv} a_v, \]

\[
Y_n := t_n - t_{n-1} = \sum_{v=0}^{n-1} \overline{t}_{nv} a_v - \sum_{v=0}^{n-1} \overline{t}_{n-1,v} a_v = \sum_{v=0}^{n} \overline{t}_{nv} a_v, \quad \text{as} \quad \overline{t}_{n-1,n} = 0.
\]

A lower triangular matrix \(T = (t_{nv})\) is called triangle if \(t_{nn} \neq 0\) for all \(n\). Clearly a triangle has a unique two sided inverse.

Concerning \(|C, I|_k\) summability, Mazhar [3] has proved the following\(n\)\:
Theorem 1.1. If
\begin{align*}
\lambda_m &= o(1), \quad \text{as } m \to \infty, \\
\sum_{n=1}^{m} n \log n |\Delta^2 \lambda_n| &= O(1), \quad \text{as } m \to \infty, \\
\sum_{i=1}^{m} \frac{k_i^k}{v} &= O(\log m) \quad \text{as } m \to \infty, 
\end{align*}
then the series \( \sum a_n \lambda_n \) is summable \( \sum_{i=1}^{k} C_i, k \geq 1 \).

Ozarslan [4], on the other hand, generalized the previous result by giving the following

Theorem 1.2. Let \( \varphi_n \) be a sequence of positive real numbers and the conditions (1.10)-(1.11) of Theorem (1.1) are satisfied. If
\begin{align*}
\sum_{v=1}^{\infty} \varphi_{v}^{k-1} \frac{|x|^k}{v} &= O(\log m) \quad \text{as } m \to \infty, \\
\sum_{n=1}^{\infty} \frac{\varphi_{n}^{k-1}}{n^{k+1}} &= O\left(\frac{\varphi_{v}^{k-1}}{v^k}\right),
\end{align*}
then the series \( \sum a_n \lambda_n \) is summable \( \varphi - [C,1]_k, k \geq 1 \).

It should be mentioned that on taking \( \varphi_n = n \) in Theorem (1.2), we get Theorem 1.1.

2. Main result

Theorem 2.1. Let \( \varphi_n, Z_n \) be sequences of positive real numbers such that \( Z_n \) is nondecreasing and the condition (1.10), is satisfied. If the sequence \( W_n \) defined by
\begin{align*}
W_v = \begin{cases} 
X_v, & v = n \\
\max_{1 \leq i \leq v} |X_i|, & v < n,
\end{cases}
X_n = \sum_{v=1}^{n} i_v a_v.
\end{align*}

satisfies the condition
\begin{align*}
\sum_{n=1}^{\infty} n^{k-1} W_n^k < \infty,
\end{align*}
and if
\begin{align*}
(i_n / t_v) & \quad \text{is nondecreasing with respect to } v, \\
n \Delta Z_n &= O(Z_n),
\end{align*}
Then the series $\sum a_n \lambda_n Z_n$ is summable $\phi - |\mathcal{T}|_k$ whenever $\sum a_n$ is summable $|T|_k$, $k \geq 1$.

**Theorem 2.2.** Let $(\varphi_n), (Z_n)$ be sequences of positive real numbers such that $(Z_n)$ is nondecreasing and the condition (1.10), (2.3) and (2.4) are satisfied. Let $A = (a_m)$, $B = (b_{iv})$ be two triangles satisfying

(2.8) $\sum_{n=1}^{\infty} |c_{iv}| < \infty, \quad 1 \leq v \leq n$, where $c_{iv} := \sum_{i=v}^{n} \hat{a}_{iv}'$

(2.9) $\sum_{n=v}^{\infty} \varphi_{iv}^{k-1} |c_{iv}| = O(v^{k-1})$

Then the series $\sum a_n \lambda_n Z_n$ is summable $\phi - |B|_k$ whenever $\sum a_n$ is summable $|A|_k$, $k \geq 1$.

**3. Lemma**

The following Lemma is needed

**Lemma 3.1.** The conditions (1.10) and (2.3) implies

(3.1) $\sum_{n=1}^{\infty} Z_n |\Delta \lambda_n| = O(1)$

(3.2) $\sum_{n=1}^{\infty} |\lambda_n| |\Delta Z_n| = O(1)$

(3.3) $n Z_n |\Delta \lambda_n| = O(1)$, as $n \to \infty$

(3.4) $Z_n |\lambda_n| = O(1)$, as $n \to \infty$

Proof. By virtue of (1.10),
\[
\sum_{n=1}^{\infty} Z_n |\Delta \lambda_n| = \sum_{n=1}^{\infty} Z_n \sum_{v=n}^{\infty} |\Delta \lambda_v| \\
= \sum_{v=1}^{\infty} |\Delta^2 \lambda_v| \sum_{n=1}^{\infty} Z_v \\
= O(1) \sum_{v=1}^{\infty} v Z_v |\Delta^2 \lambda_v| \\
= O(1).
\]

\[
\sum_{n=1}^{\infty} |\lambda_n| |\Delta Z_n| = \sum_{n=1}^{\infty} |\Delta Z_n| \sum_{v=n}^{\infty} |\Delta \lambda_v| \\
\leq \sum_{n=1}^{\infty} |\Delta Z_n| \sum_{v=n}^{\infty} |\Delta \lambda_v| \\
= O(1) \sum_{v=1}^{\infty} |\Delta \lambda_v| \sum_{n=1}^{\infty} |\Delta Z_n| \\
= O(1) \sum_{v=1}^{\infty} |\Delta \lambda_v| (Z_{v+1} - Z_v) \\
= O(1) \sum_{v=1}^{\infty} |\Delta \lambda_v| Z_v \\
= O(1).
\]

\[
n Z_n |\Delta \lambda_n| = n Z_n \left| \sum_{v=n}^{\infty} \Delta |\lambda_v| \right| \\
\leq n Z_n \sum_{v=n}^{\infty} |\Delta |\lambda_v| | \\
\leq n Z_n \sum_{v=n}^{\infty} |\Delta^2 \lambda_v| \\
= O(1) \sum_{v=n}^{\infty} v Z_v |\Delta^2 \lambda_v| \\
= O(1).
\]

\[
Z_n |\lambda_n| = Z_n \sum_{v=n}^{\infty} |\lambda_v| \\
\leq Z_n \sum_{v=n}^{\infty} |\Delta \lambda_v|
\]
\[ = O(1) \sum_{v=r}^{\infty} Z_v |\Delta \lambda_v| \]
\[ = O(1), \text{ by the first part} . \]

4. Proof of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Let
\[ T_n = \sum_{v=0}^{\infty} t_v \sum_{i=0}^{n} a_i \lambda_v Z_i = \sum_{v=0}^{\infty} a_i \lambda_v Z_v \sum_{v=0}^{n} t_v = \sum_{v=0}^{n} t_v a_v \lambda_v Z_v . \]
Therefore, we have
\[ Y_n := T_n - T_{n-1} = \sum_{v=0}^{n} t_v a_v \lambda_v Z_v . \]
By Abel's transformation,
\[ Y_n = \sum_{v=0}^{n} \left( \sum_{r=0}^{v} \hat{i}_{rv} a_r \right) \Delta (\lambda_v, Z_v) + \left( \sum_{r=0}^{n} \hat{i}_{rv} a_r \right) \lambda_v Z_n \]
\[ \leq \sum_{v=0}^{n} \sum_{r=0}^{v} \hat{i}_{rv} a_r \left| \Delta (\lambda_v, Z_v) \right| + \sum_{v=0}^{n} \hat{i}_{rv} a_r \left| \lambda_v Z_v \right| \]
\[ \leq \sum_{v=0}^{n} \sum_{r=0}^{v} \hat{i}_{rv} a_r \left( \left| \Delta \lambda_v \right| Z_v + \left| \lambda_{v+1} \right| \left| \Delta Z_v \right| \right) + \left| X_n \right| \left| \lambda_v Z_v \right| . \]
Since
\[ \left| \sum_{r=0}^{v} \hat{i}_{rv} a_r \right| = \sum_{r=0}^{v} \hat{i}_{rv} t_{rv} a_r \]
\[ \leq \frac{\hat{i}_{rv}}{t_{rv}} \max_{1 \leq v \leq n} \sum_{r=0}^{v} \hat{i}_{rv} a_r \]
\[ \leq \frac{\hat{i}_{rv}}{t_{rv}} \max |X_v| \]
\[ = \frac{\hat{i}_{rv}}{t_{rv}} W_v , \]
then, we have
\[ |Y_n| \leq \sum_{v=0}^{n} \frac{\hat{i}_{rv}}{t_{rv}} W_v \left( \left| \Delta \lambda_v \right| Z_v + \left| \lambda_{v+1} \right| \left| \Delta Z_v \right| \right) + W_n \left| \lambda_v Z_v \right| \]
\[ = Y_{n1} + Y_{n2} + Y_{n3} . \]
In order to complete the proof of the theorem, by Minkowski's theorem it is sufficient to show that
\[ \sum_{n=1}^{\infty} \varphi_{n}^{r-1} |Y_{nr}|^r < \infty , \quad r = 1, 2, 3 . \]
Making use of the Holder's inequality,
\[
\sum_{n=1}^{m} \varphi_n^{-k}|Y_n|^k = \sum_{n=1}^{m} \varphi_n^{-k} \left( \sum_{v=1}^{n-1} |T_v|^k \right) W_v |\Delta \varphi_n| Z_v \left( \sum_{v=1}^{m} |\Delta \varphi_n| Z_v \right)^{k-1}
\]
\[
\leq \sum_{n=1}^{m} \varphi_n^{-k} \sum_{v=1}^{n} |T_v|^k W_v |\Delta \varphi_n| Z_v \left( \sum_{v=1}^{m} |\Delta \varphi_n| Z_v \right)^{k-1}
\]
\[
= O(1) \sum_{v=1}^{m} \varphi_n^{-k} \left( \sum_{v=1}^{n} |T_v|^k W_v \right) |\Delta \varphi_n| Z_v \left( \sum_{v=1}^{m} |\Delta \varphi_n| Z_v \right)^{k-1}
\]
\[
= O(1) \sum_{v=1}^{m} |\Delta \varphi_n| Z_v + O(1) \sum_{v=1}^{m} v |\Delta \varphi_n| Z_v
\]
\[
= O(1) \sum_{v=1}^{m} |\Delta \varphi_{v+1}| Z_{v+1} + O(1)
\]
\[
= O(1) \sum_{v=1}^{m} |\Delta \varphi_{v+1}| Z_{v+1}
\]
\[
= O(1) .
\]
\[
\sum_{n=1}^{m} \varphi_n^{-k}|Y_n|^k = \sum_{n=1}^{m} \varphi_n^{-k} \left( \sum_{v=1}^{n-1} |T_v|^k \right) W_v |\Delta \varphi_n| |\Delta Z_v|^k
\]
\[
= O(1) \sum_{n=1}^{m} \varphi_n^{-k} \left( \sum_{v=1}^{n-1} |T_v|^k W_v \right) |\Delta \varphi_n| |\Delta Z_v|^k \left( \sum_{v=1}^{m} |\Delta \varphi_n| |\Delta Z_v|^k \right)^{k-1}
\]
\[
= O(1) \sum_{v=1}^{m} \varphi_n^{-k} \left( \sum_{v=1}^{n-1} |T_v|^k W_v \right) |\Delta \varphi_n| |\Delta Z_v|^k \sum_{n=1}^{m} \varphi_n^{-k} |T_v|^k
\]
\[
= O(1) \sum_{v=1}^{m} v^{-k} |\Delta \varphi_n| Z_v
\]
\[
= O(1) \sum_{v=1}^{m} v^{-k} W_v
\]
\[
= O(1) .
\]
\[
\sum_{n=1}^{m} \phi_n^{k-1} |Y_{n3}|^k = \sum_{n=1}^{m} \phi_n^{k-1} |W_n Z_n|^k
\]
\[
= O(1) \sum_{n=1}^{m} \phi_n^{k-1} W_n^k |\lambda_n|^k Z_n^k
\]
\[
= O(1) \sum_{n=1}^{m} n^{k-1} W_n^k
\]
\[
= O(1)
\]

**Proof of Theorem 2.2.** Let

\[
X_n = \sum_{v=0}^{n} \hat{a}_v a_v.
\]

As \( \hat{A} = (\hat{a}_v) \) is invertible, then on solving the above for \( a_n \) we obtain

\[
a_n = \sum_{v=0}^{n} \hat{a}_v' X_v.
\]

Therefore, we have

\[
Y_n = \sum_{i=0}^{n} \hat{b}_n a_i \lambda_i Z_i
\]
\[
= \sum_{i=0}^{n} \hat{b}_n \lambda_i Z_i \sum_{v=0}^{n} \hat{a}_v' X_i
\]
\[
= \sum_{v=0}^{n} X_v \sum_{i=0}^{n} \hat{b}_n \lambda_i Z_i \hat{a}_v'
\]
\[
\leq \sum_{v=0}^{n} X_v \lambda_v Z_v \sum_{i=0}^{n} \hat{b}_n \hat{a}_v'
\]
\[
= \sum_{v=0}^{n} X_v \lambda_v Z_v a_v
\]

\[
\sum_{n=1}^{m} \phi_n^{k-1} |Y_n|^k = \sum_{n=1}^{m} \phi_n^{k-1} \sum_{v=0}^{n} X_v \lambda_v Z_v a_v
\]
\[
= O(1) \sum_{n=1}^{m} \phi_n^{k-1} \sum_{v=0}^{n} |X_v|^k |\lambda_v|^k Z_v^k |a_v| \left( \sum_{v=0}^{n} |a_v| \right)^{k-1}
\]
\[
= O(1) \sum_{v=0}^{n} |X_v|^k |\lambda_v|^k Z_v^k \sum_{n=v}^{m} \phi_n^{k-1} |a_v|
\]
\[
= O(1) \sum_{v=0}^{n} v^{k-1} |X_v|^k |\lambda_v|^k Z_v^k
\]
\[
= O(1) \sum_{v=0}^{n} v^{k-1} |X_v|^k |\lambda_v|^k Z_v
\]
\[
= O(1) \sum_{v=0}^{m-1} \left( \sum_{r=1}^{v} r^{k-1} |X_r|^t \right) \Delta(\hat{\lambda}_v Z_v) + \left( \sum_{v=1}^{m} v^{k-1} |X_v|^t \right) \hat{\lambda}_m |Z_m
= O(1) \sum_{v=0}^{m-1} |\Delta \hat{\lambda}_v| Z_v + \sum_{y=0}^{m-1} |\hat{\lambda}_{v+y}| |\Delta Z_v| + O(1)
= O(1) + \sum_{y=0}^{m-1} |\hat{\lambda}_y| |\Delta Z_v|
= O(1).
\]

References


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