Approximate Solution of Weakly Singular Integral Equations by Iterative Methods

Feras M. Al Faqih
Department of Mathematics, King Faisal University, Saudi Arabia (KSA)
Postal address: AlAhssa – 31982 Alhafouf, P.O.Box 5909, Saudi Arabia (KSA)
oxfer@yahoo.com ,falfaqih@kfup.edu.sa

Abstract

In present paper we elaborated the numerical schemes of iterative methods for an approximate solution of weak singular integral equations with logarithmic Kernel. The equation is examined in a pair of spaces. The results obtained could be used for any pair of the functional spaces where the problem of finding the solution of weak singular integral equations is correctly formulated problem by Tihonovo.

Keywords: Weak Singular Integral Equation, Trigonometric Interpolating Polynomials, incorrectly formulated problem, iterative methods.

1 Introduction

The integral equations of the first kind with logarithmic kernels in the main part of the integral operator are used in different problems of mechanics, physics, chemistry and techniques [1]-[3],[6] and [7]. It is well known from the theory of such equations [8],[9] that the problem for finding the exact solution is possible only in rare particular cases and even in these cases it is necessary to calculate regular, singular and weakly singular integrals with complicated compactness.

In this connection the problem of elaboration of approximate methods for solution the weak singular integral equations of the first kind with corresponding theoretical background is important. This paper is devoted to an approximate solution by iterative methods of the integral equations with logarithmic kernel of the form
\[-\frac{1}{2\pi} \int_{0}^{2\pi} \ln \left( \sin \frac{s - \sigma}{2} \right) x(\sigma) d\sigma + \frac{1}{2\pi} \int_{0}^{2\pi} h(s, \sigma)x(\sigma) d\sigma = y(s), \quad s \in [0, 2\pi].\]  

(1.1)

Where \( h(s, \sigma), y(s) \), are known continuous \( 2\pi \)-periodic functions, \( x(\sigma) \) is an unknown function. The weak singular is to be understood as improperly one.

2 Problem Formulation

The problem of the finding the solution of the equation (1.1) is an incorrectly formulated problem. ([9]). It deals with the continuity of the weak singular integral operator \( S: X \rightarrow X \),

\[ Sx = -\frac{1}{2\pi} \int_{0}^{2\pi} \ln \left( \sin \frac{s - \sigma}{2} \right) x(\sigma) d\sigma, \quad x \in X \]

and the following continuity of the operator \( A: X \rightarrow X \), where

\[ Ax = Sx + Rx, \quad Rx \equiv \frac{1}{2\pi} \int_{0}^{2\pi} h(s, \sigma)x(\sigma) d\sigma. \]

We will study two iterative schemes for solution of the weak singular integral equations (1.1). The equation (1.1) is studied in pair of functional spaces \( \{X, Y\} \) where the problem for solving of equation (1.1) will be Correctly Formulated Problem. In our research we’ll follow by the methods from [9];

3 Numerical Schemes of Iterative Methods

Scheme 1

Let’s rewrite equation (1.1) in the equivalent form

\[ x + S^{-1} Rx = S^{-1} y, \quad (x \in X, \ S^{-1} \in X), \]

where \( S \) is a weakly singular operator with the logarithmic kernel:

\[ Sx = -\frac{1}{2\pi} \int_{0}^{2\pi} \ln \left( \sin \frac{s - \sigma}{2} \right) x(\sigma) d\sigma, \quad S^{-1} y = \frac{c_0(y)}{\ln 2} + 2 \sum_{k=-\infty}^{\infty} |k| c_k(y) e^{ikx}, \ i^2 = -1, \]

where

\[ c_k(y) = \frac{1}{2\pi} \int_{0}^{2\pi} y(\sigma) e^{-ik\sigma} d\sigma \]

are the Fourier coefficients for the function \( y(s) \) (see [10]) and
The solution of the initial equation (1.1) will be defined as the limit of the iterative sequence \( \{x_k\}_{k=0}^{\infty} \), where

\[
x_{k+1} = S^{-1}(y - Rx_k) \quad (k = 0,1,...);
\]

and \( x_0 \in X \) is an arbitrary initial approximation. The convergence of the method and the estimation of the error are established by the following

**Theorem 1.** If the regular operator \( R : X \rightarrow Y \) satisfy to the inequality

\[
q_1 \equiv \left\| S^{-1}R \right\|_{X \rightarrow X} < 1.
\]

then equation (1.1) has a unique solution \( x^* \in X \) which can be found as limit of iterative sequence (3.2):

\[
x^* = \lim_{k \rightarrow \infty} \left[ S^{-1} (y - Rx_k) \right]
\]

in the space \( X \). Here the error of the approximation at the step \( k \) is estimating by the following inequalities:

\[
\left\| x^* - x_k \right\|_X \leq q_1^k \left\| x^* - x_0 \right\|_X \leq \frac{q_1^k}{1 - q_1} \left\| x_1 - x_0 \right\|_X \quad (k = 1,2,...).
\]

**Consequence.** Let in conditions of theorem 1 the initial approximation be given by the formula

\[
x_0 = S^{-1}y, \quad y \in Y.
\]

Then the error of the approximate solution \( x_k \) is estimated by one of the following inequalities:

\[
\left\| x^* - x_k \right\|_X \leq q_1^{k+1} \left\| x^* \right\|_X ;
\]

\[
\left\| x^* - x_k \right\|_X \leq \frac{q_1^{k+1}}{1 - q_1} \left\| y \right\|_Y \quad (k = 1,2,...).
\]

**Proof.** The result of the theorem follows from general theorem of principle of the compressed reflections (see, for example, [5]). Let’s prove the consequence. Fairness of inequalities (3.5) and (3.6) is follows from the following calculations:

\[
\left\| x^* - x_k \right\|_X \leq q_1^k \left\| x^* - x_0 \right\|_X = q_1^k \left\| x^* - S^{-1}y \right\|_X =
\]

\[
= q_1^k \left\| S^{-1}y - S^{-1}Rx^* - S^{-1}y \right\|_X \leq q_1^k \left\| s^{-1}R \left\| x^* \right\|_X \quad (k = 1,2,...)
\]

and also,
Let introduce the notation \( z \equiv Sx \) \((x \in X, \ z \in Y)\). Then
\[
x = S^{-1}z \quad (x \in X, \ z \in Y).
\]

Now we will transform operator of equation (1.1) to an equivalent form, taking into account the relation (3.1), namely
\[
z + RS^{-1}z = y \quad (x \in X, \ y \in Y). \quad (3.7)
\]

We’ll define the solution of equation (3.7) as a limit of the iterative sequence \( \{z_k\}_{k=0}^{\infty} \), where
\[
z_{k+1} = y - RS^{-1}z_k \quad (k = 0,1,...); \quad (3.8)
\]

Here \( z_0 \in Y \) is an arbitrary initial approximation and the form of the inverse operator \( S^{-1} : Y \rightarrow X \) is in (3.1).

**Theorem 2.** Let the regular operator \( R : X \rightarrow Y \) satisfy the inequality (3.3). Then equation (1.1) has a unique solution \( x^* \in X \) that can be determined by the formula
\[
x^* = S^{-1}z^* ,
\]
where \( z^* \in Y \) can be found as a limit of the iterative sequence (3.8):
\[
\lim_{k \to \infty} y - RS^{-1}z_k
\]
in the space \( Y \). Here the error of the approximation \( x_k \) is estimated by one of the following inequalities:
\[
\left\| x^* - x_k \right\|_X \leq \frac{q_k}{1-q_1} \left\| x^* - x_0 \right\|_X \leq \frac{q_k}{1-q_2} \mu \left\| x_1 - x_0 \right\|_X ; \quad (k = 1,2,...),
\]

where \( x_k = S^{-1}z_k, \ q_2 = \left\| RS^{-1} \right\|_{Y \rightarrow Y}, \ \mu = \left\| S \cdot S^{-1} \right\| \).
Consequence. Let in conditions of theorem 2 the initial approximation be given by formula (3.4). Then the error of the approximation $x_k$ is estimated by the following inequality:

$$\|x^* - x_k\|_X \leq \frac{q_2^{k+1}}{1-q_2} \|y\|_Y \quad (k = 1, 2, \ldots).$$

Proof. The affirmations of the theorem and the consequence result from the arguments are brought at proof of the previous theorem, its consequence and the following inequalities:

$$\|x^* - x_k\|_X = \left\| S^{-1}(z^* - z_k) \right\|_X \leq \left\| S^{-1} \right\| \left\| z^* - z_k \right\|_Y \leq q_2^k \left\| S^{-1} \right\| \left\| z^* - z_0 \right\|_Y \leq$$

$$\leq q_2^k \left\| S \right\| \left\| S^{-1} \right\| \left\| x^* - x_0 \right\|_X \leq q_2^k \left\| S \right\| \left\| S^{-1} \right\| ^2 \left\| z^* - z_0 \right\|_Y \leq$$

$$\leq \frac{q_2^k}{1-q_2} \left\| S \right\| \left\| S^{-1} \right\| ^2 \left\| x_1 - x_0 \right\|_X;$$

$$\|x^* - x_k\|_X = \left\| S^{-1}(z^* - z_k) \right\|_X \leq \left\| S^{-1} \right\| \left\| z^* - z_k \right\|_Y \leq q_2^k \left\| S^{-1} \right\| \left\| z^* - z_0 \right\|_Y \leq$$

$$\leq \frac{q_2^k}{1-q_2} \left\| S^{-1} \right\| \left\| z_1 - z_0 \right\|_Y \leq \frac{q_2^k}{1-q_2} \left\| S \right\| \left\| S^{-1} \right\| \left\| x_1 - x_0 \right\|_X \quad (k = 1, 2, \ldots).$$

4 Quadrature Iterative Methods

The known shortages of direct and iterative methods of solving of the initial equation bring us to necessity of creation and studying of mixed methods. Such methods got the name of approximate-iterative ones. In this paper is examined the quadrature-iterative method for solving the equation (1.1). The approximate solution of weakly singular integral equation (1.1) will be presented in the form of trigonometric interpolating polynomial of the degree $n$:

$$x_n(\sigma) = \frac{2}{2n+1} \sum_{k=1}^{2n+1} a_k \Delta_n(\sigma - s_k) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left( a_k \cos ks + b_k \sin ks \right) + \frac{a_n}{2} \cos ks,$$

($a_k$, $b_k$ are real numbers) by the nodes

$$s_k = \frac{2\pi k}{2n+1} + \frac{\omega}{2n+1}, \quad k = 1, 2n+1,$$

(4.1)

Where $\Delta_n(s)$ is Dirichlet kernel of the order $n$ (see [6]).
The coefficients $a_k$ are determined by mechanical quadrature method from the system of linear algebraic equations of the order $2n + 1$

$$\frac{a_0}{2} \left[ \ln 2 + \frac{1}{2n} \sum_{l=1}^{2n} h(s_j, s_l) \right] + \sum_{k=1}^{n-1} \left[ a_k \psi_k(\cos \sigma; s_j) + b_k \psi_k(\sin \sigma; s_j) \right] +$$

$$+ \frac{a_n}{2} \psi_n(\cos \sigma; s_j) = y(s_j), \quad j = 1, 2n + 1, \quad (4.2)$$

where

$$\psi_k(\varphi(\sigma), s_j) = \frac{1}{2k} \varphi(ks_j) + \frac{1}{2n} \sum_{l=1}^{2n} h(s_j, s_l) \cdot \varphi(ks_l), \quad k = 1, n$$

and the nodes $s_j$ are defined in accordance with (4.1). This system was obtained in paper [4] when the equation (1.1) was solved by quadrature method. The system of linear algebraic equation (SLAE) (4.2) is equivalent to the operator equation

$$A_n x_n = S x_n + \mathcal{L}_{n, \omega} [IL_{n, \omega}^{\sigma} (h x_n)] = \mathcal{L}_{n, \omega} y. \quad (4.3)$$

For the large $n$, the solution of the obtained SLAE represents considerable difficulties in practice. Therefore, the solution of operator equation (4.3) (and, consequently, solution of SLAE (4.2)) will be determined by the iterative method

$$x_n^{k+1} = S^{-1} \left( y_n - R_n x_n^k \right) \quad (k = 0, 1, \ldots), \quad (4.4)$$

where $x_n^0$ is an arbitrary initial approximation from the subspace $X_n$. Here the elements $S^{-1} y_n$ and $S^{-1} R_n x_n^k$, in contrast to the elements $S^{-1} y$ and $S^{-1} R x_k$ can be calculated simply enough. For example, let

$$\varphi_n(s) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos ks + b_k \sin ks),$$

where $a_0, a_k, b_k \ (k = 1, n)$ are some coefficients; then, by virtue of (3.2) the inverse operator $S^{-1} : Y_n \to X_n$ can be represented by the following:

$$S^{-1}(\varphi_n, s) = \frac{a_0}{2 \ln 2} + 2i \sum_{k=1}^{n} (a_k - ib_k) e^{iks}.$$

**Theorem 3.** Let the inequality

$$p_n = \left\| S^{-1} R_n \right\|_{X_n \to X_s} < 1. \quad (4.5)$$

take place for $n$ (at least beginning with someone)
**Approximate solution**

Then for indicated $n$ the equation (4.3) has a unique solution $x_n^* \in X_n$ that can be found as a limit of iterative sequence (4.4) in the space $X_n$. Here, the error of the approximate solution $x_k$ is estimated by the following inequalities:

$$
\left\| x_n^* - x_n^k \right\|_{X_n} \leq p_n^k \left\| x_n^* - x_n^0 \right\|_{X_n} \leq \frac{p_n^k}{1 - p_n} \left\| x_n^1 - x_n^0 \right\|_{X_n} \quad (k = 1, 2, \ldots).
$$

**Consequence.** Let the initial approximation in conditions of theorem 3 be chosen in the form

$$
x_0^0 = S^{-1} y_n, \quad y_n \in Y_n.
$$

Then the error of the $k$th approximation is estimated by the following inequality:

$$
\left\| x_n^* - x_n^k \right\|_{X_n} \leq \frac{p_n^{k+1}}{1 - p_n} \left\| y_n \right\|_{Y_n} \quad (k = 1, 2, \ldots).
$$

The proof of the theorem and its consequence is similar to the proof of theorem 1 and its consequence.

**Theorem 4.** Let the inequality (4.5) take place for any $n$, at least beginning with someone, and equation (1.1) has the unique solution $x^* \in X$. Then for indicated $n$ the solution of equation (1.1) $x^* \in X$ can be found as the second limit in the space $X$:

$$
x^* = \lim_{n \to \infty} \lim_{k \to \infty} x_n^k
$$

for any $x_n^0 \in X_n$ and

$$
\left\| x^* - x_n^k \right\|_X \leq \left\| x^* - x_n^* \right\|_X + \left\| x_n^* - x_n^k \right\|_X \quad (k = 1, 2, \ldots).
$$

Here, the error $\left\| x^* - x_n^k \right\|_X$ of iterative method is estimated in theorem 3 and the error $\left\| x^* - x_n^* \right\|_X$ of mechanical quadrature method is estimated in [4].

Notice that the obtained theorems are general theorems for estimations of errors of the $k$th approximation of some quadrature-iterative method. We'll consider bellow an analogous theorems, which take into account the concrete form of initial equation (1.1) and also the building features of mechanical quadrature method.

Let $X = L_2[0, 2\pi]$, $Y = W_2^1[0, 2\pi]$ (see in [4]) and equation (1.1) is uniquely solvable in the space $X$ for any right-hand side $y \in Y$. 
Theorem 5. Let the inequality
\[
\frac{1}{2n + 1} \sum_{k=0}^{2n} \left\| h(s, s_k) \right\|_{1;2}^2 \leq \left( \frac{\pi}{2} + \frac{\pi}{n_0 \sqrt{3}} \right)^2.
\]
(4.6)
take place for any \( n \), at least beginning with someone.

Then for indicated \( n \) equation (4.3) has the unique solution \( x^*_n \in X_n \) that can be found as a limit of iterative sequence (4.4) in the space \( X_n \). Here, the error of the \( k \)th approximation is estimated by the following inequalities:
\[
\left\| x^*_n - x^*_n \right\|_2 \leq p^n \left\| x^*_n - x^*_n \right\|_2 \leq \frac{p^n}{1 - p^n} \left\| x^*_n - x^*_n \right\|_2 \quad (k = 1, 2, \ldots),
\]
where \( p_n = \frac{n_0 \sqrt{3}}{n_0 \sqrt{3} + 2} < 1 \).

Proof. The affirmation of the theorem follows from the results of theorem 3, taking into account the estimate brought below and the inequality
\[
R_n x_n = \mathcal{L}_{n, \omega} \left[ \mathcal{L}_{n, \omega}^q (h x_n) \right]:
\]
(4.7)
Then
\[
\left\| R_n x_n \right\|_{1;2} = \left\| \mathcal{L}_{n, \omega} \left[ \frac{1}{2n + 1} \sum_{k=0}^{2n} h(s, s_k) x_n(s_k) \right] \right\|_{1;2} \leq
\]
\[
\leq \left\| \mathcal{L}_{n, \omega} \left[ \frac{1}{2n + 1} \sum_{k=0}^{2n} h(s, s_k) x_n(s_k) \right] \right\|_{1;2} \leq
\]
\[
\leq \left\| \mathcal{L}_{n, \omega} \left[ \frac{1}{2n + 1} \sum_{k=0}^{2n} \left\| h(s, s_k) \right\|_1 \right\|_1 x_n(s_k) \right\|_{1;2} =
\]
\[
= \left\| \mathcal{L}_{n, \omega} \left[ \frac{1}{2n + 1} \sum_{k=0}^{2n} \left\| h(s, s_k) \right\|_1 x_n(s_k) \right\|_{1;2} \right\|_{1;2} \leq
\]
\[
\leq \left\| \mathcal{L}_{n, \omega} \left[ \frac{1}{2n + 1} \sum_{k=0}^{2n} \left\| h(s, s_k) \right\|_1 \left\| x_n(s_k) \right\|_1^2 \right] \right\|^{1/2} \leq \left[ \sum_{k=0}^{2n} \left\| x_n(s_k) \right\|_1^2 \right]^{1/2} =
\]
Approximate solution

\[ \leq \| \mathcal{L}_{n, \omega} \|_{\mathcal{W}_1 \to \mathcal{W}_1} \leq \frac{1}{\sqrt{2n+1}} \left[ \sum_{k=0}^{2n} \| h(s, s_k) \|_{1;2}^2 \right]^{1/2} \left[ \frac{1}{2n+1} \sum_{k=0}^{2n} | x_n(s_k) |^2 \right]^{1/2} \leq \]

\[ \leq \left( \frac{\pi}{2} + \frac{\pi}{n\sqrt{3}} \right) \frac{1}{\sqrt{2n+1}} \left[ \sum_{k=0}^{2n} \| h(s, s_k) \|_{1;2}^2 \right]^{1/2} \| x_n \|_2. \]

Theorem 5 is proved.

Remarks:

1. In conditions of theorem 5 the consequence analogous to theorem 3 and theorem 4 take place.
2. The condition of theorem 5 can be weakened, i.e.

Theorem 6. Let the following inequality

\[ p_1 = \max_{s, t \in [0, 2\pi]} | h(s, t) | < \frac{1}{2}. \]

takes place.

Then for any natural \( n \) equation \( (4.3) \) has the unique solution \( x_n^* \in X_n \) that can be found as a limit of iterative sequence \( (4.4) \) in the space \( X_n \). Here, the error of the \( k \)th approximation is estimated by the following inequalities:

\[ \| x_n^* - x^k_n \|_2 \leq p^k_1 \| x_n^* - x^0_n \|_2 \leq \frac{p^k_1}{1 - p_1} \| x_n^1 - x_n^0 \|_2 \quad (k = 1, 2, \ldots). \]

Proof. The affirmation of the theorem results from the estimates brought below, which follow from representation \( (4.7) \), taking into account results of theorem 3 and properties of Dirichlet kernel.

\[ \| R_n x_n \|_{1;2} = \left\| \frac{2}{(2n+1)^2} \sum_{j=0}^{2n} \sum_{k=0}^{2n} h(s_j, s_k) x_n(s_k) D_n(s - s_j) \right\|_{1;2} \leq \]

\[ \leq \frac{2}{(2n+1)^2} \sum_{j=0}^{2n} \sum_{k=0}^{2n} | h(s_j, s_k) | \cdot | x_n(s_k) | \cdot \| D_n(s - s_j) \|_{1,2} \leq \]

\[ \leq \frac{2}{(2n+1)^2} \left[ \sum_{j=0}^{2n} \sum_{k=0}^{2n} | h(s_j, s_k) |^2 \right]^{1/2} \left[ \sum_{j=0}^{2n} \sum_{k=0}^{2n} | x_n(s_k) |^2 \right]^{1/2} = \]
\[
\frac{2}{2n+1} \left( \sum_{j=0}^{2n} \sum_{k=0}^{2n} |h(s_j, s_k)|^2 \right)^{1/2} \left( \frac{1}{2n+1} \sum_{k=0}^{2n} \left| x_n(s_k) \right|^2 \right)^{1/2} =
\]

\[
= \frac{2}{2n+1} \left( \sum_{j=0}^{2n} \sum_{k=0}^{2n} h(s_j, s_k)^2 \right)^{1/2} \left\| x_n \right\|_2.
\]

Theorem 6 is proved.

**Conclusions:** The numerical schemes of iterative methods for approximate solution of weak singular integral equation (1.1) were elaborated. The equations were studied in pair functional spaces.

**References**


Received: October, 2008