Inverse and Direct Systems in the Category of Intuitionistic M-Fuzzy Groups

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Abstract

In this study, we first define the concept of inverse and direct systems in category of intuitionistic M-fuzzy groups. Later we examine a series of their properties. We investigate whether or not the limits of inverse and direct systems of exact sequences of intuitionistic M-fuzzy groups are exact.

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1. Introduction

The inverse (direct) limit not only is an important concept in category theory, but also plays an important role in topology, algebra, homology theory etc. Up to the present, inverse and direct systems and their limits are defined in different categories. Furthermore, a series of their properties are investigated.

Recently, intuitionistic fuzzy topological space, intuitionistic fuzzy group, intuitionistic fuzzy $M$-group, intuitionistic fuzzy module, intuitionistic fuzzy ring, intuitionistic fuzzy $H_{\nu}$-module etc. are studied by researchers. After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalizations of the Notion of fuzzy sets. The idea of “intuitionistic fuzzy set” was first published by Atanasov [2], as a generalization of the Notion of fuzzy sets. Rosenfeld [12] introduced the concept of fuzzy group. Gu [7] put forward the notion of the fuzzy group with operators, Zhan and Tan [16] defined intuitionistic fuzzy group with operators and obtain some related results. Recently, methods of homology algebra had a widespread application in fuzzy topology. Lopez-Permouth and Malik [10] introduced category of fuzzy modules which is shown the notion R-fzmod. R-fzmod has products, coproducts, kernel and cokernels but it is not an abelian category. Also, projective, injective and free fuzzy left R-modules are characterized. Zahedi and Ameri [15] defined fuzzy exact sequence in category of fuzzy modules and obtained some results about these notions. Same researchers [1] introduced the notions of fuzzy (co)homology, fuzzy exact sequence of fuzzy complexes and fuzzy homotopy. Using methods of homology algebra, we give inverse and direct systems of intuitionistic $M$-fuzzy group and investigate a series of their properties. In this study, we firstly give the concept of inverse and direct systems in the category of intuitionistic $M$-fuzzy groups and prove that their limits always exist in this category. We investigate whether or not the limits of inverse and direct systems of exact sequences of intuitionistic $M$-fuzzy groups are exact. Later, we show that direct system limit of exact sequences of intuitionistic $M$-fuzzy groups is exact. Generally, the inverse system limit of exact sequences is not exact. Then we define the notion $\lim^{(i)}$ which is first derived functor of the inverse limit functor. Finally by using the notion $\lim^{(i)}$ we prove that the inverse system limit of exact sequences of intuitionistic $M$-fuzzy groups is exact and we show that $\lim^{(i)}(G_n,\mu_n,\lambda_n) = 0$.

2. Preliminaries

In this section, we review some definitions and results which will be used later. For details, reference may be made to R. Ameri and M.M. Zahedi [1], for fuzzy modules S.R.Lopez-Permouth, D.S.Malik [10], for intuitionistic fuzzy modules B.Davvaz, W.A.Dudek, Y.B.Yun [4], for intuitionistic fuzzy $M$-groups J. Zhan and Z.Tan [16].

Firstly, let us give some necessary definitions and use the same notations as in [10]. Let $R$ be a ring (possibly without 1). A fuzzy left (right) $R$-module is defined to be a pair $(M,\mu)$ where $M$ is a left (right) $R$-module and $\mu : M \rightarrow [0,1]$ is a function satisfying

(i) $\mu(x+y) \geq \min\{\mu(x),\mu(y)\}$,
Inverse and direct systems

(ii) \( \mu(-x) = \mu(x) \).

(iii) \( \mu(0) = 1 \), and

(iv) \( \mu(rx) \geq \mu(x) \left[ \mu(xr) \geq \mu(x) \right] \)

for all \( x, y \in M \) and \( r \in R \). As is standard in the fuzzy set literature, we will refer to \( \mu \) is a modular grade function for \( M \).

Given two fuzzy \( R \)-modules \((M, \mu)\) and \((N, \eta)\) and an \( R \)-homomorphism \( f : M \to N \), we say that \( f \) is a fuzzy \( R \)-homomorphism from \((M, \mu)\) into \((N, \eta)\) if \( f \) is a fuzzy map relative to the grade functions \( \mu \) and \( \eta \), i.e. if for all \( x \in M, \eta(f(x)) \geq \mu(x) \) [9].

Lemma 2.1 [10] Let \( M \) and \( N \) be \( R \)-modules and \( f : M \to N \) be an \( R \)-homomorphism.

(i) If \((M, \mu)\) is a fuzzy \( R \)-module, there exists a modular grade function \( \mu' \) on \( N \) such that for any modular grade function \( \eta \) on \( N \), \( \bar{f} : (M, \mu) \to (N, \eta) \) if and only if \( \eta \geq \mu' \).

(ii) If \((N, \eta)\) is a fuzzy \( R \)-module, there exists a modular grade function \( \eta_\ell \) on \( M \) such that for any fuzzy \( R \)-module \((M, \mu), \bar{f} : (M, \mu) \to (N, \eta) \) if and only if \( \mu \leq \eta_\ell \).

Here, \( \mu'(x) = \sup \{ \mu(x) : f(x) = y \}, \eta_\ell(y) = \eta(f(y)) \).

Lemma 2.2 [10] (i) Given modules \( \{M_i : i \in I\} \) and \( N \) and a family of \( R \)-homomorphisms \( A = \{ f_i : M_i \to N \mid i \in I \} \), if \( \{(M_i, \mu_i) \mid i \in I\} \) are fuzzy modules, there exists a smallest grade function \( \eta = \mu^d = \mu^{\{i\}^c} \) such that, for all \( i \in I \), \( \bar{f}_i : (M_i, \mu_i) \to (N, \eta) \).

(ii) Given modules \( M \) and \( \{N_i : i \in I\} \) and \( R \)-homomorphisms \( B = \{g_i : M \to N_i \mid i \in I\} \), if \( \{(N_i, \eta_i) \} \) are fuzzy modules, then there exists a largest grade function \( \mu = \eta_\eta = \eta^{\{g_i\}} \) such that, for all \( i \in I \), \( \bar{g}_i : (M, \mu) \to (N, \eta_i) \).

Here \( \eta = \mu_\eta = \vee_{i \in I} \mu^i, \mu = \eta_B = \wedge_{i \in I} (\eta_i)_{i^c} \).

Definition 2.3. [1] A fuzzy chain complex \( \theta_c = \{\theta^c_n, \hat{\partial}_n\} \) over \( \Lambda \) is an object in \( \text{fmg}_\Lambda^\Lambda \) together with a fuzzy endomorphism \( \hat{\partial} : \theta_c \to \theta_c \) of degree -1 with \( \hat{\partial} \hat{\partial} = 0 \).

Remark 2.4. [1] Let \( \theta_c = \{\theta^c_n, \hat{\partial}_n\} \) be a fuzzy chain complex. The condition \( \hat{\partial} \hat{\partial} = 0 \) implies that \( \text{Im} \hat{\partial}_n \subseteq \ker \hat{\partial}_n, n \in Z \). Hence, we can associate with \( \theta_c \) the fuzzy graded module \( H(\theta_c) = \{H_n(\theta_c)\} \).
where \( H_n(\theta_C) = \overline{\partial}_n, \overline{\partial}_n \) is the fuzzy quotient of \( \text{ker} \overline{\partial}_n \) by \( \text{Im} \overline{\partial}_{n+1}, n \in \mathbb{Z} \). Then \( H(\theta_C)(H_n(\theta_C)) \) is called the (nth) fuzzy homology module of \( \theta_C \) (of course, if \( \Lambda = \mathbb{Z} \) we can speak of the (nth) fuzzy homology group of \( \theta_C \)).

**Theorem 2.5.** [1] For each \( n, H_n(\_): \text{FComp} \to \Lambda - \text{fz mod} \) is an additive functor.

**Definition 2.6.** [1] A fuzzy homotopy \( \tilde{\varphi} \to \tilde{\psi} : \theta_C \to \nu_D \) between two fuzzy chain maps \( \tilde{\varphi}, \tilde{\psi} : \theta_C \to \nu_D \) is a morphism of degree +1 of fuzzy graded modules \( \tilde{\Sigma} : \theta_C \to \nu_D \) such that \( \psi - \varphi = \overline{\partial}_{\Sigma} + \Sigma \overline{\partial} \), i.e., such that for \( n \in \mathbb{Z} \)

\[ \psi_n - \varphi_n = \overline{\partial}_{n+1} + n \Sigma_n + n\Sigma_{n-1} \overline{\partial}_n. \]

**Proposition 2.7.** [1] If two fuzzy chain maps \( \tilde{\varphi}, \tilde{\psi} : \theta_C \to \nu_D \) are fuzzy homotopic, then

\[ H\left(\tilde{\varphi}\right) = H\left(\tilde{\psi}\right): H(\theta_C) \to H(\nu_D). \]

**Definition 2.8.** [2] An intuitionistic fuzzy set \( A \) in a non-empty set \( X \) is an object having the form

\[ A = \{ (x, \mu_A(x), \lambda_A(x)) \mid x \in X \} , \]

where the functions \( \mu_A : X \to [0,1] \) and \( \lambda_A : X \to [0,1] \) denote the degree of membership (namely \( \mu_A(x) \)) and the degree of nonmembership (namely \( \lambda_A(x) \)) of each element \( x \in X \) to the set \( A \), respectively, and \( 0 \leq \mu_A(x) + \lambda_A(x) \leq 1 \) for all \( x \in X \). For the sake of simplicity, we shall use the symbol \( A = (\mu_A, \lambda_A) \) for the intuitionistic fuzzy set \( A = \{ (x, \mu_A(x), \lambda_A(x)) \mid x \in X \} \).

**Definition 2.9.** [4] Let \( M \) a module over a ring \( R \). An intuitionistic fuzzy set \( A = (\mu_A, \lambda_A) \) in \( M \) is called an intuitionistic fuzzy submodule of \( M \) if

1. \( \mu_A(0) = 1, \)
2. \( \min \{ \mu_A(x), \mu_A(y) \} \leq \mu_A(x - y) \) for all \( x, y \in M \),
3. \( \mu_A(x) \leq \mu_A(r x) \) for all \( x \in M \) and \( r \in R \),
4. \( \lambda_A(0) = 0, \)
5. \( \lambda_A(x - y) \leq \max \{ \lambda_A(x), \lambda_A(y) \} \) for all \( x, y \in M \),
6. \( \lambda_A(r x) \leq \lambda_A(x) \) for all \( x \in M \) and \( r \in R \).

**Definition 2.10.** [8] A group with operators is an algebraic system consisting of a group \( G \), a set \( M \) and a function defined in the product set \( M \times G \) and having values in \( G \) such that, if \( ma \) denotes the element in \( G \) determined by the element \( a \) of \( G \) and the element \( m \) of \( M \), then \( m(ab) = (ma)(mb) \) holds for all \( a, b \in G \).
and \( m \in M \). We shall use the phrases “\( G \) is an \( M \)-group” to a group with operators.

**Definition 2.11.** [16] An IFS \( A \in \{(\alpha, \beta)\} \) in a group \( G \) is called an intuitionistic fuzzy subgroup of \( G \) if

i) \( \alpha (xy) \geq \alpha (x) \wedge \alpha (y) \) and \( \beta (xy) \geq \beta (x) \vee \beta (y) \)

ii) \( \alpha (x^{-1}) \geq \alpha (x) \) and \( \beta (x^{-1}) \leq \beta (x) \)

for all \( x, y \in G \).

**Definition 2.12.** [16] Let \( G \) be an \( M \)-group and IFS \( A \in \{(\alpha, \beta)\} \) be an intuitionistic fuzzy subgroup of \( G \). If \( \alpha (mx) \geq \alpha (x) \) and \( \beta (mx) \leq \beta (x) \) for all \( x \in G \) and \( m \in M \) then IFS \( A \in \{(\alpha, \beta)\} \) is said to be an intuitionistic fuzzy subgroup with operators of \( G \). We use the phrase “\( A \in \{(\alpha, \beta)\} \) is an intuitionistic \( M \)-fuzzy subgroup of \( G \)” instead of an intuitionistic fuzzy subgroup with operators of \( G \).

### 3. Inverse system of intuitionistic fuzzy modules

Now, we firstly define some algebraic operations in the category of intuitionistic \( M \)-fuzzy group. Let \( A \in \{(\mu, \lambda)\} \) be intuitionistic fuzzy subgroup of \( M \)-group \( G \).

We denote this group by \( (G, \mu, \lambda) \). We say this group as intuitionistic \( M \)-fuzzy group.

**Definition 3.1.** \( f: (G, \mu, \lambda) \to (G', \mu', \lambda') \) is homomorphism of intuitionistic \( M \)-fuzzy groups if and only if the condition \( \mu' (f (x)) \geq \mu (x) \) and \( \lambda' (f (x)) \leq \lambda (x) \) are satisfied.

Let \( (G, \mu, \lambda) \) be intuitionistic \( M \)-fuzzy group and \( H \) be a \( M \)-group, \( f: (G, \mu, \lambda) \to H \) be a homomorphism of \( M \)-group. By using \( A \in \{(\mu, \lambda)\} \) and \( f \), we can give intuitionistic \( M \)-fuzzy group in \( H \) by

\[
\mu' (y) = \sup \{ \mu (x) \mid f (x) = y \}, \quad \lambda' (y) = \inf \{ \lambda (x) \mid f (x) = y \}.
\]

It is clear that \( (H, \mu', \lambda') \) is intuitionistic \( M \)-fuzzy group and \( f: (G, \mu, \lambda) \to (H, \mu', \lambda') \) is a homomorphism of intuitionistic \( M \)-fuzzy group.

If \( G \) is \( M \)-group and \( (H, \eta, \nu) \) is intuitionistic \( M \)-fuzzy group and \( f: G \to (H, \eta, \nu) \) is homomorphism of \( M \)-group, then we can define intuitionistic \( M \)-fuzzy group structure in \( G \) by

\[
(\eta)_{f} (x) = \eta (f (x)), \quad (\nu)_{f} (x) = \nu (f (x)).
\]
Hence \((G,(\eta)_f,(\nu)_f)\) is intuitionistic \(M\)-fuzzy group and \(f : (G,(\eta)_f,(\nu)_f) \rightarrow (H,\eta,\nu)\) is homomorphism of intuitionistic \(M\)-fuzzy group [16].

**Lemma 3.2.** Let \(G\) and \(H\) be \(M\)-groups and \(f : G \rightarrow H\) be an \(M\)-group homomorphism.

**a)** If \((G,\mu,\lambda)\) is intuitionistic \(M\)-fuzzy group, there exists a modular grade functions \((\mu^f,\lambda^f)\) on \(H\) such that for any modular grade function \(B = (\eta,\nu)\) on \(H\), \(\overline{f} : (G,\mu,\lambda) \rightarrow (H,\eta,\nu)\) is intuitionistic fuzzy homomorphism if and only if

\[ \eta \geq \mu^f, \quad \nu \leq \lambda^f \]

**b)** \((H,\eta,\nu)\) is intuitionistic \(M\)-fuzzy group, there exist a modular grade functions \((\eta^f_f,\nu^f_f)\) on \(G\) such that for any intuitionistic \(M\)-fuzzy group \((G,\mu,\lambda)\), \(\overline{f} : (G,\mu,\lambda) \rightarrow (H,\eta,\nu)\) is intuitionistic fuzzy homomorphism if and only if

\[ \mu \leq (\eta^f_f), \quad \lambda \geq (\nu^f_f). \]

**Proof.** The proof is trivial.

**Lemma 3.3.** \(a\)** Given \(M\)-groups \(\{G_a\}_{a \in \Lambda}\) and \(H\) and a family of homomorphisms \(A = \{f_a : G_a \rightarrow H\}_{a \in \Lambda}\) if \((G_a,\mu_a,\lambda_a)\) are intuitionistic \(M\)-fuzzy groups, then there exists a smallest grade functions \(\eta = \mu^d = \mu^{f_a}\), \(\nu = \lambda^d = \lambda^{f_a}\) such that, for all \(\alpha \in \Lambda\), \(f : (G_a,\mu_a,\lambda_a) \rightarrow (H,\eta,\nu)\) is intuitionistic fuzzy homomorphism.

**b)** Given \(M\)-groups \(G\) and \(\{H_a\}_{a \in \Lambda}\) and a family of homomorphisms \(B = \{g_a : G \rightarrow H_a\}_{a \in \Lambda}\), if \((H_a,\eta_a,\nu_a)\) are intuitionistic \(M\)-fuzzy groups, then there exists a largest grade functions \(\mu = \eta_B = \eta_{(a)}, \lambda = \nu_B = \nu_{(a)}\) such that, for all \(\alpha \in \Lambda\), \(f : (G,\mu,\lambda) \rightarrow (H_a,\eta_a,\nu_a)\) is intuitionistic fuzzy homomorphism.

**Proof.** \(a\)** Let \(\eta = \mu^d = \vee_{a \in \Lambda} \mu^f_a\), \(\nu = \lambda^d = \wedge_{a \in \Lambda} \lambda^f_a\)

\[ \mathrm{b)} \ \text{Let} \ \mu = \eta_B = \wedge_{a \in \Lambda} (\eta_a)_{g_a}, \ \lambda = \nu_B = \vee_{a \in \Lambda} (\nu_a)_{g_a}. \]

By using this lemma, we define subgroup, quotient group, product and coproduct operations in the category of intuitionistic \(M\)-fuzzy groups.

If \((G,\mu,\lambda)\) is intuitionistic \(M\)-fuzzy group and \(H \subset G\) is subgroup, then \((H,\mu|_H,\lambda|_H)\) is intuitionistic fuzzy subgroup of \((G,\mu,\lambda)\).

If \((G,\mu,\lambda)\) is intuitionistic \(M\)-fuzzy group and \(p : G \rightarrow G/\sim\) is canonical homomorphism, then \((G/\sim,\mu_p,\lambda_p)\) is quotient group of \((G,\mu,\lambda)\). Hence, for each homomorphism of intuitionistic \(M\)-fuzzy group \(f : (G,\mu,\lambda) \rightarrow (H,\eta,\nu)\),
intuitionistic $M$-fuzzy group $(\ker f, \mu | \ker f, \lambda | \ker f)$ and $(H | mf, \eta_p, \nu_p)$ are obtained, where $p : H \to H | \text{Im} f$.

If $\{(G_a, \mu_a, \lambda_a)\}_{a \in \Lambda}$ is a family of intuitionistic $M$-fuzzy groups, then we define product of this families by $\left( \prod_{a \in \Lambda} G_a, \mu_A, \lambda_A \right)$, where $A = \left\{ \pi_a : \prod_{a \in \Lambda} G_a \to G_a \right\}_{a \in \Lambda}$ are the usual projection maps. Coproduct of this families is $\left( \sum_{a \in \Lambda} G_a, \mu^\#, \lambda^\# \right)$, where $B = \left\{ i_a : G_a \to \sum_{a \in \Lambda} G_a \right\}$ are the usual injections.

The following theorem is easily proved.

**Theorem 3.4.** The category of intuitionistic $M$-fuzzy groups has zero objects, sums, products, kernels and cokernels.

We denote the category of intuitionistic $M$-fuzzy group by $\text{IFMGr}$.

**Definition 3.5.** Any functor $D : \Lambda^{op} \to \text{IFMGr}$ ($D : \Lambda \to \text{IFMGr}$), where $\Lambda$ is directed set (considered as a category), is called the inverse (direct) system of intuitionistic $M$-fuzzy groups, the limit of $D$ is called limit of inverse (direct) system [3].

Let

$$(G, \mu, \lambda) = \left( \left\{ (G_a, \mu_a, \lambda_a) \right\}_{a \in \Lambda}, \left\{ p^a : (G_a, \mu_a, \lambda_a) \to (G, \mu, \lambda) \right\}_{a \in \Lambda} \right) \quad (1)$$

be inverse system of intuitionistic $M$-fuzzy groups. $A = \left\{ \pi_a : \prod_{a \in \Lambda} G_a \to G_a \right\}_{a \in \Lambda}$ be a family of projections and $\left( \prod_{a \in \Lambda} G_a, \mu_A, \lambda_A \right)$ be the direct product of the intuitionistic $M$-fuzzy groups. Then we get the intuitionistic $M$-fuzzy group

$$\left( \lim_{\leftarrow} G_a, \mu, \lambda \right) = \left( \lim_{\leftarrow} G_a, \mu|_a, \lambda|_a \right).$$

**Theorem 3.6.** Every inverse system in representation (1) has a limit in the category $\text{IFMGr}$, and this limit is equal to the intuitionistic $M$-fuzzy subgroup

$$\left( \lim_{\leftarrow} G_a, \mu|_a, \lambda|_a \right).$$

**Proof.** It suffices to show that, there exists a unique homomorphism of intuitionistic $M$-fuzzy groups $\overline{\psi} : (H, \eta, \nu) \to \left( \lim_{\leftarrow} G_a, \mu|_a, \lambda|_a \right)$ which has the following commutative diagram:
Here, for every intuitionistic $M$-fuzzy groups $(H,\eta,\nu)$ and $\alpha \prec \alpha'$, \(\overline{\varphi}_\alpha : (H,\eta,\nu) \rightarrow (G_a,\mu_a,\lambda_a)\) is the family of homomorphism of intuitionistic $M$-fuzzy groups which makes up the following commutative diagram:

\[
\begin{array}{ccc}
(H,\eta,\nu) & \xrightarrow{\overline{\varphi}_\alpha} & (G_a,\mu_a,\lambda_a) \\
\overline{\varphi}_{\alpha'} & & \overline{\alpha'} \\
\downarrow & & \downarrow \\
(G_{\alpha'},\mu_{\alpha'},\lambda_{\alpha'}) & \xrightarrow{\overline{\varphi}_\alpha} & (G_a,\mu_a,\lambda_a)
\end{array}
\]

and also \(\pi_a : \left(\varphi_a \left(\lim_{\alpha} G_a,\mu_a\right) \left| \varphi_a \left(\lim_{\alpha} G_a,\lambda_a\right) \left| \varphi_a \left(\lim_{\alpha} G_a\right)\right) \rightarrow (G_a,\mu_a,\lambda_a)\right)\) is canonical projection. We define \(\psi : H \rightarrow \lim_{\alpha} G_a\) as the homomorphism of $M$-groups such that for every $x \in N$,

\[\psi(x) = \langle \varphi_a(x) \rangle_{\alpha \in \land} .\]

Then, is the \(\overline{\psi} : (H,\eta,\nu) \rightarrow \left(\lim_{\alpha} G_a,\mu_a\right) \left| \nu \left(\lim_{\alpha} G_a\right)\right)\) an homomorphism of intuitionistic $M$-fuzzy groups? Since \(\varphi_a : (H,\eta,\nu) \rightarrow (G_a,\mu_a,\lambda_a)\) is the homomorphism of intuitionistic $M$-fuzzy groups for every $\alpha \in \land$, the condition

\[\mu_a \left(\varphi_a(x)\right) \geq \eta \left(x\right), \lambda_a \left(\varphi_a(x)\right) \leq \nu \left(x\right)\]

is satisfied, for $\forall x \in H$. Therefore, we obtain the condition

\[\mu_a \left(\left\{\varphi_a(x)\right\}\right) = \land_{\alpha \in \land} \mu_a \left(\varphi_a(x)\right) \geq \eta \left(x\right), \lambda_a \left(\left\{\varphi_a(x)\right\}\right) = \lor_{\alpha \in \land} \lambda_a \left(\varphi_a(x)\right) \leq \nu \left(x\right) .\]

Since $\psi$ is unique homomorphism, $\overline{\psi}$ is also unique homomorphism.

It is clear that $\lim$ is a functor from the category of inverse system of intuitionistic $M$-fuzzy groups to the category of intuitionistic $M$-fuzzy groups.
Let \((G, \mu, \lambda)\) intuitionistic \(M\)-fuzzy group exist. For any \(\alpha \in [0,1]\), the set \(U(\mu, \alpha) = \{x \in G \mid \mu(x) \geq \alpha\}\) \((L(\mu, \alpha) = \{x \in G \mid \mu(x) \leq \alpha\}\) is called an upper (lower) \(\alpha\)-level cut of \(\mu\) [16].

If \((G, \mu, \lambda) = \{(G_i, \mu_i, \lambda_i)\}_{i \in I}, \{p'_i\}_{i \in I}\) is inverse system of intuitionistic \(M\)-fuzzy groups, then for any \(\alpha \in \text{Im} \mu_i \cap \text{Im} \lambda_i, \forall i \in I\) the systems
\[
\{U(\mu_i, \alpha)\}_{i \in I} \quad \text{and} \quad \{(L(\lambda_i, \alpha))\}_{i \in I}
\]
are inverse systems of \(M\)-groups.

**Theorem 3.7.** Let \((G, \mu, \lambda) = \{(G_i, \mu_i, \lambda_i)\}_{i \in I}, \{p'_i\}_{i \in I}\) inverse system of intuitionistic \(M\)-fuzzy groups, then
\[
\lim_{\rightarrow} U(\mu_i, \alpha) = U\left(\mu_i \lim_{\rightarrow} G_i, \alpha\right) \quad \text{and} \quad \lim_{\rightarrow} U(\lambda_i, \alpha) = U\left(\lambda_i \lim_{\rightarrow} G_i, \alpha\right)
\]
are satisfied.

**Proof.** The proof is trivial.

**Definition 3.8.** A sequence
\[
\ldots \rightarrow (G_{n+1}, \mu_{n+1}, \lambda_{n+1}) \xrightarrow{f_{n+1}} (G_n, \mu_n, \lambda_n) \xrightarrow{f_n} (G_{n+1}, \mu_{n+1}, \lambda_{n+1}) \rightarrow \ldots
\]
of intuitionistic \(M\)-fuzzy groups is said to be fuzzy exact if and only if
\[
\left(\mu_n \mid \text{Im} f_{n+1}, \lambda_n \mid \text{Im} f_{n+1}\right) = \left(\mu_n \mid \text{ker} f_{n+1}, \lambda_n \mid \text{ker} f_{n+1}\right), \quad \text{for all} \; n \in \mathbb{Z}.
\]

It is clear that if the following sequence
\[
\ldots \rightarrow G_{n-1} \xrightarrow{h_n} G_n \xrightarrow{h_n} G_{n+1} \rightarrow \ldots
\]
is exact sequence of \(M\)-groups, then the sequence of intuitionistic \(M\)-fuzzy groups
\[
\ldots \rightarrow (G_{n-1}, \mu_{n-1}, \lambda_{n-1}) \xrightarrow{h_{n-1}} (G_n, \mu_n, \lambda_n) \xrightarrow{h_n} (G_{n+1}, \mu_{n+1}, \lambda_{n+1}) \rightarrow \ldots
\]
is exact, where \(\mu_n = \chi_{[0]}\), \(\lambda_n = 1 - \mu_n\).

For each intuitionistic fuzzy homomorphism \(\tilde{f} : (G, \mu, \lambda) \rightarrow (H, \eta, \nu)\), the following sequence
\[
\tilde{0} \rightarrow (\text{ker} f, \mu | \text{ker} f, \lambda | \text{ker} f) \xrightarrow{\tilde{i}} (G, \mu, \lambda) \xrightarrow{\tilde{r}} (H, \eta, \nu) \xrightarrow{\tilde{p}}
\]
is exact, where \(i\) is inclusion mapping, \(p\) is canonical mapping.

If the sequence of intuitionistic \(M\)-fuzzy groups in (2) is exact, then the sequence of \(M\)-groups in (3) is exact. But generally, inverse of this is not true. Homomorphism of \(M\)-groups can not be homomorphism of intuitionistic \(M\)-fuzzy groups [15].

**Example 3.9.** Let \(G\) be \(M\)-group. Define the fuzzy grade functions \(\mu, \lambda, \eta\) and \(\nu\) as follows:
\[
\mu = \chi_{[0]}, \quad \lambda = 1 - \chi_{[0]}, \quad \eta(x) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x \neq 0 \end{cases}, \quad \nu(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } x \neq 0 \end{cases}
\]

Then the sequence of intuitionistic \(M\)-fuzzy groups
is exact, but the sequence of $M$-groups

$$0 \to G \to G \to G \to 0$$

is not exact.

Is the limit of an exact sequence of inverse system of intuitionistic $M$-fuzzy groups exact?

**Example 3.10.** Let $G_n = Z, G'_n = Z, G''_n = Z_2$ be modules an $Z$ ring $\forall n \in N$. Then,

$$G = \{G_n\}_{n \in N}, \{p^{m+1}_n (m) = 3m\}, \quad G' = \{G'_n\}_{n \in N}, \{q^{m+1}_n (m) = 3m\},$$

$$G'' = \{G''_n\}_{n \in N}, \{p^{m+1}_n (m) = [m]\}$$

are inverse systems of $Z$-groups and

$$f = \{f_n : G'_n \to G_n | f_n (m) = 2m\}, \quad g = \{g_n : G_n \to G''_n | g_n (m) = [m]\}$$

are morphisms of inverse systems. The following sequence

$$0 \to G' \to G \to G'' \to 0$$

is short exact sequence of inverse systems of $Z$-groups. Then the following sequence

$$0 \to (G', \nu', \lambda') \to (G_n, \mu_n, \lambda_n) \otimes (G'_n, \mu'_n, \lambda'_n) \to 0$$

is also short exact sequence of intuitionistic $Z$-fuzzy groups, where

$$\mu_n = (\chi (0))_{G}, \lambda_n = 1 - \mu_n, \quad \mu_n' = (\chi (0))_{G'}, \lambda_n' = 1 - \mu_n', \quad \mu_n'' = (\chi (0))_{G''}, \lambda_n'' = 1 - \lambda_n''$$

Hence the sequence

$$0 \to (G', \nu', \lambda') \to (G, \mu, \lambda) \otimes (G'_n, \mu', \lambda') \to 0$$

is short exact sequence of inverse systems of intuitionistic $Z$-fuzzy groups. Taking the limits of this sequence, the sequence

$$0 \to 0 \to 0 \to (Z_2, \mu'', \lambda'') \to 0$$

is not exact.

A category of chain complexes (cochain complexes) is defined in the category of $IFMGr$ similarly as [1].

**Definition 3.11.** An intuitionistic fuzzy chain complexes $(G, \mu, \lambda) = \{(G_n, \mu_n, \lambda_n, \partial_n)\}_{n \in \mathbb{Z}}$ is an object in $IFMGr$ together with an intuitionistic fuzzy endomorphism $\overline{\partial} : (G, \mu, \lambda) \to (G, \mu, \lambda)$ of degree -1 with $\overline{\partial} \circ \overline{\partial} = 0$.

A morphism of intuitionistic fuzzy chain complexes $\overline{\phi} : (G, \mu, \lambda) \to (H, \eta, \nu)$ is a morphism $\overline{\phi} = \{\overline{\phi}_n : (G_n, \mu_n, \lambda_n) \to (H_n, \eta_n, \nu_n)\}_{n \in \mathbb{Z}}$ which has degree 0 such that $\overline{\phi}_n \circ \overline{\partial}_n = \overline{\partial}_n \circ \overline{\phi}_n$, where $\overline{\partial}$ denotes the fuzzy differential in $(H, \eta, \nu)$.

**Definition 3.12.** Let $(G, \mu, \lambda) = \{(G_n, \mu_n, \lambda_n, \partial_n)\}_{n \in \mathbb{Z}}$ be an intuitionistic fuzzy chain complex. The condition $\overline{\partial} \circ \overline{\partial} = 0$ implies that $\text{Im} \overline{\partial}_{n-1} \subseteq \ker \overline{\partial}_n, n \in \mathbb{Z}$. Hence, we can
associate with \((G, \mu, \lambda)\) the grade intuitionistic \(M\)-fuzzy group

\[ H((G, \mu, \lambda)) = \left\{ H_n(G, \mu, \lambda) \right\}, \]

where \( H_n(G, \mu, \lambda) = \left( \ker \partial_n, \mu_n \right| \ker \partial_n, \lambda_n \right) \left/ \left( \operatorname{Im} \partial_{n+1}, \mu_n \right| \operatorname{Im} \partial_{n+1}, \lambda_n \right) \). Then \( H((G, \mu, \lambda)) \) is called the intuitionistic \(M\)-fuzzy homology group of \((G, \mu, \lambda)\). Duality, it can be defined cochain complex and cohomology group.

Let \( \Phi, \Psi : (G, \mu, \lambda) \to (H, \eta, \nu) \) be two morphisms of intuitionistic fuzzy chain complexes.

**Definition 3.13.** An intuitionistic fuzzy homotopy \( \Sigma : (M, \mu, \lambda) \to (N, \eta, \nu) \) between \( \Phi \) and \( \Psi \) is a morphism of degree \(+1\) such that

\[ \Psi - \Phi = \partial \circ \Sigma + \Sigma \circ \partial. \]

We say that \( \Phi, \Psi \) are intuitionistic fuzzy homotopic, if there exists an intuitionistic fuzzy homotopy \( \Sigma \).

The following theorem can be easily proved.

**Theorem 3.14.** The intuitionistic fuzzy homotopy relation is an equivalence relation and intuitionistic \(M\)-fuzzy homology (cohomology) groups are invariant with respect to this relation.

We get inverse system in (1). We define the following homomorphism of \(M\)-groups:

\[ d : \prod \{ G_\alpha \} \to \prod \{ G_\alpha \} \]

by the formula:

\[ d \left( \{ x_\alpha \} \right) = \{ x_\alpha - p_\alpha (x_\alpha') \}_{\alpha \neq \alpha'}. \]

Here, is the \( d : \prod \left( \prod G_\alpha, \mu_\alpha, \lambda_\alpha \right) \to \prod \left( \prod G_\alpha, \mu_\alpha, \lambda_\alpha \right) \) an homomorphism of intuitionistic \(M\)-fuzzy groups?

\[ \mu_\alpha (d \left( \{ x_\alpha \} \right) = \mu_\alpha \left( \{ x_\alpha - p_\alpha (x_\alpha') \} \right) = \bigwedge \mu_\alpha \left( x_\alpha - p_\alpha (x_\alpha') \right), \]

\[ \lambda_\alpha (d \left( \{ x_\alpha \} \right) = \lambda_\alpha \left( \{ x_\alpha - p_\alpha (x_\alpha') \} \right) = \bigvee \lambda_\alpha \left( x_\alpha - p_\alpha (x_\alpha') \right). \]

Since \( \mu_\alpha \left( p_\alpha (x_\alpha') \right) \geq \mu_\alpha \left( x_\alpha' \right) \) and \( \lambda_\alpha \left( p_\alpha (x_\alpha') \right) \leq \lambda_\alpha \left( x_\alpha' \right), \)

\[ \mu_\alpha \left( d \left( \{ x_\alpha \} \right) \right) = \bigwedge \min \{ \mu_\alpha \left( x_\alpha \right), \lambda_\alpha \left( p_\alpha (x_\alpha') \right) \} = \bigwedge \mu_\alpha \left( x_\alpha \right) \wedge \lambda_\alpha \left( x_\alpha' \right), \]

and
\[ \lambda_\alpha \left( \{ x_\alpha \} \right) \leq \vee \max \{ \lambda_\alpha \left( x_\alpha \right), \lambda_\beta \left( x_\beta \right) \} = \vee \left( \lambda_\alpha \left( x_\alpha \right) \vee \lambda_\beta \left( x_\beta \right) \right) = \vee \lambda_\alpha \left( x_\alpha \right) = \lambda_\alpha \left( \{ x_\alpha \} \right) \]

Then \( \overline{d} \) is a homomorphism of intuitionistic \( \mathcal{M} \) -fuzzy groups. Therefore \((\ker d, \mu_\alpha) | \ker d, \lambda_\alpha | \ker d \) and \((\text{co ker} d, (\mu_\alpha)_p^\sim, (\lambda_\alpha)_p^\sim)\) are defined.

For inverse system of \( \mathcal{M} \) -groups \( \left\{ \left\{ G_\alpha \right\}_{\alpha \in \alpha}, \left\{ p_\alpha^\sim \right\}_{\alpha < \alpha'} \right\} \), \( \lim^{(1)} G_\alpha = \prod_\alpha G_\alpha / \text{Im} d \).

If \( \pi = \prod_\alpha G_\alpha \rightarrow \lim^{(1)} G_\alpha \) is the canonical homomorphism, we can define intuitionistic \( \mathcal{M} \) -fuzzy group by \( \left( \lim^{(1)} G_\alpha, (\mu_\alpha)^\pi, (\lambda_\alpha)^\pi \right) \).

**Definition 3.15.** \( \left( \lim^{(1)} G_\alpha, (\mu_\alpha)^\pi, (\lambda_\alpha)^\pi \right) \) is called “first derived functor” of the inverse system of intuitionistic \( \mathcal{M} \) -fuzzy groups given (1).

**Proposition 3.16.** \( \lim^{(1)} \) is a functor.

**Proof.** For this reason, it suffices to show that for each the morphism
\[
\overline{f} = \left( \rho : B \rightarrow A, \left( \overline{f}_\beta : (G_\rho(\beta), \mu_\rho(\beta), \lambda_\rho(\beta)) \rightarrow (H_\beta, \eta_\beta, \nu_\beta) \right)_{\beta \in B} \right),
\]
\[
\lim^{(1)} \overline{f} : \left( \lim^{(1)} G_\alpha, (\mu_\alpha)^\pi, (\lambda_\alpha)^\pi \right) \rightarrow \left( \lim^{(1)} H_\beta, (\eta_\beta)^\pi, (\nu_\beta)^\pi \right)
\]
\[
\text{is the morphism of intuitionistic } \mathcal{M} \text{-fuzzy groups. Since}
\]
\[
(\mu_\alpha)^\pi \left( x + \text{im} d \right) = \sup_{\text{z \in \text{im} d}} \mu_\alpha \left( x + z \right) \leq \sup_{\text{z \in \text{im} d}} \eta_\alpha \left( f \left( x + z \right) \right) = \sup_{\text{z \in \text{im} d}} \eta_\alpha \left( f \left( x \right) + f \left( z \right) \right)
\]
\[
= \sup_{y \in f \left( x \right) + y} \eta_\alpha \left( f \left( x \right) + y \right) = (\eta_\alpha)^\pi \left( \lim^{(1)} f \left( x + \text{im} d \right) \right)
\]
\[
(\lambda_\alpha)^\pi \left( x + \text{im} d \right) = \inf_{\text{z \in \text{im} d}} \lambda_\alpha \left( x + z \right) \geq \inf_{\text{z \in \text{im} d}} \nu_\alpha \left( f \left( x + z \right) \right) = \inf_{\text{z \in \text{im} d}} \nu_\alpha \left( f \left( x \right) + f \left( z \right) \right)
\]
\[
= \inf_{y \in f \left( x \right) + y} \nu_\alpha \left( f \left( x \right) + y \right) = (\nu_\alpha)^\pi \left( \lim^{(1)} f \left( x + \text{im} d \right) \right)
\]
\[
\lim^{(1)} \text{ is a functor.}
\]

Let us consider the following intuitionistic fuzzy cochain complex
\[
\rightarrow 0 \rightarrow \left( \prod_\alpha G_\alpha, \mu_\alpha, \lambda_\alpha \right) \rightarrow \left( \prod_\alpha G_\alpha, \mu_\alpha, \lambda_\alpha \right) \rightarrow 0.
\]

Intuitionistic \( \mathcal{M} \) -fuzzy cohomology group of this complex are \( \ker \overline{d} \) and \( \text{co ker} \overline{d} \).

**Lemma 3.17.** \( \lim^{(1)} (G_\alpha, \mu_\alpha, \lambda_\alpha) = \ker \overline{d} \) and \( \lim^{(1)} (G_\alpha, \mu_\alpha, \lambda_\alpha) = \text{co ker} \overline{d} \).

**Proof.** The proof of lemma is trivial.

We accept natural numbers set which is index set of inverse system.

**Theorem 3.18.** Let the sequence
\[
(\mu_1, \lambda_1) \leftarrow p_1 \rightarrow (\mu_2, \lambda_2) \leftarrow p_2 \rightarrow \ldots
\]
be inverse sequence of intuitionistic $M$-fuzzy groups. For each infinite
subsequence of this sequence, $\lim_{\kappa}^{(1)}$ doesn’t change.

**Proof.** Let $S = \{i, j, k, \ldots\}$ be infinite subsequence of natural numbers $\Box$. From
Lemma 3.17, $\lim_{\kappa}^{(1)}$ is defined by the following homomorphism of intuitionistic
$M$-fuzzy groups as appropriate subsequence $S$

$$\overline{d} : \prod_{\kappa \in S} G_{\kappa}, \wedge_{\kappa \in S} \mu_{\kappa}, \vee_{\kappa \in S} \lambda_{\kappa} \rightarrow \prod_{\kappa \in S} G_{\kappa}, \wedge_{\kappa \in S} \mu_{\kappa}, \vee_{\kappa \in S} \lambda_{\kappa}.$$  

We may define $f_0, f_1 : \prod_{\kappa \in S} G_{\kappa} \rightarrow \prod_{\kappa \in N} G_{\kappa}$ homomorphisms of $M$-groups with this
formula:

$$f_0(x_i, x_j, x_k, \ldots) = (p_1^i(x_i), p_2^j(x_j), \ldots, p_{i-1}^i(x_i), x_i, p_{i+1}^i(x_j), \ldots, p_{j-1}^j(x_j), x_j, \ldots)$$

$$f_1(x_i, x_j, x_k, \ldots) = (0, 0, \ldots, x_i, 0, \ldots, x_j, 0, \ldots, x_k, 0, \ldots).$$

Also,

$$\left(\bigwedge_{\kappa \in S} \mu_{\kappa}\right) \left( (p_1^i(x_i), \ldots, p_{i-1}^i(x_i), x_i, p_{i+1}^i(x_j), \ldots, p_{j-1}^j(x_j), x_j, \ldots) \right)$$

$$\quad = \mu_i \left( p_1^i(x_i) \right) \wedge \ldots \wedge \mu_{i-1} \left( p_{i-1}^i(x_i) \right) \wedge \mu_i \left( x_i \right) \wedge \mu_{i+1} \left( p_{i+1}^i(x_j) \right) \wedge \ldots \wedge \mu_j \left( x_j \right) \wedge \ldots$$

$$\geq \left[ \mu_i \left( x_i \right) \wedge \ldots \wedge \mu_i \left( x_i \right) \wedge \mu_i \left( x_i \right) \right] \wedge \left[ \mu_j \left( x_j \right) \wedge \ldots \wedge \mu_j \left( x_j \right) \right] \wedge \ldots$$

$$= \mu_i \left( x_i \right) \wedge \mu_j \left( x_j \right) \wedge \ldots = \bigwedge_{\kappa \in S} \mu_{\kappa} \left( x_i \right),$$

and

$$\left(\bigvee_{\kappa \in S} \lambda_{\kappa}\right) \left( (0, 0, \ldots, x_i, 0, \ldots, x_j, 0, \ldots) \right) = \lambda_i (0) \vee \ldots \vee \lambda_i (0) \vee \ldots \vee \lambda_j (x_j) \vee \ldots$$

$$\quad = \lambda_i (x_i) \vee \lambda_j (x_j) \vee \ldots = \bigvee_{\kappa \in S} \lambda_{\kappa} \left( x_i \right).$$

Then $\overline{f_0}, \overline{f_1} : \prod_{\kappa \in S} G_{\kappa}, \wedge_{\kappa \in S} \mu_{\kappa}, \vee_{\kappa \in S} \lambda_{\kappa} \rightarrow \prod_{\kappa \in N} G_{\kappa}, \wedge_{\kappa \in N} \mu_{\kappa}, \vee_{\kappa \in N} \lambda_{\kappa}$ are homomorphisms
of intuitionistic $M$-fuzzy groups. It is clear that the following diagram is
commutative:
\[
\left( \prod_{s \in S} G_s, \wedge_{s \in S} \mu_s, \vee_{s \in S} \lambda_s \right) \to \left( \prod_{n \in N} G_n, \wedge_{n \in N} \mu_n, \vee_{n \in N} \lambda_n \right)
\]

\[
\left( \prod_{s \in S} G_s, \wedge_{s \in S} \mu_s, \vee_{s \in S} \lambda_s \right) \to \left( \prod_{n \in N} G_n, \wedge_{n \in N} \mu_n, \vee_{n \in N} \lambda_n \right)
\]

\[\text{i.e., } \left\{ \overline{f}_0, \overline{f}_1 \right\} \text{ are morphisms of intuitionistic fuzzy cochain complexes.}
\]

Now, let us define \( g_0, g_1 : \prod_{n \in N} G_n \to \prod_{s \in S} G_s \) homomorphisms with this formula:

\[g_0(x_1, x_2, x_3, \ldots) = (x_1, x_1, x_2, x_3, \ldots)\]

\[g_1(x_1, x_2, x_3, \ldots) = (x_1 + p_1^{i+1}(x_{i+1}) + \ldots + p_{i-1}^{j-1}(x_{j-1}), x_j + p_j^{i+1}(x_{j+1}) + \ldots + p_{j-1}^{k-1}(x_{k-1}), \ldots)\]

For

\[\big( \wedge_{s \in S} \mu_s \big)(x_i, x_j, x_k, \ldots) = \mu_i(x_i) \wedge \mu_j(x_j) \wedge \ldots \geq \wedge_{n \in N} \mu_n(x_n)\]

\[\big( \vee_{s \in S} \lambda_s \big)(x_i, x_j, x_k, \ldots) = \lambda_i(x_i) \vee \lambda_j(x_j) \vee \ldots \leq \vee_{n \in N} \lambda_n(x_n),\]

and

\[\left( \wedge_{s \in S} \mu_s \right)\bigg((x_i + p_1^{i+1}(x_{i+1}) + \ldots + p_{i-1}^{j-1}(x_{j-1}), x_j + \ldots + p_j^{k-1}(x_{k-1}), \ldots)\bigg)\]

\[\geq \min\left\{ \mu_i(x_i), \mu_j(x_j), \ldots, \mu_{n, \in N}(x_n) \right\} \wedge \min\left\{ \lambda_i(x_i), \lambda_j(x_j), \ldots, \lambda_{n, \in N}(x_n) \right\} \wedge \ldots \geq \wedge_{m \in S} \mu_m(x_m) \geq \wedge_{n \in N} \mu_n(x_n)\]

\[\left( \vee_{s \in S} \lambda_s \right)\bigg((x_i + p_1^{i+1}(x_{i+1}) + \ldots + p_{i-1}^{j-1}(x_{j-1}), x_j + \ldots + p_j^{k-1}(x_{k-1}), \ldots)\bigg)\]

\[\leq \max\left\{ \lambda_i(x_i), \lambda_j(x_j), \ldots, \lambda_{n, \in N}(x_n) \right\} \vee \max\left\{ \lambda_i(x_i), \lambda_{i+1}(x_{i+1}), \ldots, \lambda_{j-1}(x_{j-1}) \right\} \vee \ldots \leq \lambda_m(x_m) \leq \vee_{n \in N} \lambda_n(x_n)\]

Thus, \( \overline{g}_0, \overline{g}_1 : \prod_{n \in N} G_n, \wedge_{n \in N} \mu_n, \vee_{n \in N} \lambda_n \to \prod_{s \in S} G_s, \wedge_{s \in S} \mu_s, \vee_{s \in S} \lambda_s \) are homomorphisms of intuitionistic \( M \)-fuzzy groups and \( \overline{d} \circ \overline{g}_0 = \overline{g}_1 \circ \overline{d} \) are satisfied. i.e., \( \left\{ \overline{g}_0, \overline{g}_1 \right\} \) are homomorphisms of intuitionistic fuzzy cochain complexes. It is clear that
Inverse and direct systems

\[
g_n \circ f_o = g_1 \circ f_1 = 1_{\coprod_{\alpha \in \Omega, \mu \in \Lambda, \nu \in \Delta}}.
\]

Hence, we give \( D: \prod_{n \in \mathbb{N}} G_n \rightarrow \prod_{n \in \mathbb{N}} G_n \) homomorphism of \( M \)-groups with this formula:

\[
D(x_1, x_2, x_3, \ldots) = (x_1 + p_1^0(x_2) + \ldots + p_1^{i-1}(x_{i-1}), x_2 + p_2^0(x_3) + \ldots + p_2^{i-1}(x_{i-1}), \ldots, x_{i-1}, 0, x_{i+1} + p_{i+1}^0(x_{i+2}) + \ldots + p_{i+1}^{j-1}(x_{j-1}), x_{i+2} + \ldots + p_{i+2}^{j-1}(x_{j-1}), 0, \ldots)
\]

For, \( \bigwedge_{n \in \mathbb{N}} \mu_n(x_i) + \bigwedge_{n \in \mathbb{N}} \mu_n(0) + \bigwedge_{i \in \mathbb{N}} \bigwedge_{k \leq j} \mu_n(x_k) \bigwedge_{i \in \mathbb{N}} \bigwedge_{k \leq j} \mu_n(x_k) = \bigwedge_{n \in \mathbb{N}} \mu_n(x_i), \)

\[
\bigvee_{n \in \mathbb{N}} \lambda_n(x_i) + \bigvee_{n \in \mathbb{N}} \lambda_n(0) + \bigvee_{i \in \mathbb{N}} \bigvee_{k \leq j} \lambda_n(x_k) \bigvee_{i \in \mathbb{N}} \bigvee_{k \leq j} \lambda_n(x_k) \leq \bigvee_{n \in \mathbb{N}} \lambda_n(x_i), \]

\[\bar{D}: \left( \prod_{n \in \mathbb{N}} G_n, \bigwedge_{n \in \mathbb{N}} \mu_n, \bigwedge_{n \in \mathbb{N}} \lambda_n \right) \rightarrow \left( \prod_{n \in \mathbb{N}} G_n, \bigwedge_{n \in \mathbb{N}} \mu_n, \bigwedge_{n \in \mathbb{N}} \lambda_n \right)\] is a homomorphism of intuitionistic \( M \)-fuzzy groups. By using simplicity of calculation, it is shown that \( \bar{D} \) is a intuitionistic fuzzy chain homotopy between \( f_o \circ g_o \) and \( f_1 \circ g_1 \) homomorphisms. Then the following intuitionistic \( M \)-fuzzy cohomology groups of intuitionistic fuzzy cochain complexes

\[
0 \rightarrow \left( \prod_{n \in \mathbb{N}} G_n, \bigwedge_{n \in \mathbb{N}} \mu_n, \bigvee_{n \in \mathbb{N}} \lambda_n \right) \overset{\partial}{\rightarrow} \left( \prod_{n \in \mathbb{N}} G_n, \bigwedge_{n \in \mathbb{N}} \mu_n, \bigwedge_{n \in \mathbb{N}} \lambda_n \right) \rightarrow 0
\]

\[
0 \rightarrow \left( \prod_{s \in S} G_s, \bigwedge_{s \in S} \mu_s, \bigvee_{s \in S} \lambda_s \right) \overset{\partial}{\rightarrow} \left( \prod_{s \in S} G_s, \bigwedge_{s \in S} \mu_s, \bigwedge_{s \in S} \lambda_s \right) \rightarrow 0
\]

are quasi isomorphic [1]. Since \( \lim_{\to}^{(\cdot)} \) is first cohomology module, the theorem is proved.
Since \( \lim(G_n, \mu_n, \lambda_n) = \ker d \) and \( p_n^{n+1}(x_{n+1}) = x_n \) is satisfied for each \( \{x_n\} \in \lim G_n \),

\[ \mu_n(x_n) = \mu_n(\mu_n^{n+1}(x_{n+1})) \geq \mu_{n+1}(x_{n+1}), \quad \lambda_n(x_n) = \lambda_n(\mu_n^{n+1}(x_{n+1})) \leq \lambda_{n+1}(x_{n+1}) \]

i.e., for each \( \{x_n\} \in \ker d \), \( \{\mu_n(x_n)\} \) is a decreasing sequence, \( \{\lambda_n(x_n)\} \) is an increasing sequence.

**Theorem 3.19.** For \( \forall \{x_n\} \in \ker d \), if \( \lim_{n \to \infty} \mu_n(x_n) = 0 \) or \( \lim_{n \to \infty} \lambda_n(x_n) = 1 \) and the following diagram is a short exact sequence of inverse system of intuitionistic \( M \)-fuzzy groups

\[
\begin{array}{ccccccccc}
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

then the sequence

\[
0 \to \lim(G_n, \mu_n, \lambda_n) \to \lim(G_n, \mu_n, \lambda_n) \to \lim(G_n, \mu_n, \lambda_n) \to 0
\]

is exact.

**Proof.** For inverse system of intuitionistic \( M \)-fuzzy groups \( \{(G_n, \mu_n, \lambda_n)\}_{n \in \mathbb{N}} \),

\[
C = 0 \to \prod_{n \in \mathbb{N}} G_n, \mu_n, \lambda_n \to \prod_{n \in \mathbb{N}} G_n, \mu_n, \lambda_n \to 0 \to \ldots
\]

is a cochain complex of intuitionistic \( M \)-fuzzy groups.

\[
H^\alpha(C) = \lim(G_n, \mu_n, \lambda_n), \quad H^\beta(C) = \lim(G_n, \mu_n, \lambda_n), \quad H^g(C) = 0, \quad k \geq 2 \quad (4)
\]

are intuitionistic fuzzy cohomology \( M \)-groups of this complexes. Similarly, for the inverse system of intuitionistic \( M \)-fuzzy groups \( \{(G'_n, \mu'_n, \lambda'_n)\} \) and \( \{(G''_n, \mu''_n, \lambda''_n)\} \), we can constitute the following intuitionistic fuzzy cochain complexes

\[
C' = 0 \to \prod_{n \in \mathbb{N}} G'_n, \mu'_n, \lambda'_n \to \prod_{n \in \mathbb{N}} G'_n, \mu'_n, \lambda'_n \to 0 \to \ldots
\]

\[
C'' = 0 \to \prod_{n \in \mathbb{N}} G''_n, \mu''_n, \lambda''_n \to \prod_{n \in \mathbb{N}} G''_n, \mu''_n, \lambda''_n \to 0 \to \ldots
\]

It is clear that intuitionistic \( M \)-fuzzy homology group of this complexes is the form in (4). From the condition of this theorem, the following sequence

\[
0 \to C' \to C \to C'' \to 0
\]

is short exact sequence of cochain complexes of intuitionistic \( M \)-fuzzy groups. But generally, the following sequence of cohomology modules of this sequence
0 \rightarrow H^0(C') \rightarrow H^\theta(C) \rightarrow H^\theta(C') \rightarrow \cdots 
\rightarrow H^i(C') \rightarrow H^i(C) \rightarrow H^i(C') \rightarrow H^2(C') \rightarrow 
is not exact, because \overline{d} is usually not homomorphism of intuitionistic \textit{M}-fuzzy group [1]. Since \( H^\theta(C') = \ker d^n \) and \( \lim_{n \to \infty} \mu_n^*(x_n^*) = 0 \left( \lim_{n \to \infty} \lambda_n^*(x_n^*) = 1 \right) \), grade function \( \mu^*(\lambda^*) \) of intuitionistic \textit{M}-fuzzy group \( (H^\theta(C'), \mu^*, \lambda^*) \) is equal to grade function \( \mu_0 (\lambda^* \text{ is equal to grade function } 1) \) [10]. Thus \overline{d} is homomorphism of intuitionistic \textit{M}-fuzzy group. Therefore the sequence \( \overline{ \lim } (G'_{n+1}, \mu'_{n+1}, \lambda'_{n+1}) \rightarrow \overline{ \lim } (G'_{n}, \mu'_{n}, \lambda'_{n}) \rightarrow \overline{ \lim } (G'_{n+1}, \mu'_{n+1}, \lambda'_{n+1}) \rightarrow \overline{ \lim } (G'_{n}, \mu'_{n}, \lambda'_{n}) \rightarrow \overline{ \lim } (G'_{n}, \mu'_{n}, \lambda'_{n}) \rightarrow 0. 

**Lemma 3.20.** Given the following inverse system of intuitionistic \textit{M}-fuzzy groups 
\( (G_1, \mu_1, \lambda_1) \leftarrow \overline{\lim} (G_2, \mu_2, \lambda_2) \leftarrow \overline{\lim} (G_3, \mu_3, \lambda_3) \leftarrow \cdots \) (5) 
if each homomorphisms \( \varphi_n \) are fuzzy epimorphisms, then \( \lim (G_n, \mu_n, \lambda_n) = 0. \)

**Proof.** The proof is obvious, since 
\( \overline{d} : \prod_{n=1}^{\infty} (G_n, \mu_n, \lambda_n) \rightarrow \prod_{n=1}^{\infty} (G_n, \mu_n, \lambda_n) \) 
is an intuitionistic fuzzy epimorphism.

**Definition 3.21.** Given inverse system of intuitionistic \textit{M}-fuzzy groups (5), for every integer \( n \), if there exists \( m \geq n \) such that 
\( \text{Im}[(G_i, \mu_i, \lambda_i) \rightarrow (G_n, \mu_n, \lambda_n)] = \text{Im}[(G_n, \mu_n, \lambda_n) \rightarrow (G_n, \mu_n, \lambda_n)] \) \((\forall i \geq m)\), 
then it is said that the inverse system in (5) satisfies the condition Mittag-Lefler.

**Theorem 3.22.** If the inverse system in (5) satisfies the condition Mittag-Lefler, then 
\( \lim (G_n, \mu_n, \lambda_n) = 0. \)

**Proof.** Let us denote by \( G'_n = \text{Im} \varphi'_n \), for large \( i \). Then from condition of the theorem, the homomorphism \( \varphi_{n\leq i} \) carries the module \( G'_{n+1} \) to \( G'_n \). Then \( \varphi_{n\leq i} \) is an epimorphism. Thus for large \( i \), the homomorphisms 
\( \overline{\varphi}_n : (G'_{n+1}, \mu_{n\leq i}, \lambda_{n\leq i}) \rightarrow (G'_n, \mu_{n\leq i}, \lambda_{n\leq i}) \)
are epimorphisms. Then by using Lemma 3.20, we have \( \lim (G'_n, \mu'_n, \lambda'_n) = 0. \)
Here \( \mu'_n = \mu_{n\leq i}, \lambda'_n = \lambda_{n\leq i} \). Let us consider the following sequence of the inverse system of quotient intuitionistic \textit{M}-fuzzy groups 
\( \left\{ \frac{G_i}{G'_i}, \mu_i, \lambda_i \right\} \leftarrow \left\{ \frac{G_2}{G'_2}, \mu_2, \lambda_2 \right\} \leftarrow \cdots \) (6)
For every $n$, there exists $m > n$ such that the homomorphism $\frac{G_n}{G_m} \rightarrow \frac{G_{n'}}{G_m}$ is zero homomorphism. Then $\lim_{n} \left( \frac{G_n}{G_n}, \tilde{\mu}_n, \tilde{\lambda}_n \right) = 0$ [1]. Consequently, the limit of inverse system in (6) is equal to 0. Therefore, $\lim_{n} \left( \frac{G_n}{G_n}, \tilde{\mu}_n, \tilde{\lambda}_n \right) = 0$ as well. Then let us consider the following short exact sequence of inverse system in category of $IFMGr$

$$0 \rightarrow \left\{ \left( G'_n, \mu'_n, \lambda'_n \right) \right\} \rightarrow \left\{ \left( G_n, \mu_n, \lambda_n \right) \right\} \rightarrow \left\{ \left( \frac{G_n}{G_n}, \tilde{\mu}_n, \tilde{\lambda}_n \right) \right\} \rightarrow 0 \quad (7)$$

Granting that $\lim_{n} \left( \frac{G_n}{G_n}, \tilde{\mu}_n, \tilde{\lambda}_n \right) = 0$, we can apply to Theorem 3.19 for the sequence (7), we have the exact sequence

$$0 \rightarrow \lim_{n} \left( G'_n, \mu'_n, \lambda'_n \right) \rightarrow \lim_{n} \left( G_n, \mu_n, \lambda_n \right) \rightarrow \lim_{n} \left( \frac{G_n}{G_n}, \tilde{\mu}_n, \tilde{\lambda}_n \right) \rightarrow \quad (8)$$

Since $\lim_{n} \left( G'_n, \mu'_n, \lambda'_n \right) = 0$, $\lim_{n} \left( \frac{G_n}{G_n}, \tilde{\mu}_n, \tilde{\lambda}_n \right) = 0$ and $\lim_{n} \left( G_n, \mu_n, \lambda_n \right) = 0$, then the sequence (8) would look like

$$0 \rightarrow \lim_{n} \left( G'_n, \mu'_n, \lambda'_n \right) \rightarrow \lim_{n} \left( G_n, \mu_n, \lambda_n \right) \rightarrow 0 \rightarrow 0 \rightarrow \lim_{n} \left( G_n, \mu_n, \lambda_n \right) \rightarrow 0 \rightarrow 0.$$

This proves that $\lim_{n} \left( G_n, \mu_n, \lambda_n \right) = 0$.

4. Direct system of intuitionistic $M$-fuzzy groups

Now we will investigate the direct system of intuitionistic $M$-fuzzy groups. Let

$$\left( \overline{G}, \overline{\mu}, \overline{\lambda} \right) = \left\{ \left( G_a, \mu_a, \lambda_a \right) \right\}_{a \in A}, \left\{ \overline{p}_a : \left( G_a, \mu_a, \lambda_a \right) \rightarrow \left( G_{a'}, \mu_{a'}, \lambda_{a'} \right) \right\}_{a < a'} \quad (9)$$

be direct system of intuitionistic $M$-fuzzy groups, $\left( \oplus_{a} G_a, \mu^\beta, \lambda^\beta \right)$ be intuitionistic $M$-fuzzy group and $\pi : \oplus_{a} G_a \rightarrow \lim_{a} G_a$ be canonical epimorphism. Then we get the intuitionistic $M$-fuzzy group $\left\{ \lim_{a} G_a, \left( \mu^\beta \right)^\epsilon, \left( \lambda^\beta \right)^\epsilon \right\}$ [10].
Theorem 4.1. Every direct system in representation (9) has limit on the category of \( \text{IFMGr} \) and this limit is equal to the intuitionistic \( M \)-fuzzy group \( \left( \lim_{\alpha} G_{\alpha}, (\mu^\alpha)^\pi, (\lambda^\beta)^\pi \right) \).

Proof. It suffices to show that, there exists a unique homomorphism of intuitionistic \( M \)-fuzzy groups \( \overline{\psi} : \left( \lim_{\alpha} G_{\alpha}, (\mu^\alpha)^\pi, (\lambda^\beta)^\pi \right) \rightarrow (H, \eta, \nu) \) which makes commutative following diagram:

\[
\begin{array}{ccc}
(G_{\alpha}, \mu_{\alpha}, \lambda_{\alpha}) & \xrightarrow{\varphi_{\alpha}} & (H, \eta, \nu) \\
\xrightarrow{\pi_{\alpha}} \downarrow & & \downarrow \psi \\
\lim_{\alpha} \left( G_{\alpha}, (\mu^\alpha)^\pi, (\lambda^\beta)^\pi \right) & & \\
\end{array}
\]

where \( \overline{\varphi} = \left\{ \varphi_{\alpha} : (G_{\alpha}, \mu_{\alpha}, \lambda_{\alpha}) \rightarrow (H, \eta, \nu) \right\}_{\alpha \in \land} \) is the family of homomorphisms of intuitionistic \( M \)-fuzzy groups which makes commutative following diagram:

\[
\begin{array}{ccc}
(G_{\alpha}, \mu_{\alpha}, \lambda_{\alpha}) & \xrightarrow{\varphi_{\alpha}} & (H, \eta, \nu) \\
\xrightarrow{\pi_{\alpha}} \downarrow & & \downarrow \psi \\
(G_{\alpha'}, \mu_{\alpha'}, \lambda_{\alpha'}) & & \\
\end{array}
\]

and also \( \overline{\iota}_{\alpha} : (G_{\alpha}, \mu_{\alpha}, \lambda_{\alpha}) \rightarrow \left( \bigoplus_{\alpha} G_{\alpha}, \mu^\alpha, \lambda^\beta \right) \) are usual injections and \( \pi_{\alpha} = \pi \circ \iota_{\alpha} \).

For every \( x \in \lim_{\alpha} G_{\alpha} \), there exists \( x_{\alpha} \in G_{\alpha} \) such that \( \pi_{\alpha} (x_{\alpha}) = x \). If \( \pi_{\alpha} (x_{\alpha'}) = x \) for each \( x_{\alpha'} \in G_{\alpha'} \), then \( \varphi_{\alpha} (x_{\alpha'}) \) is equal to \( \varphi_{\alpha} (x_{\alpha}) \). We define the homomorphism \( \psi : \lim_{\alpha} G_{\alpha} \rightarrow H \) is defined by \( \psi (x) = \varphi_{\alpha} (x_{\alpha}) \).

Now, we can check that \( \overline{\psi} \) is the homomorphism of intuitionistic \( M \)-fuzzy groups. For each \( x \in \lim_{\alpha} G_{\alpha} \), let \( \pi \circ \iota_{\alpha} (x_{\alpha}) = x \) be. Here,

\[
\begin{align*}
(\mu^\alpha)^\pi (x) &= \sup \left\{ \vee_{\alpha} \mu_{\alpha} \right\} (x) = \sup \left\{ \vee_{\alpha} \mu_{\alpha} (x_{\alpha}) : \pi_{\alpha} (x_{\alpha}) = x \right\} \\
(\lambda^\beta)^\pi (x) &= \inf \left\{ \wedge_{\alpha} \lambda_{\alpha} \right\} (x) = \inf \left\{ \wedge_{\alpha} \lambda_{\alpha} (x_{\alpha}) : \pi_{\alpha} (x_{\alpha}) = x \right\}.
\end{align*}
\]

Therefore,

\[
\eta (\psi (x)) = \eta (\varphi_{\alpha} (x_{\alpha})) \geq \mu_{\alpha} (x_{\alpha}), \ \ \nu (\psi (x)) = \eta (\varphi_{\alpha} (x_{\alpha})) \leq \lambda_{\alpha} (x_{\alpha}).
\]

Since this inequality is satisfied for each \( x_{\alpha} \) which satisfies \( \pi_{\alpha} (x_{\alpha}) = x \), we write the inequality.
From the definition, it is obvious that Figure 1 is commutative. We can easily show that \( \lim \) is a functor from the category of direct system of intuitionistic \( M \)-fuzzy groups to the category of intuitionistic \( M \)-fuzzy groups. Let

\[
\overrightarrow{G} = \left\{ (G_a, \mu_a, \lambda_a) \right\}_{a \in A}, \quad \overrightarrow{G'} = \left\{ (G'_a, \mu'_a, \lambda'_a) \right\}_{a \in A}, \quad \overrightarrow{G^*} = \left\{ (G'^*, \mu'^*, \lambda'^*) \right\}_{a \in A},
\]

be direct system of intuitionistic \( M \)-fuzzy groups, and the sequence

\[
\overrightarrow{G} \rightarrow \overrightarrow{G'} \rightarrow \overrightarrow{G^*} \quad (10)
\]

be exact sequence of this system.

**Theorem 4.2.** If the direct limit functor is applied the sequence in (10), then the sequence

\[
\lim\alpha (G'_a, \mu'_a, \lambda'_a) \rightarrow \lim\alpha (G_a, \mu_a, \lambda_a) \rightarrow \lim\alpha (G'^*, \mu'^*, \lambda'^*)
\]

is exact as well.

**Proof.** Let the sequence (10) be exact. Then the ordinary sequence of \( M \)-groups

\[
G_a \rightarrow G_a \rightarrow G'^*_a
\]

is exact sequence for \( \forall \alpha \in \mathcal{E} [15] \). Hence, the sequence

\[
\left\{ G'_a \right\}_{\alpha} \rightarrow \left\{ G_a \right\}_{\alpha} \rightarrow \left\{ G'^*_a \right\}_{\alpha}
\]

is exact sequence of the direct system of ordinary \( M \)-groups. Therefore the limit of this exact sequence

\[
\lim\alpha G'_a \rightarrow \lim\alpha G_a \rightarrow \lim\alpha G'^*_a \quad (11)
\]

is also exact. For the following sequence of intuitionistic \( M \)-fuzzy groups

\[
\left( \lim\alpha G'_a, (\mu'^*)^x, (\lambda'^*)^x \right) \rightarrow \left( \lim\alpha G_a, (\mu^*)^x, (\lambda^*)^x \right) \rightarrow \left( \lim\alpha G'^*_a, (\mu'^*)^x, (\lambda'^*)^x \right)
\]

\[
(\mu^*)^x \lim\alpha f_a = (\mu'^*)^x \ker \lim\alpha g_a, \quad (\lambda^*)^x \lim\alpha f_a = (\lambda'^*)^x \ker \lim\alpha g_a
\]

are true, because sequence (11) is exact.

From Theorem 4.2, we obtain following conclusion.

**Conclusion 4.3.** The functor of direct limit defenses monomorphism and epimorphism in the category of intuitionistic \( M \)-fuzzy groups. Now, we want to constitute direct system of chain complexes. Let \( I \) be directed set, for \( \forall i \in I \)

\[
C(i) = \left( (G^{(i)}_n, \mu^{(i)}_n, \lambda^{(i)}_n), \overrightarrow{\partial}_n : (G_n(i), \mu_n(i), \lambda_n(i)) \rightarrow (G_{n-1}(i), \mu_{n-1}(i), \lambda_{n-1}(i)) \right)_n
\]
be chain complexes of intuitionistic \(M\)-fuzzy groups and for \(\forall i < j, f_{ij} : C(i) \to C(j)\) be morphism of chain complexes and \(\{C(i), f_{ij}\}\) be direct system of chain complexes.

**Theorem 4.4.** The limit of intuitionistic homology groups of direct system of chain complexes of intuitionistic fuzzy groups is quasi isomorphic to the intuitionistic homology groups of the limit of this direct system, i.e.,

\[
\lim_i H_n \left( \lim_i C(i) \right) \cong \lim_i H_n \left( C(i) \right).
\]

**Proof.** The proof of this theorem is done by using Conclusion 4.3. Hence,

\[
\lim_i H_n \left( C(i) \right) = \lim_i \left( \ker \partial_n(i) \bigg|_{\lim_i \mu_n(i)} , \mu_n(i) , \lambda_n(i) \right) \\
\approx \lim_i \left( \ker \partial_n(i) , \mu_n \bigg|_{\ker \partial_n(i)} , \lambda_n \bigg|_{\ker \partial_n(i)} \right) \bigg|_{\lim_i \lim_i \mu_n(i)} \\
\approx \ker \lim_i \partial_n(i) \bigg|_{\lim_i \lim_i \mu_n(i)} = H_n \left( \lim_i C(i) \right).
\]

The proof is completed.

**References**


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