High Degree Ruled Surfaces

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Abstract. Let $C$ be a smooth genus $g \geq 2$. Here we describe the set of all degree $d$ scrolls on $C$ if $d \geq 6g - 4$.

Mathematics Subject Classification: 14H60; 14J26; 14N05

Keywords: scroll; surface scroll; rank two vector bundle; ruled surface

1. Introduction

Fix an integer $g \geq 2$. Let $C$ be a smooth and connected projective curve of genus $g$ defined over an algebraically closed field $\mathbb{K}$ with char$(\mathbb{K}) = 0$. For all integers $m > 2 > 0$ let $G(2, m)$ denote the Grassmanian of all $(m - 2)$-dimensional linear subspaces of $\mathbb{K}^m$. For any vector space $W$ and any integer $m > 0$ let Grass$(m, W)$ denote the set of all $m$-dimensional linear subspaces of $W$. Let $E$ be a rank 2 vector bundle on $C$ and $V \in$ Grass$(m, H^0(C, E))$ such that $V$ spans $E$. The universal property of the Grassmannian $G(2, m)$ gives that the pair $(E, V)$ induces a morphism $h_{E, V} : C \to G(2, m)$. There is a natural isomorphism between $H^0(C, E)$ and $H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$. Moreover (up to this isomorphism), a linear subspace of $H^0(C, E)$ spans $E$ if and only if it spans $\mathcal{O}_{\mathbb{P}(E)}(1)$. Thus the pair $(E, V)$ induces a morphism $f_{E, V} : \mathbb{P}(E) \to \mathbb{P}^{m-1}$ such that $f_{E, V}(\mathcal{O}_{\mathbb{P}(E)}(1)) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$. From the point of view of scrolls in $\mathbb{P}^{m-1}$ it is natural to make the further assumption that the morphism $f_{E, V}$ is birational onto its image. For several results on ruled surface when the base curve as general moduli (not the case studied here), see [2] and [3]. In section 2 we extend a part of [1] to the case in which $C$ has not general moduli, i.e. we describe the irreducible components of the set of all degree $d \geq 6g - 4$ ruled surfaces $S \subset \mathbb{P}^{d+1-2g}$ with $C$ as the normalization of a general hyperplane section. For any rank 2 vector bundle $E$ on $C$ let Grass$(m, H^0(C, E))'$ denote the set of all $m$-dimensional subspaces $V$ of $H^0(C, E)$ such that $V$ spans $E$ and $f_{E, V}$ is birational onto its image. Grass$(m, H^0(C, E))'$ is always an open

\footnote{The author was partially supported by MIUR and GNSAGA of INdAM (Italy).}
subset of Grass($m, H^0(C, E)$). In each case we will use we will check the nonemptyness of it. In section 2 we prove the following result.

**Theorem 1.** Fix integers $g, d, m$ such that $g \geq 2$, $d \geq 6g - 4$ and $4 \leq m \leq d + 2 - 2g$. Let $C$ be a smooth curve of genus $g$. Let $gon(C)$ denote the gonality of $C$. Let $H[C, d, m]$ denote the set of all pairs $(E, V)$, where $E$ is a rank 2 vector bundle on $C$ and $V \in Grass(m, H^0(C, E))'$. For all integers $x, y$ such and $1 \leq x \leq g$ and $gon(C) \leq y \leq 2g - 2$ set $B(C, y, x) := \{L \in \text{Pic}^d(C) : h^1(C, L) = x$ and $L$ is spanned}. Let $T(C, y, x)$ denote the set of all irreducible components of $B(C, y, x)$. Set $H[C, d, m] := \{(E, V) \in H[C, d, m] : h^1(C, E) = 0\}$ (the non-special vector bundles) and $S[C, d, m] := \{(E, V) \in H[C, d, m] : E \cong O_C \oplus L$ for some $L \in \text{Pic}^d(C)\}$ (the set of all pairs such that $f_{E,V}(\mathbb{P}(E))$ is a cone). For any $T \in T(C, y, x)$ set $A[C; T, y, x] := \{(R \oplus L, V) : R \in T, L \in \text{Pic}^{d-y}(C), V \in \text{Grass}(m, H^0(C, M))'\}$. The algebraic set $H[C, d, m]$ is the disjoint union of the irreducible locally closed subsets $H[C, d, m]'$, $S[C, d, m]$ and all sets $A[C; T, y, x]$. The closure $\overline{H[C, d, m]'}$ of $H[C, d, m]'$ in $H[C, d, m]$ is an irreducible component of $H[C, d, m]$ with dimension $4g - 3 + m(d + 2 - 2g - m)$.

**Remark 1.** $H[C, d, m]'$ (and hence $H[C, d, m]$) is not an algebraic scheme. The set of all rank 2 vector bundles on $C$ with fixed degree is only an Artin stack (even if we restrict to its open subset formed by the non-special spanned vector bundles such that $f_{E,H^0(C,E)}$ is birational onto its image or it is an embedding). It is an irreducible Artin stack of dimension $4g - 3$. If $d \geq 4g - 3$ (as in our case) an open and dense subset of the stack $\mathcal{V}_C(2, d)$ of rank 2 vector bundles on $C$ with degree $d$ is formed by the set $U(C; 2, d)$ of all stable vector bundles. Since $d \geq 4g - 4$, $h^1(C, E) = 0$ for every $E \in U(C; 2, d)$. Since $d \geq 4g - 3$, every $E \in U(C; 2, d)$ is spanned. Since $d \geq 4g - 2$ the morphism $f_{E,H^0(C,E)}$ is an embedding for every $E \in U(C; 2, d)$.

**Remark 2.** By definition of gonality $B(C, gon(C), 1) \neq \emptyset$ and $B(C, gon(C), x) = \emptyset$ for all $x \geq 2$. Obviously, $B(C, 2g - 2, 1) = \{\omega_C\}$ and $B(C, 2g - 2, x) = \emptyset$ for all $x \geq 2$. There are curves for which there are integers $y$ such that $gon(C) < y < 2g - 2$ and $B(C, y, x) = \emptyset$ for all integers $x > 0$ (e.g. take $C$ hyperelliptic and $y$ odd). Assume $B(C, y, x) \neq \emptyset$ and fix any $R \in B(C, y, x)$ and any $L \in \text{Pic}^{d-y}(C)$. By assumption $R$ is spanned. Since $d - y \geq d - 2g + 2 \geq 2g + 1, L$ is very ample. Set $E := R \oplus L$. Since $h^1(C, R) = x$ and $h^1(C, L) = 0$, $h^0(C, E) = d + 2 - 2g + x$. Hence Grass($m, H^0(C, E)$) is an integral variety of dimension $m(d + 2 - 2g + x - m)$ (a very large integer with respect to $d$ and $4g - 3$). Since $L$ is very ample, $f_{E,H^0(C,E)}$ is birational onto its image (see Lemma 1 in section 2). Hence $f_{E,V}$ is birational onto its image for a general $V \in Grass(m, H^0(C, E))$ (see Lemma 4 in section 2). Thus for each $T \in B(C, y, x)$ the set $A[C; T, y, x]$ is an irreducible algebraic set of dimension $\dim(T) + g + m(d + 2 - 2g + x - m)$. 
2. Proof of Theorem 1

For any spanned vector bundle $E$ on $C$ set $h_E := h_{E,H^0(C,E)}$ and $f_E := f_{E,H^0(C,E)}$.

We lift from [1] the following observation.

**Remark 3.** Let $C$ be a smooth genus $g$ curve and $E$ a rank 2 vector bundle on $C$. Set $d := \deg(E)$ and $s := s(E)$. Notice that $d \equiv s \pmod{2}$. The integer $s$ is often called the degree of stability of $E$. Let $L$ be a maximal degree line subbundle of $E$. Hence $E/L \in \text{Pic}(C)$, $\deg(L) = (d-s)/2$, $\deg(E/L) = (d+s)/2$ and $E$ is an extension of $E/L$ by $L$. Thus $h^1(C,E) = 0$ if $(d-|s|)/2 \geq 2g-1$, i.e. if $d \geq 4g-2 + |s|$. Since $s(F) \geq 2 - 2g$ for any indecomposable rank 2 vector bundle $F$ on $C$, we get $h^1(C,F) = 0$ for any indecomposable rank 2 vector bundle on $C$ such that $\deg(F) \geq 6g-4$. Thus if $g \geq 2$ and $d \geq 6g-4$, then either $(E,V) \in H^0[C,d,m]$ or $E$ is decomposable.

**Remark 4.** Let $E$ be a spanned rank 2 vector bundle on $C$. Assume $k := h^0(C,E) \geq 4$ and that the morphism $f_A : \mathbb{P}(A) \to \mathbb{P}^{k-1}$ is birational onto its image. Fix an integer $m$ such that $4 \leq m \leq k$. Let $V$ be a general element of $\text{Grass}(m,H^0(C,E))$. A dimensional count gives that $V$ spans $A$ (here a general 3-dimensional linear subspace would be sufficient). Since $f_{A,V} : \mathbb{P}(A)$ is a general linear projection of the surface $f_A(\mathbb{P}(A))$ into $\mathbb{P}^{m-1}$ and $m \geq 4$, $f_{A,V}$ is birational onto its image. A similar statement is obviously true for line bundles. If $m \geq 6$ (resp. $m \geq 5$) we see in the same way that $f_{A,V}$ is an embedding (resp. unramified) if and only if $f_A$ is an embedding (resp. unramified).

**Lemma 1.** Assume that $V$ spans $A$ and that $\dim(V) \geq 4$. The morphism $f_{A,V}$ is birational onto its image if and only if the morphism $h_{A,V}$ is birational onto its image. If $h_{A,V}$ has degree $a$, then $f_{A,V}$ is a generically finite morphism of degree $a$.

**Proof.** The morphism $f_{A,V}$ is constant if and only if $A$ is trivial and hence if and only if $h_{A,V}$ is constant. Hence we may assume that the maps $f_{A,V}$ and $h_{A,V}$ are not constant. Since the domain of $h_{A,V}$ is a curve, $h_{A,V}$ has a finite degree $a \geq 0$. $h_{A,V}$ is birational onto its image if and only if $a = 1$. Hence it is sufficient to prove that $f_{A,V}$ is a generically finite morphism of degree $a$. Let $C'$ be the normalization of $h_{A,V}(C)$. The curve $C'$ is a smooth curve and there are a degree $a$ morphism $u : C \to C'$, a vector bundle $A'$ on $C'$ and a linear subspace $V' \subseteq H^0(C',A')$ such that $\dim(V') = \dim(V)$, $V'$ spans $A'$, $u^*(A') \cong A$ $u^*(V') = V$ (up to the previous isomorphism) and $h_{A,V} = h_{A',V'} \circ u$. Obviously, there is a degree $a$ morphism $v : \mathbb{P}(A) \to \mathbb{P}(A')$ such that $f_{A,V} = v \circ f_{A',V'}$. Hence it is sufficient to prove the “if” part of the first assertion of the lemma. Assume that $h_{A,V}$ is birational onto its image, but that $f_{A,V}$ is not birational onto its image. Fix a general $P \in C$. Notice that $f_{A,V}(\mathbb{P}(A\{P\}))$ is a line. The birationality of $h_{A,V}$ implies that there is no $Q \in C$ such that $Q \neq P$ and $f_{A,V}(\mathbb{P}(A\{Q\})) = f_{A,V}(\mathbb{P}(A\{P\}))$. 


Hence every line of the ruling of $f_{A,V}(\mathbb{P}(A))$ intersects the line $f_{A,V}(\mathbb{P}(A\{\{P\}\}))$. Varying $P$ in $C$ we get a contradiction, because we assumed $\dim(V) \geq 4$. 

**Proof of Theorem 1.** $E$ gives a cone if and only if it has $\mathcal{O}_C$ as a factor. If $E$ is indecomposable, then $h^1(C, E) = 0$ (Remark 3), i.e. $(E, V) \in H[C, d, m]'$ for every $V \in \text{Grass}(m, H^0(C, E))'$. Let $E \cong R \oplus L$ be a decomposable and spanned vector bundle such that $\deg(E) = d$, $h^1(C, E) > 0$ and $E$ has no trivial factor. Let $R$ be a factor of $E$ such that $h^1(C, R) > 0$. Set $y := \deg(R)$ and $x := h^1(C, R)$. Since $R \neq \mathcal{O}_C$, $\text{gon}(C) \leq m \leq 2g - 2$. Since $\deg(L) = d - m > 2g - 2$, $h^1(C, L) = 0$. Hence $h^1(C, E) = x$. Since $d - m \geq 2g + 1$, every degree $d - m$ line bundle is very ample. By Lemma 1 $f_{R \oplus M}$ is birational onto its image for every $M \in \text{Pic}^{(d-m)}(C)$. Remark 4 gives that the same is true for a general $W \in \text{Grass}(m, H^0(C, R \oplus M))$. Thus Theorem 1 gives a stratification of $H[C, d, m]$ with irreducible strata. The last assertion follows from the semicontinuity theorem for cohomology and the irreducibility of $U(C; 2, d)$, because $\dim(U(C; 2, d)) = 4g - 3$.

**References**


Received: September, 2008