(n, g(x))-Clean Rings

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Abstract

A ring $R$ is called $n$-clean ring if every element of $R$ can be written as a sum of an idempotent and $n$ units in $R$. Let $C(R)$ be the center of a ring $R$ and $g(x)$ be a fixed polynomial in $C(R)[x]$. An element is called $g(x)$-clean if $r = u + s$ where $u$ is a unit of $R$ and $g(s) = 0$ and $R$ is $g(x)$-clean if every element is $g(x)$-clean. An element $x \in R$ is called $(n, g(x))$-clean if $x = u_1 + u_2 + \cdots + u_n + s$, where $g(s) = 0$ and $u_1, u_2, ..., u_n$ are units in $R$ and $R$ is called $(n, g(x))$-clean ring if every element is $(n, g(x))$-clean. The class of clean rings, $n$-clean ring, and $g(x)$-clean rings is a proper subset of the class of $(n, g(x))$-clean rings. In this paper, we investigate some properties of $(n, g(x))$-clean rings.

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1 INTRODUCTION

Throughout this paper, $R$ denotes an associative ring with unity, $U(R)$ its group of units, $\sqrt{0}$ its nilradical, $Id(R)$ its set of idempotents, and $J(R)$ its Jacobson radical.

A ring $R$ is called clean if every element of $R$ can be written as a sum of a unit and an idempotent. This definition was introduced by Nicholson [5]. According to Xiao and Tong [7], an element $x$ of a ring $R$ is called $n$-clean if $x = u_1 + u_2 + \cdots + u_n + e$ where $e \in Id(R)$, $u_i \in U(R)$, and $n$ is a positive integer. The ring $R$ is called $n$-clean if every element of $R$ is $n$-clean for some fixed positive integer $n$.

Let $C(R)$ be the center of a ring $R$ and $g(x)$ be a fixed polynomial in $C(R)[x]$. Camillo and Simón [1] defined $R$ to be $g(x)$-clean ring if each $x \in R$ has the form $x = a + s$ where $a$ is a unit of $R$ and $g(s) = 0$. 

Nicholson and Zhou [4] showed that $\text{End}(R M)$ is $g(x)$-clean where $R M$ is a semisimple left $R$-module and $g(x) \in (x - a)(x - b)C(R)[x]$ with $a, b \in C(R)$ and $b, b - a \in U(R)$. Also Fan and Yang [2] investigated $g(x)$-clean rings and obtained several important results. In this paper, we will extend the definition of $g(x)$-clean ring to obtain a larger class of rings, $(n, g(x))$-clean rings.

2 $(n, g(x))$-CLEAN RINGS

Let $C(R)$ be the center of a ring $R$ and $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $r \in R$ is called $g(x)$-clean [1], if $r = s + u$ where $g(s) = 0$ and $u$ is a unit of $R$ and $R$ is $g(x)$-clean if every element of $R$ is $g(x)$-clean.

**Definition 2.1.** Let $n$ be a positive integer and let $g(x)$ be a fixed polynomial in $C(R)[x]$. An element $\alpha \in R$ is called $(n, g(x))$-clean if $\alpha = u_1 + u_2 + \cdots + u_n + s$, where $g(s) = 0$ and $u_1, u_2, ..., u_n$ are units in $R$. A ring is called an $(n, g(x))$-clean ring if every element in $R$ is $(n, g(x))$-clean.

Clearly, clean rings are $(1, x^2 - x)$-clean rings, $n$-clean rings are $(n, x^2 - x)$-clean rings, $g(x)$-clean rings are $(1, g(x))$-clean rings.

**Example 2.2.** Let $R$ be the ring of all $3 \times 3$ upper triangular matrices over $\mathbb{Z}_2$. Since

\[
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},
\]

where \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) are units in $R$, and

\[
\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ is } (2, x^2 + x^3)-\text{clean.}
\]

**Example 2.3.** Let $G$ be a cyclic group of order 3, then the group ring $\mathbb{Z}_7[G]$ is 2-clean ring by Xiao and Tong [6] but $\mathbb{Z}_7[G]$ is not clean by Han and Nicholson [3]. So $\mathbb{Z}_7[G]$ is $(2, x^2 - x)$-clean ring which is not $(1, x^2 - x)$-clean ring. Thus, we obtain an example which is $(2, x^2 - x)$-clean ring but not $(x^2 - x)$-clean ring.

**Proposition 2.4.** Let $g(x)$ be a fixed polynomial in $C(R)[x]$. The following two conditions are equivalent:
(1) Every element \( \alpha \in R \) has the form \( \alpha = u_1 + u_2 + \cdots + u_n + s \), where \( g(s) = 0 \) and \( u_1, u_2, \ldots, u_n \) are units in \( R \).

(2) Every element \( \alpha \in R \) has the form \( \alpha = u_1 + u_2 + \cdots + u_n - s \), where \( g(s) = 0 \) and \( u_1, u_2, \ldots, u_n \) are units in \( R \).

**Proof.** (1) \( \Rightarrow \) (2) Let \( \alpha \in R \). Write \( -\alpha = v_1 + v_2 + \cdots + v_n + s \), where \( g(s) = 0 \) and \( v_1, v_2, \ldots, v_n \) are units in \( R \). Then \( \alpha = (v_1 - v_2) + \cdots + (v_n - s) \) where \( -(v_i) \in U(R) \) \( 1 \leq i \leq n \), \( g(s) = 0 \).

(2) \( \Rightarrow \) (1) Let \( \alpha \in R \). Write \( -\alpha = v_1 + v_2 + \cdots + v_n - s \), where \( g(s) = 0 \) and \( v_1, v_2, \ldots, v_n \) are units in \( R \). Then \( \alpha = (v_1 - v_2) + \cdots + (v_n + s) \) where \( -(v_i) \in U(R) \) \( 1 \leq i \leq n \), \( g(s) = 0 \).

Recall that an element \( x \in R \) is called periodic if there exist integers \( n, m \) with \( n > m \geq 1 \) such that \( x^n = x^m \). Ye [8], an element \( r \) of a ring \( R \) is called semiclean if \( r = a + u \), where \( a \) is periodic and \( u \) is a unit in \( R \).

If an element \( r \in R \) is semiclean, then \( r = a + u \), where \( u \) is a unit and \( a^k = a^l \), for some positive integers \( k, l \) \( k > l \geq 1 \). So, \( r \) is an \((1, x^k - x^l)\)-clean.

**Lemma 2.5.** (Ye [8]): Every periodic element in a ring \( R \) is clean.

**Proposition 2.6.** Let \( n, m \) be two positive integers, \( m > 1 \). If the ring \( R \) is \((n, x^m - x)\)-clean ring, then \( R \) is \((n+1)\)-clean ring.

**Proof.** Let \( r \in R \) then \( r = u_1 + u_2 + \cdots + u_n + s \), and \( g(s) = s^m - s = 0 \). Since \( s \) is a periodic element in \( R \), \( s \) is clean. So \( s = v + e \), where \( v \in U(R) \) and \( e \in Id(R) \). Hence \( r = u_1 + u_2 + \cdots + u_n + v + e \), where \( u_1, u_2, \ldots, u_n, v \in U(R) \), \( e \in Id(R) \). Thus \( R \) is \((n+1)\)-clean ring.

For each positive integer \( n \), let \( U_n(R) \) denote the set of elements of \( R \) that can be written as a sum of no more than \( n \) units of \( R \).

**Proposition 2.7.** Let \( n, m \) be two positive integers. If \( R \) is an \((n, a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0)\)-clean ring, \( a_0 \in U(R) \) then \( R = U_{n+1}(R) \).

**Proof.** Let \( \alpha \in R \). Then \( \alpha = u_1 + u_2 + \cdots + u_n + s \), where \( a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s = 0 \) and \( u_1, u_2, \ldots, u_n \) are units in \( R \). So \( (a_m s^{m-1} + a_{m-1} s^{m-2} + \cdots + a_1) s = -a_0 \in U(R) \). Therefore \( s \in U(R) \). Hence \( R = U_{n+1}(R) \).

**Proposition 2.8.** Let \( f : R \rightarrow S \) be a ring epimorphism. If \( R \) is an \((n, g(x))\)-clean ring, then \( S \) is an \((n, h(g(x)))\)-clean ring, where \( h \) is a map from \( C(R)[x] \) to \( C(S)[x] \) such that \( h(\sum a_i x^i) = \sum f(a_i) x^i \).
Proof. Let \( g(x) = \sum_{i=0}^{m} a_i x^i \in C(R) [x] \), then \( h(g(x)) = h(\sum_{i=0}^{m} a_i x^i) = \sum_{i=0}^{m} f(a_i)x^i \in C(S) [x] \). For each \( s \in S \) there exists \( r \in R \) such that \( f(r) = s \). Since \( R \) is an \((n, g(x))\)-clean ring, \( r = d + u_1 + u_2 + \cdots + u_n \), where \( g(d) = 0 \) and \( u_1, u_2, \ldots, u_n \) are units in \( R \). So \( 0 = f(g(d)) = f(a_0) + f(a_1)f(d) + \cdots + f(a_m)(f(d))^m = h(g(f(d))) \). Now \( s = f(r) = f(d) + f(u_1) + \cdots + f(u_n) \), where \( f(u_i) \in U(S) \quad (1 \leq i \leq n) \) and \( h(g(f(d))) = 0 \). Hence \( S \) is an \((n, h(g(x)))\)-clean ring.

Let \( R[[x]] \) be the ring of all formal power series over a ring \( R \). Fan and Yang [2] have shown that the ring \( R[[x]] \) is \((n, g(x))\)-clean ring if and only if \( R \) is \((g(x))\)-clean. We extend this result to \((n, g(x))\)-clean ring.

**Proposition 2.9.** Let \( n \) be a positive integer. The ring \( R[[x]] \) is \((n, g(x))\)-clean ring if and only if \( R \) is \((n, g(x))\)-clean ring.

**Proof.** Suppose that \( R[[x]] \) is an \((n, g(x))\)-clean ring. By proposition 2.8, \( R \cong R[[x]]/(x) \) is an \((n, g(x))\)-clean ring. Conversely, suppose \( R \) is an \((n, g(x))\)-clean ring. Let \( f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]] \). Write \( a_0 = u_1 + u_2 + \cdots + u_n + s \), where \( g(s) = 0 \) and \( u_1, u_2, \ldots, u_n \) are units in \( R \). Then \( f = s + (u_1 + \sum_{i=1}^{\infty} a_i x^i) + u_2 + \cdots + u_n \), where \( (u_1 + \sum_{i=1}^{\infty} a_i x^i) \in U(R[[x]]), \quad u_i \in U(R) \subseteq U(R[[x]]) \quad (2 \leq i \leq n) \), \( g(s) = 0 \). Thus \( R[[x]] \) is \((n, g(x))\)-clean ring.

**Proposition 2.10.** If \( R \) is any commutative ring with unity, then the polynomial ring \( R[x] \) is not \((n, g(x))\)-clean ring.

**Proof.** Since \( R \) is a commutative ring with unity, \( U(R[x]) = \{a_0 + a_1 x + \cdots + a_k x^k \mid a_0 \in U(R), a_1, \ldots, a_k \in \sqrt{0}\} \). If \( x \) is \((n, g(x))\)-clean, we may let \( x = (u_1 + a_{11} x + \cdots + a_{1k} x^k) + (u_2 + a_{21} x + \cdots + a_{2k} x^k) + \cdots + (u_n + a_{n1} x + \cdots + a_{nk} x^k) + s \), where \( g(s) = 0, \) \( u_1, u_2, \ldots, u_n \) are units in \( R \) and \( a_{ij} \in \sqrt{0} \subseteq J(R) \quad (1 \leq i \leq n, 1 \leq j \leq k_i) \). Then \( 1 = \sum_{i=1}^{n} a_{i1} \in J(R) \), which is a contradiction. Thus \( R[x] \) is not \((n, g(x))\)-clean ring.

**Proposition 2.11.** A direct product \( R = \Pi R_\alpha \) of rings \( \{R_\alpha\} \) is an \((n, g(x))\)-clean ring if and only if each \( R_\alpha \) is an \((n, g(x))\)-clean ring.

**Proof.** \((\Rightarrow)\) This follows from Proposition 2.8.

\((\Leftarrow)\) Suppose each \( R_\alpha \) is an \((n, g(x))\)-clean ring. Let \( x = (x_\alpha) \in \Pi R_\alpha \). For each \( \alpha, \) \( x_\alpha = u_1^\alpha + u_2^\alpha + \cdots + u_n^\alpha + s_\alpha \), where \( u_1^\alpha, u_2^\alpha, \ldots, u_n^\alpha \) are units in \( R_\alpha \)
and \( g(s_\alpha) = 0 \). Then \( x = u^1 + u^2 + \cdots + u^n + s \), where \( u^i = (u^i)_\alpha \in U(\Pi R_\alpha) \) (\( 1 \leq i \leq n \)) and \( g(s) = g((s_\alpha)) = 0 \). Hence \( R = \Pi R_\alpha \) is an \((n, g(x))\)-clean ring.

References


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