Abstract

In this paper, we generalise Peiffer elements for groups given in [9] to higher dimensions of pro-$C$ group case giving systematic ways of generating them.

Mathematics Subject Classification: 18G50, 18G30, 55P10

Keywords: Profinite Groups, Simplicial Groups, Peiffer elements

1 Introduction

The theory of profinite and in particularly pro-$p$ groups has several times provided significant results for the theory of presentations of finite groups. The most remarkable example of this is, of course, the Golod-Šafarevič Theorem (c.f. [10]).

Profinite and pro-$C$ simplicial groups occupy a place somewhere between homological group theory, homotopy theory, algebraic $K$-theory and algebraic geometry.

R. Brown and J.-L. Loday [2], noted that if the second dimension $G_2$ of a simplicial group, $G$, is generated by degenerate elements, that is, elements coming from lower dimensions, then the image of the second term, $\partial_2 NG_2$, of the Moore complex, $(NG, \partial)$, of $G$ by the differential $\partial$ is

$$[\text{Ker}d_1, \text{Ker}d_0]$$

where the square brackets as usual denote the commutator subgroup.

In this paper, we generalise Peiffer elements for commutative algebra given in [1] and for groups given in [9] to higher dimensions of pro-$C$ group case giving
systematic ways of generating them. The methods we use are based on ideas of Conduche, [5] and techniques developed by Carrasco and Cegarra [4].

Terminology

In this paper \( \mathcal{C} \) will denote a class of finite groups which is closed under the formation of subgroups, homomorphic images, finite products and which contains at least one non-trivial group.

2 Definitions and notation

A pro-\( \mathcal{C} \) simplicial group \( P \) consists of pro-\( \mathcal{C} \) groups \( \{ P_n \} \) together with continuous face and degeneracy maps \( d_i = d^n_i : P_n \to P_{n-1}, 0 \leq i \leq n, (n \neq 0) \) and \( s_i = s^n_i : P_n \to P_{n+1}, 0 \leq i \leq n \), satisfying the usual simplicial identities given in [6] and also [7], [8].

For the ordered set \( [n] = \{0 < 1 < \ldots < n\} \), let \( \alpha^n_i : [n+1] \to [n] \) be the increasing surjective map given by

\[
\alpha^n_i(j) = \begin{cases} 
  j & \text{if } j \leq i, \\
  j-1 & \text{if } j > i.
\end{cases}
\]

Let \( S(n, n-l) \) be the set of all monotone increasing surjective maps from \( [n] \) to \( [n-1] \). This can be generated from the various \( \alpha^n_i \) by composition. The composition of these generating maps satisfies the rule \( \alpha_j \alpha_i = \alpha_{i-1} \alpha_j \) with \( j < i \). This implies that every element \( \alpha \in S(n, n-l) \) has a unique expression as \( \alpha = \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_l} \) with \( 0 \leq i_1 < i_2 < \ldots < i_l \leq n \), where the indices \( i_k \) are the elements of \( [n] \) at which \( \alpha(i) = \alpha(i+1) \). We thus can identify \( S(n, n-l) \) with the set \( \{(i_1, \ldots, i_l) : 0 \leq i_1 < \ldots < i_l \leq n-1 \} \). In particular the single element of \( S(n, n) \), defined by the identity map on \( [n] \), corresponds to the empty 0-tuple (\( () \)) denoted by \( \emptyset \). Similarly the only element of \( S(n, 0) \) is \( (n-1, n-2, \ldots, 0) \). For all \( n \geq 0 \), let

\[
S(n) = \bigcup_{0 \leq l \leq n} S(n, n-l).
\]

We say that \( \alpha = (i_l, \ldots, i_1) < \beta = (j_m, \ldots, j_1) \) in \( S(n) \)

if \( i_1 = j_1, \ldots, i_k = j_k \) but \( i_{k+1} > j_{k+1} \) \( (k > 0) \)

or

if \( i_1 = j_1, \ldots, i_l = j_l \) and \( l < m \).

This makes \( S(n) \) an ordered set. For instance, the orders of \( S(2), S(3) \) and \( S(4) \) are respectively:

\[
S(2) = \{\emptyset < (1) < (0) < (1,0)\},
\]
Peiffer pairings in the Moore complex

\[ S(3) = \{ \emptyset < (2) < (1) < (2, 1) < (0) < (2, 0) < (1, 0) < (2, 1, 0) \}, \]
\[ S(4) = \{ \emptyset < (3) < (2) < (3, 2) < (1) < (3, 1) < (2, 1) < (3, 2, 1) < (0) < (3, 0) < (2, 0) < (3, 2, 0) < (1, 0) < (3, 1, 0) < (2, 1, 0) < (3, 2, 1, 0) \}. \]

If \( \alpha, \beta \in S(n) \) we define \( \alpha \cap \beta \) to be the set of indices which belong to both \( \alpha \) and \( \beta \).

The Moore complex \( \mathbf{NP} \) of a pro-\( \mathcal{C} \) simplicial group \( \mathbf{P} \) is defined to be the normal chain complex \((\mathbf{NP}, \partial)\) with

\[ \mathbf{NP}_n = \bigcap_{i=0}^{n-1} \text{Ker} d_i \]

and with continuous differential \( \partial_n : \mathbf{NP}_n \to \mathbf{NP}_{n-1} \) induced from \( d_n \) by restriction. Its homology gives the homotopy groups of the pro-\( \mathcal{C} \) simplicial group.

### 3 The semidirect decomposition of a pro-\( \mathcal{C} \) simplicial group

The fundamental idea behind this can be found in Conduche [5]. A detailed investigation of this for the case of simplicial groups is given in Carrasco and Cegarra [4].

**Lemma 3.1.** Let \( \mathbf{P} \) is a pro-\( \mathcal{C} \) simplicial group. Then \( P_n \) can be decomposed as a semidirect product:

\[ P_n \cong \text{Ker} d_0^n \rtimes s_0^{n-1}(P_{n-1}). \]

**Proof.** The isomorphism can be defined as follows:

\[ \theta : P_n \longrightarrow \text{Ker} d_0^n \rtimes s_0^{n-1}(P_{n-1}) \]
\[ p \longmapsto (ps_0d_0p^{-1}, s_0d_0p). \]

\( \square \)

Since we have the isomorphism \( P_n \cong \text{Ker} d_0^n \rtimes s_0^{n-1}(P_{n-1}) \), we can repeat this process as often as necessary to get each of the \( P_n \) as a multiple semidirect product of degeneracies of terms in the Moore complex. Let \( \mathbf{K} \) be the simplicial group defined by

\[ K_n = \text{Ker} d_0^{n+1}, d_i^m = d_i^{m+1} |_{\text{Ker} d_0^{n+1}} \text{ and } s_i^p = s_i^{n+1} |_{\text{Ker} d_0^{n+1}}. \]
Applying Lemma 1 to $P_{n-1}$ and to $K_{n-1}$, gives

$$P_n \cong \ker d_0 \rtimes s_0 P_{n-1} = \ker d_0 \rtimes s_0(\ker d_0 \rtimes s_0 P_{n-2}) = K_{n-1} \rtimes (s_0 \ker d_0 \rtimes s_0 s_0 P_{n-2}).$$

Since $K$ is a simplicial group, we have the following

$$\ker d_0^n = K_{n-1} \cong \ker d_0^k \rtimes s_0^k K_{n-2} = (\ker d_1 \cap \ker d_0) \rtimes s_1 \ker d_0$$

and this enables us to write

$$P_n = ((\ker d_1^n \cap \ker d_0^n) \rtimes s_1(\ker d_0^{n-1})) \rtimes (s_0(\ker d_0^{n-1}) \rtimes s_0 s_0(P_{n-2})).$$

Thus we can decompose $P_n$ as follows:

**Proposition 3.2.** If $P$ is a pro-$C$ simplicial group, then for any $n \geq 0$

$$P_n \cong (NP_n \rtimes s_{n-1}^n NP_{n-1}) \rtimes \ldots \rtimes s_i^n \{((NP_{n-1} \rtimes s_{n-2}^i NP_{n-2}) \rtimes \ldots \rtimes s_j^{n-2}(NP_{n-2} \rtimes s_{n-3}^j NP_{n-3}) \rtimes \ldots \rtimes s_j^{n-1}(NP_{n-1} \times s_j^1 NP_j))\}.$$

The bracketing and the order of terms in this multiple semidirect product are generated by the sequence:

$$P_1 \cong NP_1 \rtimes s_0^0 NP_0,$$

$$P_2 \cong (NP_2 \rtimes s_0^1 NP_1) \rtimes (s_1^1 NP_2 \rtimes s_0^1 s_0^0 NP_0),$$

$$P_3 \cong ((NP_3 \rtimes s_0^2 NP_2) \rtimes (s_2^2 NP_3 \rtimes s_0^2 s_0^1 NP_1) \rtimes (s_0^2 s_1^1 NP_1) \rtimes (s_1^2 s_0^1 NP_1 \rtimes s_0^2 s_0^1 s_0^0 NP_0))$$

and

$$P_4 \cong (((NP_4 \rtimes s_0^3 NP_3) \rtimes (s_3^3 NP_4 \rtimes s_0^3 s_0^2 NP_2)) \rtimes (s_0^3 s_0^2 s_0^1 NP_1) \rtimes (s_1^3 s_0^2 s_0^1 s_0^1 NP_1)) \rtimes (((s_3^3 s_3^2 s_2^2 NP_2) \rtimes (s_1^3 s_0^2 NP_2 \rtimes s_0^3 s_0^2 NP_1)) \rtimes (s_0^3 s_0^2 s_0^1 s_0^0 NP_0)).$$

and correspond to the order in $S(n)$ where the term corresponding to $\alpha = (i_l, \ldots, i_1) \in S(n)$ is $s_\alpha(NP_{n-\#\alpha}) = s_{i_l \ldots i_1}(NP_{n-\#\alpha}) = s_{i_l} \ldots s_{i_1}(NP_{n-\#\alpha})$, where $\#\alpha = l$. Hence any element $x \in P_n$ can be written in the form

$$x = y \prod_{\alpha \in S(n)} s_\alpha(x_\alpha) \text{ with } y \in NP_n \text{ and } x_\alpha \in NP_{n-\#\alpha}.$$

### 4 Higher order Peiffer elements

We will define closed normal subgroup $N_n$ of $P_n$. Let $G(n)$ be a set consisting of pairs of elements $(\alpha, \beta)$ from $S(n)$ with $\alpha \cap \beta = \emptyset$, and $\alpha \prec \beta$, where
Peiffer pairings in the Moore complex 981

\[ \alpha = (i_r, \ldots, i_1), \beta = (j_s, \ldots, j_1) \in S(n). \] We write \( \# \alpha = r \), i.e. the length of the string \( \alpha \). The linear continuous morphisms that we will need,

\[ \{ F_{\alpha, \beta} : NP_{n-\# \alpha} \times NP_{n-\# \beta} \rightarrow NP_n : (\alpha, \beta) \in G(n), \ n \geq 0 \} \]

are given as composites \( F_{\alpha, \beta} = g_\mu(s_\alpha \times s_\beta) \) where

\[ s_\alpha = s_{i_r} \ldots s_{i_1} : NP_{n-\# \alpha} \rightarrow P_n, \ s_\beta = s_{j_s} \ldots s_{j_1} : NP_{n-\# \beta} \rightarrow P_n, \]

\( p : P_n \rightarrow NP_n \) is defined by composite continuous projections \( g = g_{n-1} \ldots g_0 \), where \( g_j(z) = zs_jd_j(z)^{-1} \) with \( j = 0, 1, \ldots, n-1 \) and \( \mu : P_n \times P_n \rightarrow P_n \) is given by the commutator. Thus

\[ F_{\alpha, \beta}(x_\alpha, y_\beta) = (1s_{n-1}d_{n-1}^{-1}) \ldots (1s_0d_0^{-1})[s_\alpha(x_\alpha), s_\beta(y_\beta)]. \]

We define the closed normal subgroup \( N_n \) to be that generated by elements of the form \( F_{\alpha, \beta}(x_\alpha, y_\beta) \) where \( x_\alpha \in NP_{n-\# \alpha} \) and \( y_\beta \in NP_{n-\# \beta} \).

The idea for the construction of \( N_n \) and the use of the structure maps came from examining the thesis of Carrasco [3], see also Carrasco and Cegarra, [4].

Let \( P \) be a pro-\( \mathcal{C} \) simplicial group with Moore complex \( NP \) and for \( n > 1 \), let \( D_n \) be the closed normal subgroup generated by the degenerate elements in dimension \( n \). If \( P_n = D_n \), then

\[ \partial_n(NP_n) = \partial_n(N_n) \quad \text{for all } n > 1, \]

where \( N_n \) is a closed normal subgroup in \( P_n \) generated by a fairly small explicitly given set of elements, see below.

**Theorem 4.1.** If \( n = 2, 3 \) or \( 4 \), then the image of the Moore complex of the pro-\( \mathcal{C} \) simplicial group \( P \) can be given in the form

\[ \partial_n(NP_n) = \prod_{I,J} [K_I, K_J] \]

where the square brackets denote the closed commutator subgroup and \( \emptyset \neq I, J \subset [n-1] = \{0, 1, \ldots, n-1\} \) with \( I \cup J = [n-1] \), and where

\[ K_I = \bigcap_{i \in I} \text{Ker} d_i \quad \text{and} \quad K_J = \bigcap_{j \in J} \text{Ker} d_j. \]

**References**


Received: August, 2008