Hyper Y-Domination in Bipartite Graphs

V. Swaminathan

Research Coordinator
Ramanujan Research Centre
S.N.College, Madurai, India.

Y. B. Venkatakrishnan

Department of Mathematics
SASTRA University, Tanjore, India
venkatakris2@maths.sastra.edu

Abstract

In this paper, we define hyper Y-dominating set in bipartite graph and give the bipartite theory of vertex-edge dominating set and edge-vertex dominating set.

Mathematics Subject Classification: 05C69

Keywords: Bipartite graph, Hyper Y-dominating set, edge-vertex and vertex-edge dominating sets

1. INTRODUCTION

All graphs considered here are undirected. All the notations not defined in this paper can be found in [1]. The bipartite theory of graphs were introduced in [4,5] and a parameter called X-domination number of a bipartite graph was defined. Let $G = (X, Y, E)$ be a bipartite graph with $|X| = p$ and $|Y| = q$. Two vertices $u$ and $v$ in $X$ are X-adjacent if they have a common adjacent vertex $y \in Y$. Let $u \in X$ and $\Delta_r = \max \{|N_r(u)|/ u \in X\}$ where the X-neighbor set $N_r(u)$ is defined as $N_r(u) = \{v \in X/ u \text{ and } v \text{ are X-adjacent}\}$.

A subset $D \subseteq X$ is an X-dominating set if every $x \in X$ is X-adjacent to some vertex in $D$. The minimum cardinality of a X-dominating set is called X-domination number and is denoted by $\gamma_X(G)$. 
We say a vertex \( x \in X \) hyper Y-dominates \( y \in Y \) if \( y \in N(x) \) or \( y \in N(N_Y(x)) \). A subset \( S \subseteq X \) is a hyper Y-dominating set if every \( y \in Y \) is hyper Y-dominated by a vertex of \( S \). The minimum cardinality of a hyper Y-dominating set is called hyper Y-domination number and is denoted by \( \gamma_{hY}(G) \).

2. BOUNDS ON \( \gamma_{hY}(G) \).

**Theorem 2.1:** In a bipartite graph, every X-dominating set is a hyper Y-dominating set, but not conversely.

**Proof:** Let \( D \subseteq X \) be a X-dominating set. \( \forall x \in X - S, \exists u \in D \) such that \( u \) and \( x \) are adjacent. Every \( y \in Y \) is adjacent to a element of \( D \) or adjacent to a vertex X-adjacent to a vertex of \( D \). Hence, \( D \) is a hyper Y-dominating set. \( \square \)

Conversely, consider the graph

\[
\begin{array}{c}
 a \quad b \quad c \quad d \\
 1 \quad 2 \quad 3 \\
\end{array}
\]

\( S = \{b\} \) is hyper Y-dominating set but not X-dominating set. \( \square \)

**Corollary 2.1.1:** In a bipartite graph \( G \), \( \gamma_{hY} \leq \gamma_X \).

**Theorem 2.2:** In a bipartite graph \( G \), \( \gamma_{hY} \geq \frac{|Y|}{\Delta \Delta_Y} \).

**Proof:** A vertex \( x \in X \) can hyper Y-dominate at most \( \Delta_Y \Delta \) vertices. Hence, \( \gamma_{hY} \geq \frac{|Y|}{\Delta \Delta_Y} \). \( \square \)

**Corollary 2.2.1:** In a bipartite graph \( G \), if every vertex in \( X \) is of degree 2 then \( \gamma_{hY} \geq \frac{|Y|}{2(\Delta - 1)\Delta} \).

**Proof:** If every vertex in \( X \) is of degree 2 then a vertex in \( X \) can have at most \( 2(\Delta - 1) \) X-neighbors. Hence, \( \gamma_{hY} \geq \frac{|Y|}{2(\Delta - 1)\Delta} \). \( \square \)
Corollary 2.2.2: In a bipartite graph $G$, if every vertex in $Y$ is of degree 2 then 
$\gamma_{hY} \geq \frac{|Y|}{\Delta^2}$.

Proof: If every vertex in $Y$ is of degree 2 then a vertex in $X$ can have at most $\Delta$ X-neighbors. Hence, $\gamma_{hY} \geq \frac{|Y|}{\Delta^2}$. □

3. HYPER Y-INDEPENDENT SET:

A subset $S \subseteq X$ is called a hyper Y-independent set if every $y \in Y$ satisfies one of the conditions (i) $y \notin N(x) \forall x \in S$ or (ii) There exists a neighbor of $y$ say $x \in S$ such that $N_y(x) \subset S$.

The maximum cardinality of a hyper Y-independent set is called hyper Y-independence number and is denoted by $\beta_{hY}(G)$.

[4,5] A subset $S \subseteq X$ is called a hyper independent set if $N(y) \subset S, \forall y \in Y$. The maximum cardinality of a hyper independent set is called hyper independence number and is denoted by $\beta_h(G)$.

Theorem 3.1: In a bipartite graph $G$, every hyper independent set is hyper Y-independent but not conversely.

Proof: Let $S$ be a hyper independent set. $N(y) \subset S$ for every $y \in Y$. Equivalently $y \notin N(x)$ for every $x \in S$ or there exists a neighbor of $y$ say $x \in S$ such that $N_y(x) \subset S$. Hence, $S$ is hyper Y-independent set.

Converse need not be true. Consider the graph

$S = \{a\}$ is hyper Y-independent set but not hyper independent set. □

Corollary 3.1.1: In a bipartite graph $G$, $\beta_h \leq \beta_{hY}$.

4. GALLAI TYPE THEOREM:

Theorem 4.1: In a bipartite graph $G$, $S$ is a hyper Y-dominating set if and only if $X - S$ is hyper Y-independent set.
Proof: $S$ is a hyper $Y$-dominating set. Every $y \in Y$ is adjacent to a vertex of $S$ or $y \in N(N_y(x))$ for some $x \in S$. Equivalently, $y \notin N(u)$ where $u \in X - S$ or $\exists u \in X - S$ adjacent to $y$ such that $N_y(u) \subseteq (X - S)$. Therefore, $X - S$ is hyper $Y$-independent set.

Conversely, $D$ is a hyper $Y$-independent set. For every $y \in Y$ one of the condition is satisfied (i) $y \notin N(x) \forall x \in D$ or (ii) There exists a neighbor of $y$ say $x \in D$ such that $N_y(x) \subseteq D$. Equivalently, $y \notin N(u)$ for some $u \in X - D$ or there exists $u \in X - D$ such that $N(N_y(u))$ contains $y$. $X - D$ is hyper $Y$-dominating set. $\square$

Corollary 4.1.1: In a bipartite graph $G$, $\gamma_{hv}(G) + \beta_{hv}(G) = |X|$.

5. BIPARTITE THEORY OF GRAPHS

[4,5] suggests that given any problem, say P, on an arbitrary graph $G$, there is very likely a corresponding problem Q on a bipartite graph $G^1$, such that a solution for Q provides a solution for P.

Various bipartite graphs can be constructed from an arbitrary graph $G$, some of them are defined as in [4,5]. The graph $VE(G) = (V,E,F)$ is defined by the edges $F=\{(u,e)/e=(u,v) \in E\}$. Let $V^1$ be a copy of the vertices $V$ of $G$. The graph $VV(G) = (V,V^1,E^1)$ is defined by the edges $E^1=\{(u,v^1)/(u,v) \in E\}$ and the graph $VV^+ = (V, V^1,E^+)$ contains the edges $E^1$ of the graph $VV$ together with the edges $\{(u,u^1)/u \in V\}$.

[3] Given an arbitrary graph $G=(V,E)$, a vertex $u \in V(G)$ ve-dominates an edge $vw \in E(G)$ if (i) $u = v$ or $u = w$ (u incident to $vw$) or (ii) $uv$ or $uw$ is an edge in $G$. A set $S \subseteq V(G)$ is a vertex-edge dominating set if for all edges $e \in E(G)$, there exists a vertex $v \in S$ such that $v$ dominates $e$. The minimum cardinality of a ve-dominating set of $G$ is called the vertex-edge domination number and is denoted as $\gamma_{ve}(G)$.

[3] An edge $e = uv \in E(G)$ ev-dominates a vertex $w \in V(G)$ if (i) $u = w$ or $v = w$ (w is incident to $e$) or (ii) $uw$ or $vw$ is an edge in $G$. (w is adjacent to $u$ or $v$). A set $S \subseteq E(G)$ is an edge-vertex dominating set if for all vertices $v \in V(G)$, there exists an edge $e \in S$ such that $e$ dominates $v$. The minimum cardinality of a ev-dominating set of $G$ is called the edge-vertex domination number and is denoted as $\gamma_{ev}(G)$.

We give the bipartite equivalent of vertex-edge and edge-vertex dominating sets.

Theorem 5.1: For any graph $G$, $\gamma_{hv}(VE(G)) = \gamma_{ve}(G)$.

Proof: Let $D$ be a $\gamma_{ve}$ set of graph $G$. Every edge in $G$ is incident with a vertex in $D$ or adjacent to a edge incident with a vertex in $D$. In $VE(G)$, every vertex
in \( Y \) is adjacent to a vertex of \( D \) or adjacent to a vertex \( X \)-adjacent to a vertex of \( D \). \( D \) is a hyper \( Y \)-dominating set in \( \text{VE}(G) \). Therefore, 
\[
\gamma_{HY}(\text{VE}(G)) \leq |S| = \gamma_{ve}(G).
\]
Conversely, let \( S \) be a \( \gamma_{HY}(\text{VE}(G)) \) set. Every vertex \( y \in Y \) is incident with a vertex in \( S \) or incident with a vertex \( X \)-adjacent to a vertex in \( S \). In graph \( G \), every edge is incident with a vertex in \( S \) or incident with a vertex adjacent to a vertex of \( S \). \( S \) is a vertex edge dominating set. Therefore, 
\[
\gamma_{ve} \leq |S| = \gamma_{HY}(\text{VE}(G)).
\]
Hence, \( \gamma_{HY}(\text{VE}(G)) = \gamma_{ve}(G) \). \( \square \)

**Theorem 5.2:** For any graph \( G \), \( \gamma_{HY}(\text{EV}(G)) = \gamma_{ve}(G) \).

**Proof:** Let \( D \) be a \( \gamma_{ve}(G) \) set of graph \( G \). Every vertex in \( G \) is incident with an edge in \( D \) or incident with an edge adjacent to an edge in \( D \). In \( \text{EV}(G) \), every vertex in \( Y \) is adjacent to a vertex in \( D \) or adjacent to a vertex which is \( X \)-adjacent to a vertex in \( D \). Therefore \( D \) is hyper \( Y \)-dominating set of \( \text{EV}(G) \).
\[
\gamma_{HY}(\text{EV}(G)) \leq |D| = \gamma_{ve}(G).
\]
Conversely, let \( S \) be a \( \gamma_{HY}(\text{EV}(G)) \) set of \( \text{EV}(G) \). Every vertex in \( Y \) is adjacent to a vertex in \( S \) or adjacent to a vertex \( X \)-adjacent to a vertex of \( S \). In \( G \), every vertex in \( G \) is incident with an edge in \( S \) or incident with an edge adjacent to a vertex in \( S \). Therefore, \( S \) is an edge vertex dominating set. Hence, \( \gamma_{ev}(G) \leq |S| = \gamma_{HY}(\text{EV}(G)). \) Therefore, \( \gamma_{HY}(\text{EV}(G)) = \gamma_{ev}(G) \). \( \square \)

We construct a new graph \( G_{13} \) from a given graph \( G \) as follows, \( G_{13} \) has the same vertex set as \( G \) and two vertices in \( G_{13} \) are adjacent iff they are at a distance one or three in \( G \).

**Theorem 5.3:** For any graph \( G \), \( \gamma_{HY}(\text{VV}(G)) = \gamma_{t}(G_{13}) \).

**Proof:** Let \( S \) be a \( \gamma_{t}(G_{13}) \) set of graph \( G_{13} \). For every \( v \in V \) there exists \( u \in S \) such that \( u \) and \( v \) are adjacent. In \( G \), \( d(u,v) = 1 \) or \( 3 \). In graph \( \text{VV} \), \( u \in S \) hyper \( Y \)-dominates \( v \). Therefore, \( S \) is a hyper \( Y \)-dominating set.
\[
\gamma_{HY}(\text{VV}(G)) \leq |S| = \gamma_{t}(G_{13}) \).
\]
Conversely, \( D \) is a \( \gamma_{HY}(\text{VV}(G)) \) set. Every \( v \in Y \) is hyper \( Y \)-dominated by a vertex in \( D \). In graph \( G \), \( d(u,v) = 1 \) or \( 3 \). Therefore, \( u \) and \( v \) are adjacent in \( G_{13} \). Hence, \( D \) is a total dominating set in \( G_{13} \).
\[
\gamma_{t}(G_{13}) \leq |D| = \gamma_{HY}(\text{VV})
\]
Therefore, \( \gamma_{HY}(\text{VV}(G)) = \gamma_{t}(G_{13}) \). \( \square \)

**Theorem 5.4:** For any graph \( G \), \( \gamma_{HY}(\text{VV}^*) = \gamma_{s3}(G) \).
Proof: Let $S$ by a $\gamma_{hY}(VV^+)$ set. Every vertex in $Y$ is adjacent to a vertex of $S$ or adjacent to a vertex $X$-adjacent to a vertex of $S$. In graph $G$, $d(v, S) \leq 3$ for every vertex $v \in V - S$. Hence, $S$ is a distance-3 dominating set.

$\gamma_{\leq 3}(G) \leq |S| = \gamma_{hY}(VV^+)$. Conversely, $D$ is a $\gamma_{\leq 3}(G)$ set. For every $u \in V - S$ there exists a vertex $v \in S$ such that $d(u, v) \leq 3$. In graph $VV^+(G)$, every vertex in $Y$ is adjacent to a vertex of $D$ or adjacent to a vertex $X$-adjacent to a vertex of $D$. Therefore, $D$ is a hyper Y-dominating set. Hence, $\gamma_{hY}(VV^+) \leq |D| = \gamma_{\leq 3}(G)$. Therefore, $\gamma_{hY}(VV^+) = \gamma_{\leq 3}(G)$. □

Corollary 2.2.1 and 2.2.2 gives the following

**Theorem 5.5:** For any graph $G$, $\gamma_{we} \geq \frac{|E|}{\Delta^2}$ and $\gamma_{ev} \geq \frac{|V|}{2\Delta(\Delta - 1)}$.

**REFERENCES**


Received: October, 2008