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Some Results in Generalized n-Inner Product Spaces

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Abstract

The aim of this paper is to prove parallelogram law, polarization identity in Generalized n-inner product spaces defined by K. Trencevski and R. Malceski [5] which is generalization of n-inner product spaces introduced by A. Misiak [1]. Also we discuss the notion of strong and weak convergence in Generalized n-inner product spaces and study the relation between them.

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A. Misiak [1] has introduced an n-normed linear space by the following definition

Definition 1.1. Let $n \in N$ (natural numbers) and $X$ be a real linear space of dimension greater than or equal to $n$. A real valued function $\| \cdot , \ldots , \cdot \|$ on $X \times X \ldots \times X = X^n$ satisfying the following four properties:

1. $\|x_1, x_2, \ldots, x_n\| = 0$ if any only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

2. $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,

3. $\|x_1, x_2, \ldots, a x_n\| = |a| \|x_1, x_2, \ldots, x_n\|$, for any $a \in R$ (real),

4. $\|x_1, x_2, \ldots, x_{n-1}, y + z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$

is called an n-norm on $X$ and the pair $(X, \| \cdot , \ldots , \cdot \|)$ is called n-normed linear space. Although in this paper we only consider real vector spaces, the results of this paper can easily be generalized for complex vector spaces. A. Misiak [1] introduced an n-inner product in the following form:

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Definition 1.2. Assume that $n$ is a positive integer and $V$ is a real vector space such that $\dim X \geq n$ and $(\cdot \cdot \cdot \cdot \cdot)$ is a real function defined on $\mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ such that:

1. $(x_1, x_1 | x_2, \ldots, x_n) \geq 0$, for any $x_1, x_2, \ldots, x_n \in X$ and $(x_1, x_1 | x_2, \ldots, x_n) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors;

2. $(a, b | x_1, \ldots, x_{n-1}) = (\varphi(a), \varphi(b) | \pi(x_1), \ldots, (x_{n-1}))$, for any $a, b, x_1, x_2, \ldots, x_{n-1} \in X$ and for any bijection $\pi : \{x_1, x_2, \ldots, x_{n-1}\} \rightarrow \{x_1, x_2, \ldots, x_{n-1}\}$ and $\varphi : \{a, b\} \rightarrow \{a, b\}$;

3. If $n > 1$, then $(x_1, x_1 | x_2, \ldots, x_n) = (x_2, x_2 | x_1, x_3, \ldots, x_n)$, for any $x_1, x_2, \ldots, x_n \in X$;

4. $(\alpha a, b | x_1, \ldots, x_{n-1}) = \alpha (a, b | x_1, \ldots, x_{n-1})$, for any $a, b, x_1, \ldots, x_{n-1} \in X$ and any scalar $\alpha \in \mathbb{R}$;

5. $(a + a_1, b | x_1, \ldots, x_{n-1}) = (a, b | x_1, \ldots, x_{n-1}) + (a_1, b | x_1, \ldots, x_{n-1})$, for any $a, b, a_1, x_1, x_2, \ldots, x_n \in X$.

Then $(\cdot \cdot \cdot \cdot \cdot)$ is called the $n$-inner product and $(X, (\cdot \cdot \cdot \cdot \cdot)$ is called the $n$-inner product space.

This $n$-inner product induces an $n$-norm by

$$\|x_1, \ldots, x_n\| = \sqrt{(x_1, x_1 | x_2, \ldots, x_n)}.$$ 

K. Trenceveski and R. Malceski [5] has introduced following definition of generalized $n$-inner product as:

Definition 1.3. Assume that $n$ is a positive integer, $X$ is a real vector space such that $\dim X \geq n$ and $(\cdot \cdot \cdot \cdot \cdot)$ is a real function on $X^{2n}$ such that

(3.1) $\langle a_1, \ldots, a_n | a_1, \ldots, a_n \rangle > 0$ if $a_1, \ldots, a_n$ are linearly independent vectors,

(3.2) $\langle a_1, \ldots, a_n | b_1, \ldots, b_n \rangle = \langle b_1, \ldots, b_n | a_1, \ldots, a_n \rangle$
    for any $a_1, \ldots, a_n, b_1, \ldots, b_n \in X$

(3.3) $\langle \lambda a_1, \ldots, a_n | b_1, \ldots, b_n \rangle = \lambda \langle a_1, \ldots, a_n | b_1, \ldots, b_n \rangle$
    for any scalar $\lambda \in \mathbb{R}$ and any $a_1, \ldots, a_n, b_1, \ldots, b_n \in X$,

(3.4) $\langle a_1, \ldots, a_n | b_1, \ldots, b_n \rangle = -\langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} | b_1, \ldots, b_n \rangle$
    for any odd permutation $\sigma$ in the set $\{1, \ldots, n\}$
    and any $a_1, \ldots, a_n, b_1, \ldots, b_n \in X$.
(3.5) \[ \langle a_1 + c, a_2, \ldots, a_n | b_1, \ldots, b_n \rangle = \langle a_1, a_2, \ldots, a_n | b_1, \ldots, b_n \rangle + \langle c, a_2, \ldots, a_n | b_1, \ldots, b_n \rangle \]
for any \( a_1, \ldots, a_n, b_1, \ldots, b_n, c \in X. \)

(3.6) If \( \langle a_1, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n | b_1, \ldots, b_n \rangle = 0 \) for each \( i \in \{1, 2, \ldots, n\} \), then \( \langle a_1, \ldots, a_n | b_1, \ldots, b_n \rangle = 0 \) for arbitrary vectors \( a_2, \ldots, a_n \).

Then the function \( \langle \bullet, \ldots, \bullet | \bullet, \ldots, \bullet \rangle \) is called an generalized \( n \)-inner product and the pair \( (X, \langle \bullet, \ldots, \bullet | \bullet, \ldots, \bullet \rangle) \) is called an generalized \( n \)-inner product space. In the special case if we consider only such pairs of sets \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) which differ for at most one vector, for example \( a_1 = a, b_1 = b \) and \( a_2 = b_2 = x_1, \ldots, a_n = b_n = x_{n-1} \), then by putting

\[ (a, b|x_1, \ldots, x_{n-1}) = (a, x_1, \ldots, x_{n-1}|b, x_1, \ldots, x_{n-1}) \]

we obtain an \( n \)-inner product.

**Example 1.4 ([5])**. Let \( X \) be a space with inner product \( \langle \bullet \bullet \rangle \) then

\[
\langle a_1, \ldots, a_n | b_1, \ldots, b_n \rangle = \begin{bmatrix} \langle a_1 | b_1 \rangle & \langle a_1 | b_2 \rangle & \cdots & \langle a_1 | b_n \rangle \\ \langle a_2 | b_1 \rangle & \langle a_2 | b_2 \rangle & \cdots & \langle a_2 | b_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_n | b_1 \rangle & \langle a_n | b_2 \rangle & \cdots & \langle a_n | b_n \rangle \end{bmatrix}
\]
defines an generalized \( n \)-inner product on \( X \).

K. Trenceveski and R. Malceski [5] proved Cauchy-Schwarz inequality in generalized \( n \)-inner product on \( X \) as :

\[
\langle a_1, \ldots, a_n | b_1, \ldots, b_n \rangle^2 \leq \langle a_1, \ldots, a_n | a_1, \ldots, a_n \rangle \langle b_1, \ldots, b_n | b_1, \ldots, b_n \rangle
\]

The generalized \( n \)-inner product on \( X \) induces an \( n \)-norm by

\[ \|x_1, \ldots, x_n\| = \sqrt{\langle x_1, \ldots, x_n | x_1, \ldots, x_n \rangle} \]
and it is the same \( n \)-norm induced by Definition 2.1. Let \( (X, \langle \bullet, \ldots, \bullet | \bullet, \ldots, \bullet \rangle) \) be Generalized \( n \)-inner product space and \( \|\bullet, \ldots, \bullet\| \) be the induced \( n \)-norm. We observe that if \( x_k \rightarrow x \), then

\[ \|x_k, z_2, \ldots, z_n\| \rightarrow \|x, z_2, \ldots, z_n\| \]
for every \( z_2, \ldots, z_n \in X \). This tells us that \( n \)-norm \( \|\bullet, \ldots, \bullet\| \) is continuous in first variable. And by property (1.2) of \( n \)-norms \( \|\bullet, \ldots, \bullet\| \) is continuous in each variable.
2 CONTINUITY OF GENERALIZED $n$-INNER PRODUCT

Next if $x_{1k} \to y_1, x_{2k} \to y_2, \ldots, x_{nk} \to y_n$, then by property (3.5) and Cauchy-Schwarz inequality for generalized $n$-inner product we have,

$$|\langle x_{1k}, x_{2k}, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle - \langle y_1, y_2, \ldots, y_n|z_1, z_2, \ldots, z_n \rangle|$$

$$= |\langle x_{1k}, x_{2k}, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle - \langle y_1, x_{2k}, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle + \langle y_1, x_{2k}, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle - \langle y_1, y_2, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle|$$

$$+ |\langle y_1, y_2, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle - \langle y_1, y_2, \ldots, y_n|z_1, z_2, \ldots, z_n \rangle|$$

$$\leq |\langle x_{1k} - y_1, x_{2k}, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle + \langle y_1, x_{2k} - y_2, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle + \langle y_1, y_2, \ldots, x_{nk} - y_n|z_1, z_2, \ldots, z_n \rangle|$$

as $k \to 0$ Since we know that $n$-norm $\|\cdot, \ldots, \cdot\|$ is continuous in each variable.

Hence $\langle x_{1k}, x_{2k}, \ldots, x_{nk}|z_1, z_2, \ldots, z_n \rangle \to \langle y_1, y_2, \ldots, y_n|z_1, z_2, \ldots, z_n \rangle$

This shows that $\langle \cdot, \ldots, \cdot|\cdot, \ldots, \cdot \rangle$ is continuous in first $n$-variable and hence by the property (3.2) and (3.4) we get $\langle \cdot, \ldots, \cdot|\cdot, \ldots, \cdot \rangle$ is continuous in each variable. Now, we give the Parallelogram law and Polarization identity in generalized $n$-inner product space.

**Proposition 2.1** (Parallelogram law in generalized $n$-inner product space).

$$\|x + y, x_2, \ldots, x_n\|^2 + \|x - y, x_2, \ldots, x_n\|^2$$

$$= 2\|x, x_2, \ldots, x_n\|^2 + 2\|y, x_2, \ldots, x_n\|^2$$
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Proof.

\[ ||x + y, x_2, \ldots, x_n||^2 + ||x - y, x_2, \ldots, x_n||^2 = \langle x + y, x_2, \ldots, x_n | x + y, x_2, \ldots, x_n \rangle + \langle x - y, x_2, \ldots, x_n | x - y, x_2, \ldots, x_n \rangle \]
\[ = \langle x, x_2, \ldots, x_n | x + y, x_2, \ldots, x_n \rangle + \langle y, x_2, \ldots, x_n | x + y, x_2, \ldots, x_n \rangle + \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle - \langle y, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle \]
\[ = \langle x, x_2, \ldots, x_n | x, x_2, \ldots, x_n \rangle + \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle + \langle y, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle - \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle \]
\[ = 2 \langle x, x_2, \ldots, x_n | x, x_2, \ldots, x_n \rangle + 2 \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle = 2 ||x, x_2, \ldots, x_n||^2 + 2 ||y, x_2, \ldots, x_n||^2. \]

Hence,

\[ ||x + y, x_2, \ldots, x_n||^2 + ||x - y, x_2, \ldots, x_n||^2 = 2 ||x, x_2, \ldots, x_n||^2 + 2 ||y, x_2, \ldots, x_n||^2 \]

Proposition 2.2 (Polarization Identity in generalized n-inner product space).

\[ 4\langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle = ||x + y, x_2, \ldots, x_n||^2 - ||x - y, x_2, \ldots, x_n||^2 \]

Proof.

\[ ||x + y, x_2, \ldots, x_n||^2 - ||x - y, x_2, \ldots, x_n||^2 \]
\[ = \langle x + y, x_2, \ldots, x_n | x + y, x_2, \ldots, x_n \rangle - \langle x - y, x_2, \ldots, x_n | x - y, x_2, \ldots, x_n \rangle \]
\[ = \langle x, x_2, \ldots, x_n | x + y, x_2, \ldots, x_n \rangle + \langle y, x_2, \ldots, x_n | x + y, x_2, \ldots, x_n \rangle - \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle - \langle y, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle \]
\[ = \langle x, x_2, \ldots, x_n | x, x_2, \ldots, x_n \rangle + \langle y, x_2, \ldots, x_n | x, x_2, \ldots, x_n \rangle + \langle y, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle - \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle \]
\[ = 2 \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle + 2 \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle = 4 \langle x, x_2, \ldots, x_n | y, x_2, \ldots, x_n \rangle \]
3 STRONG AND WEAK CONVERGENCE IN GENERALIZED n-INNER PRODUCT SPACE

Now, we discuss the strong and weak convergence let \((X, \langle \cdot, \cdot, \cdot, \cdot \rangle)\) be an 
generalized \(n\)-inner product space and \(\|\cdot, \cdot, \cdot, \cdot\|\) be the induced \(n\)-norm. A 
sequence \(\langle x_k \rangle\) in \(X\) is said to converge strongly to a point \(x \in X\) whenever 
\(\|x_k - x, a_2, \ldots, a_n\| \to 0\) for every \(a_1, \ldots, a_n \in X\). In such a case, we 
write \(x_k \to x\). Meanwhile, \(\langle x_k \rangle\) is said to converge weakly to \(x\) whenever 
\(\langle x_k - x, a_2, \ldots, a_n|b_1, \ldots, b_n\rangle \to 0\) for every \(a_2, \ldots, a_n, b_1, \ldots, b_n \in X\). Clearly 
if \(\langle x_k \rangle\) and \(\langle y_k \rangle\) converges strongly/weakly to \(x\) and \(y\) respectively, then, for any 
\(\alpha, \beta \in R\), \(\alpha x_k + \beta y_k\) converges strongly/weakly to \(\alpha x + \beta y\). Now, we show that a 
sequence cannot converges weakly to two distinct point as follows:

**Proposition 3.1.** If \(\langle x_k \rangle\) converges weakly to \(x\) and \(x'\) simultaneously, then \(x = x'\).

**Proof.** By hypothesis and Property (3.5) of generalized \(n\)-inner products, we have

\[
\langle x_k, a_2, \ldots, a_n|b_1, \ldots, b_n\rangle \to \langle x, a_2, \ldots, a_n|b_1, \ldots, b_n\rangle
\]

and

\[
\langle x_k, a_2, \ldots, a_n|b_1, \ldots, b_n\rangle \to \langle x', a_2, \ldots, a_n|b_1, \ldots, b_n\rangle
\]

for every \(a_2, \ldots, a_n, b_1, \ldots, b_n \in X\). By the uniqueness of the limit of a sequence of 
real numbers, we must have

\[
\langle x, a_2, \ldots, a_n|b_1, \ldots, b_n\rangle = \langle x', a_2, \ldots, a_n|b_1, \ldots, b_n\rangle
\]

\[
\langle x - x', a_2, \ldots, a_n|b_1, \ldots, b_n\rangle = 0
\]

for every \(a_2, \ldots, a_n, b_1, \ldots, b_n \in X\). In particular, by taking \(b_1 = x - x'\) and \(a_i = b_i\) 
for each \(i = 2, \ldots, n\). We obtain \(\|x - x', a_2, \ldots, a_n\| = 0\) for every \(a_2, \ldots, a_n \in X\). 
By property (2.1) of \(n\)-norms and elementary linear algebra, this can only happen if 
\(x - x' = 0\) or \(x = x'\).

The next proposition says that the strong convergence implies the weak 
convergence in Generalized \(n\)-inner product spaces.

**Proposition 3.2.** If \(\langle x_k \rangle\) converges strongly to \(x\), then it converges weakly to \(x\).

**Proof.** By the Cauchy-Schwarz inequality we have

\[
|\langle x_k - x, a_2, \ldots, a_n|b_1, b_2, \ldots, b_n\rangle| \leq \langle x_k - x, a_2, \ldots, a_n|b_1, b_2, \ldots, b_n\rangle \cdot
\langle b_1, b_2, \ldots, b_n|b_1, b_2, \ldots, b_n\rangle
\]

\[
\leq \|x_k - x, a_2, \ldots, a_n\| \cdot \|b_1, b_2, \ldots, b_n\|
\]

for every \(a_2, \ldots, a_n, b_1, \ldots, b_n \in X\).

Since \(\langle x_k \rangle\) converges strongly to \(x\), thus \(\|x_k - x, a_2, \ldots, a_n\| \to 0\) as \(k \to 0\) for every 
\(a_2, \ldots, a_n \in X\) so, \(\langle x_k - x, a_2, \ldots, a_n|b_1, b_2, \ldots, b_n\rangle \to 0\) for every \(a_2, \ldots, a_n \in X\).
Corollary. A sequence cannot converge strongly to two distinct points. But weakly convergent sequences do not converge strongly, we give a counter example.

Example 1. Let \((X, \langle \cdot, \cdot \rangle)\) be a separable Hilbert space of infinite dimension and \(\langle e_k \rangle\) indexed by \(N\), be an orthonormal basis for \(X\). Then, for each \(x\) and \(b \in X\), we have \(\sum_k \langle x, e_k \rangle \langle x, b \rangle = \langle x, b \rangle\). In particular, if \(x = z\), then we have Parseval’s identity \(\sum_k \|x\|^2 = \|x\|^2\) where \(\|\cdot\| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}\) denotes the induced norm.

Now \(X\) with the standard generalized \(n\)-inner product \(\langle \cdot, \cdot, \cdots, \cdot, \cdots \rangle\) Then, for each \(x, a_3, \ldots, a_n, b_1, \ldots, b_n \in X\), we have the following identity

\[
\sum_k \langle x, b_1, \ldots, b_{n-1} | e_k, b_1, \ldots, b_{n-1} \rangle^2 = \sum_k \langle x, e_k | b_1, \ldots, b_{n-1} \rangle^2 = \|x, b_1, \ldots, b_{n-1}\|^2 \|b_1, \ldots, b_{n-1}\|_{n-1}^2.
\]

Where \(\|\cdot, \cdots, \cdot\|_{n-1}\) denotes the standard \((n-1)\)-norm on \(X\).

For \(n = 2\) we see,

\[
\sum_k \langle x, b | e_k, b \rangle^2 = \sum_k \langle x, e_k | b \rangle^2 = \sum_k [\|x, e_k\|^2 - \langle x, b \rangle \langle b, e_k \rangle]^2 = \sum_k [\|x, e_k\|^2 - 2\langle x, e_k \rangle \langle b, e_k \rangle \|b\|^2 + \langle x, b \rangle^2 \langle b, e_k \rangle^2] = \|x\|^2 \|b\|^2 - 2\langle x, b \rangle^2 \|b\|^2 + \langle x, b \rangle^2 \|b\|^2 = \|x, b\|^2 \|b\|^2.
\]

Now, the counter example is \(\langle e_k \rangle\). Because of Parseval’s identity, we must have \(\langle x, b_1, \ldots, b_n | e_k, b_1, \ldots, b_n \rangle \to 0\) for every \(x, b_1, \ldots, b_n \in X\). i.e, \(\langle e_k \rangle\) converges weakly to 0. Now for \(k \in N\) and \(b_1, \ldots, b_n \in X\) denote by \(e_k^*\) the orthogonal projection of \(e_k\) on spanned by \(b_1, \ldots, b_n\). Then one may observe that \(\|e_k - e_k^*\| \to 1\).

Hence \(\|e_k, b_1, \ldots, b_n\| = \|e_k - e_k^*\| \|b_1, \ldots, b_n\|_{n-1} \to \|b_1, \ldots, b_n\|_{n-1} \neq 0\) where \(b_1, \ldots, b_{n-1}\) are linearly independent. This shows \(\langle e_k \rangle\) does not converge strongly to 0 in \(X\). But in the finite dimensional case, both convergence are equivalent.

References


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