A Numerical Method for Solving Nonlinear Integral Equations

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Abstract

In this paper, an iterative scheme based on the homotopy analysis method (HAM) has been used to solve nonlinear integral equations. To check the numerical method, it is applied to solve different test problems with known exact solutions and the numerical solutions obtained confirm the validity of the numerical method and suggest that it is an interesting and viable alternative to existing numerical methods for solving the problem under consideration. Convergence is also observed.

Mathematics Subject Classification: 65H20, 65H05, 45G05, 45G10

Keywords: Homotopy Analysis Method; Nonlinear Integral Equations; Numerical Method

1 Introduction

Integral equations play an important role in many branches of linear and nonlinear functional analysis and their applications in the theory of elasticity, engineering, mathematical physics, potential theory, electrostatics and radiative heat transfer problems. Therefore, many different methods are used to obtain the solution of the linear and nonlinear integral equations.

Some different valid methods for solving integral equation have been developed in the last years [1–8]. In [2], Brunner et al., introduced a class of methods depending on some parameters to obtain the numerical solution of Abel integral equation of the second kind. The linear multistep methods were
applied in [7], to obtain the numerical solution of a singular nonlinear Volterra integral equation. Also, in [8], Kilbas and Saigo used an asymptotic method to obtain numerically the solution of nonlinear Abel–Volterra integral equation. In [9], Orsi used a Product Nyström method, as a numerical method, to obtain the solution of nonlinear Volterra integral equation, when its kernel takes a logarithmic and Carleman forms. Variational iteration method [13], Homotopy perturbation method [5-6] and Adomian decomposition method [1] are effective and convenient for solving integral equations.

The homotopy analysis method (HAM) [9-12] is a general analytic approach to get series solutions of various types of nonlinear equations, including algebraic equations, ordinary differential equations, partial differential equations, differential-difference equation. More importantly, different from all perturbation and traditional non-perturbation methods, the HAM provides us a simple way to ensure the convergence of solution series, and therefore, the HAM is valid even for strongly nonlinear problems.

The HAM is based on homotopy, a fundamental concept in topology and differential geometry. Briefly speaking, by means of the HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping, and an auxiliary parameter is used to ensure the convergence of solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. By means of this kind of freedom, a complicated nonlinear problem can be transferred into an infinite number of simpler, linear sub-problems, as shown by Liao and Tan [12].

Until recently, the application of the homotopy analysis method in nonlinear problems has been devoted by scientists and engineers, because this method is to continuously deform a simple problem easy to solve into the difficult problem under study.

In this paper, we use the HAM for nonlinear integral equations such that

\[ y(x) = g(x) + \int_a K(x, t)f(y(t))dt, \]

where the upper limit may be either variable or fixed, the kernel of the integral \(K(x, t)\) and \(g(x)\) are known functions, \(f(y)\) is a known function of \(y\) and \(y(x)\) is unknown function that will be determined.

Some examples are tested, and the obtained results suggest that newly improvement technique introduces a promising tool and powerful improvement for solving integral equations.
2 Description of the Method

Consider

\[ N[y] = y(x) - g(x) - \int_a K(x, t) f(y(t)) dt = 0, \quad (1) \]

where \( N \) is an operator, \( y(x) \) is unknown function and \( x \) the independent variable. Let \( y_0(x) \) denote an initial guess of the exact solution \( y(x) \), \( h \neq 0 \) an auxiliary parameter, \( H(x) \neq 0 \) an auxiliary function, and \( L \) an auxiliary linear operator with the property \( L[r(x)] = 0 \) when \( r(x) = 0 \). Then using \( q \in [0, 1] \) as an embedding parameter, we construct such a homotopy

\[ (1 - q)L[\phi(x; q) - y_0(x)] - qhH(x)N[\phi(x; q)] = \hat{H}[\phi(x; q); y_0(x), H(x), h, q]. \quad (2) \]

It should be emphasized that we have great freedom to choose the initial guess \( y_0(x) \), the auxiliary linear operator \( L \), the non-zero auxiliary parameter \( h \), and the auxiliary function \( H(x) \).

Enforcing the homotopy (2) to be zero, i.e.,

\[ \hat{H}[\phi(x; q); y_0(x), H(x), h, q] = 0 \]

we have the so-called zero-order deformation equation

\[ (1 - q)L[\phi(x; q) - y_0(x)] = qhH(x)N[\phi(x; q)]. \quad (3) \]

When \( q = 0 \), the zero-order deformation equation (3) becomes

\[ \phi(x; 0) = y_0(x), \quad (4) \]

and when \( q = 1 \), since \( h \neq 0 \) and \( H(x) \neq 0 \), the zero-order deformation equation (3) is equivalent to

\[ \phi(x; 1) = y(x). \quad (5) \]

Thus, according to (4) and (5), as the embedding parameter \( q \) increases from 0 to 1, \( \phi(x; q) \) varies continuously from the initial approximation \( y_0(x) \) to the exact solution \( y(x) \). Such a kind of continuous variation is called deformation in homotopy.

By Taylor’s theorem, \( \phi(x; q) \) can be expanded in a power series of \( q \) as follows

\[ \phi(x; q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m \quad (6) \]
where

\[ y_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} \Big|_{q=0}. \quad (7) \]

If the initial guess \( y_0(x) \), the auxiliary linear parameter \( L \), the nonzero auxiliary parameter \( h \), and the auxiliary function \( H(x) \) are properly chosen so that the power series (6) of \( \phi(x; q) \) converges at \( q = 1 \). Then, we have under these assumptions the solution series

\[ y(x) = \phi(x; 1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \quad (8) \]

For brevity, define the vector

\[ \mathbf{y}_n(x) = \{ y_0(x), y_1(x), y_2(x), \ldots, y_n(x) \}. \quad (9) \]

According to the definition (7), the governing equation of \( y_m(x) \) can be derived from the zero-order deformation equation (3). Differentiating the zero-order deformation equation (3) \( m \) times with respective to \( q \) and then dividing by \( m! \) and finally setting \( q = 0 \), we have the so-called \( m \)th-order deformation equation

\[ L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)R_m(\mathbf{y}_{m-1}(x)), \quad y_m(0) = 0, \quad (10) \]

where

\[ R_m(\mathbf{y}_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x; q)]}{\partial q^{m-1}} \Big|_{q=0} \quad (11) \]

and

\[ \chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}. \]

Note that the high-order deformation equation (10) is governing by the linear operator \( L \), and the term \( R_m(\mathbf{y}_{m-1}(x)) \) can be expressed simply by (11) for any nonlinear operator \( N \).

According to the definition (11), the right-hand side of equation (10) is only dependent upon \( y_{m-1}(x) \). Therefore \( y_m(x) \) can be easily gained, especially by means of computational software such as MATLAB. The solution \( y(x) \) given by the above approach is dependent of \( L, h, H(x) \), and \( y_0(x) \). Thus, unlike all previous analytic techniques, the convergence region and rate of solution series given by the above approach might not be uniquely determined.
Here, we rigorous definitions and then give some properties of the homotopy-derivative. These properties are useful to deduce the high-order deformation equations and provide us with a simple and convenient way to apply the HAM to nonlinear problems.

Let $\phi$ be a function of the homotopy-parameter $q$, then

$$D_m(\phi) = \frac{1}{(m)!} \frac{\partial^m \phi(x; q)}{\partial q^m}|_{q=0}$$

is called the $m$th-order homotopy-derivative of $\phi$, where $m \geq 0$ is an integer.

According to the Leibnitz’s rule for derivatives and using the induction, one can show the following properties of the homotopy-derivative.

**Theorem 1** For homotopy-series

$$\phi(x; q) = \sum_{m=0}^{\infty} y_m(x)q^m, \quad \psi(x; q) = \sum_{k=0}^{\infty} v_k(x)q^k,$$

where $m, l, n$ and $0 \leq k \leq m$ are positive integers, then

1. For $p \geq 0$, a positive integer, it holds

   $$D_m(\phi^p) = \sum_{r_1=0}^{m} y_{m-r_1} \sum_{r_2=0}^{r_1} y_{r_1-r_2} \sum_{r_3=0}^{r_2} y_{r_2-r_3} \cdots \sum_{r_{pij}=0}^{r_{pij-3}} y_{r_{pij-3}-r_{pij-2}} \sum_{r_{pij-1}=0}^{r_{pij-2}} y_{r_{pij-2}-r_{pij-1}}.$$

2. $D_m(f\phi + g\psi) = fD_m(\phi) + gD_m(\psi)$.
3. $D_m(\phi) = y_m$.
4. $D_m(q^k\phi) = D_{m-k}(\phi)$.
5. $D_m(\phi\psi) = \sum_{i=0}^{m} D_i(\phi) D_{m-i}(\psi) = \sum_{i=0}^{m} D_i(\psi) D_{m-i}(\phi)$.
6. $D_m(\phi^n\psi^l) = \sum_{i=0}^{m} D_i(\phi^n) D_{m-i}(\psi^l) = \sum_{i=0}^{m} D_i(\psi^l) D_{m-i}(\phi^n)$.
7. it holds the recurrence formulas

   $$D_0(e^\phi) = e^{y_0},$$

   $$D_m(e^\phi) = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) D_k(e^\phi) D_{m-k}(\phi), \quad \text{for } m \geq 1.$$

8. it holds the recurrence formulas

   $$D_0(\sin \phi) = \sin(y_0),$$

   $$D_m(\sin \phi) = \sum_{k=0}^{m-1} \left(1 - \frac{k}{m}\right) D_k(\cos \phi) D_{m-k}(\phi), \quad \text{for } m \geq 1.$$
it holds the recurrence formulas

\[ D_0(\cos \phi) = \cos(y_0), \]
\[ D_m(\cos \phi) = -\sum_{k=0}^{m-1} \left( 1 - \frac{k}{m} \right) D_k(\sin \phi) D_{m-k}(\phi), \quad \text{for } m \geq 1. \]

3 Computational procedure

In this section we will use the HAM approach to consider nonlinear integral equations of the type:

\[ y(x) = g(x) + \int_a K(x, t)[y(t)]^p dt, \quad (12) \]

where the upper limit may be either variable or fixed, \( p \) is a positive integer, the kernel \( k(x, t) \) and \( g(x) \) are known functions, whereas \( y \) is to be determined.

Let

\[ N[y] = y(x) - g(x) - \int_a K(x, t)[y(t)]^p dt, \]

The corresponding mth-order deformation equation (10) reads

\[ L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x) R_{m-1}(\overrightarrow{y}_{m-1}(x)), \quad (13) \]
\[ y_m(0) = 0, \]

where

\[ R_{m-1}(\overrightarrow{y}_{m-1}(x)) = y_{m-1} - (1 - \chi_m)g - \int_a K(x, t) R_{m-1}(\phi^p) dt \]

and

\[ R_m(\phi^p) = \sum_{r_1=0}^{m} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{r_2} \sum_{r_{p-3}=0}^{r_{p-2}} \sum_{r_{p-2}=0}^{r_{p-1}} \sum_{r_{p-1}=0}^{r_p} y_{m-r_1-r_2-r_3} \cdots y_{r_{p-3}-r_{p-2}} y_{r_{p-2}-r_{p-1}} y_{r_p}. \]

To obtain a simple iteration formula for \( y_m(x) \), choose \( Ly = y \) as an auxiliary linear operator, as a zero-order approximation to the desired function \( y(x) \), the solution \( y_0(x) = g(x) \), is taken, the nonzero auxiliary parameter \( h \)
and the auxiliary function $H(x)$, can be taken as $h = -1$ and $H(x) = 1$. This is substituted into (13) to obtain

$$y_0(x) = g(x),$$
$$y_m(x) = \int_a^b K(x, t)R_{m-1}(\phi^p)dt, \quad m = 1, 2, 3, \ldots.$$ 

The corresponding homotopy-series solution is given by

$$y(x) = \sum_{m=0}^{\infty} y_m(x) \quad (14)$$

### 4 Convergence Analysis

**Theorem 2** The integral equation

$$y(x) = g(x) + \int_a^b K(x, t)f(y(t))dt, \quad (15)$$

with the kernel $K(x, t)$ satisfies $|K(x, t)| < M$ for all $(x, t) \in [a, b] \times [a, b]$, $g(x)$ is a given continuous function defined on $[a, b]$ and $f(y)$ is Lipschitz continuous with $|f(y) - f(z)| \leq L|y - z|$, has a unique solution whenever $0 < \alpha < 1$, where, $\alpha = LM(b - a)$.

**Proof.** Consider the space $C[a, b]$ of all continuous functions defined on the interval $[a, b]$ with metric $d$ given by $d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|$. Obviously, (15) can be written $y = Ty$ where

$$Ty(x) = g(x) + \int_a^b K(x, t)f(y(t))dt.$$ 

Let $y_1$ and $y_2$ be two different solutions to (15) then

$$|Ty_1(x) - Ty_2(x)| = \left| \int_a^b K(x, t)[f(y_1) - f(y_2)]dt \right|$$
$$\leq \int_a^b |K(x, t)||f(y_1) - f(y_2)| dt$$
$$\leq LMd(y_1, y_2) \int_a^b dt$$
$$\leq \alpha d(y_1, y_2).$$
Since $0 < \alpha < 1$, $T$ becomes a contraction and the Banach’s fixed point theorem completes the proof.

As the function $f(x) = x^p$ is Lipschitz continuous, the integral equation (12) has a unique solution.

**Theorem 3** Let $S(x) = \sum_{n=0}^{\infty} y_n(x)$. Then for $k \geq 2$, where $k$ is an integer,

$$\sum_{m=0}^{\infty} R_m(\phi^k) = S^k(x).$$

**Proof.** The proof by induction on the $k$. From Theorem 1, for $k = 2$, we have

$$\sum_{m=0}^{\infty} R_m(\phi^2) = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} y_{m-j}y_j \right).$$

Thus

$$\sum_{m=0}^{\infty} R_m(\phi^2) = \sum_{j=0}^{\infty} \left( \sum_{m=j}^{\infty} y_{m-j}y_j \right) = \sum_{j=0}^{\infty} \left( \sum_{m=j}^{\infty} y_{m-j} \right) = \sum_{j=0}^{\infty} y_j \sum_{m=j}^{\infty} y_{m-j} = S^2.$$

Put $\phi^{k+1} = \phi^k \phi^1$, with the help of Theorem 1, we obtain

$$\sum_{m=0}^{\infty} R_m(\phi^{k+1}) = \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} y_{m-j}R_j(\phi^k) \right) = \sum_{j=0}^{\infty} \left( \sum_{m=j}^{\infty} y_{m-j}R_j(\phi^k) \right) = \sum_{j=0}^{\infty} R_j(\phi^k) \sum_{m=j}^{\infty} y_{m-j} = S^k S = S^{k+1}.$$

This ends the proof.

**Theorem 4** As long as the series (14) converges, it must be the exact solution of the integral equation (12).
Proof. If the series (14) converges, we can write
\[ S(x) = \sum_{m=0}^{\infty} y_m(x), \]
and it holds that
\[ \lim_{m \to \infty} y_m(x) = 0. \]  (16)

We can verify that
\[ \sum_{m=1}^{n} \left[ y_m(x) - \chi_m y_{m-1}(x) \right] = y_1 + (y_2 - y_1) + \cdots + (y_n - y_{n-1}) = y_n(x), \]
which gives us, according to (16),
\[ \sum_{m=1}^{\infty} \left[ y_m(x) - \chi_m y_{m-1}(x) \right] = \lim_{n \to \infty} y_n(x) = 0. \]  (17)

Furthermore, using (17) and the definition of the linear operator \( L \), we have
\[ \sum_{m=1}^{\infty} L \left[ y_m(x) - \chi_m y_{m-1}(x) \right] = L \left[ \sum_{m=1}^{\infty} \left[ y_m(x) - \chi_m y_{m-1}(x) \right] \right] = 0. \]

In this line, we can obtain that
\[ \sum_{m=1}^{\infty} L \left[ y_m(x) - \chi_m y_{m-1}(x) \right] = hH(x) \sum_{m=1}^{\infty} \mathcal{R}_{m-1}(\nabla^m y_{m-1}(x)) = 0, \]
which gives, since \( h \neq 0 \) and \( H(x) \neq 0 \), that
\[ \sum_{m=1}^{\infty} \mathcal{R}_{m-1}(\nabla^m y_{m-1}(x)) = 0. \]  (18)

Substituting \( \mathcal{R}_{m-1}(\nabla^m y_{m-1}(x)) \) into the above expression, recall Theorem 3, and
simplifying it, we have

\[ \sum_{m=1}^{\infty} \Re_{m-1}(\overrightarrow{y}_{m-1}(x)) = \sum_{m=1}^{\infty} [y_{m-1} - (1 - \chi_m)g - \int_a^b K(x, t)\Re_{m-1}(\phi^p)dt] \]

\[ = \sum_{m=0}^{\infty} y_m(x) - g(x) - \int_a^b K(x, t) \sum_{m=1}^{\infty} \Re_{m-1}(\phi^p)dt \]

\[ = \sum_{m=0}^{\infty} y_m(x) - g(x) - \int_a^b K(x, t) [\sum_{m=0}^{\infty} y_m(t)]^p dt \]

\[ = S(x) - g(x) - \int_a^b K(x, t) [S(t)]^p dt \] (19)

From (18) and (19), we have

\[ S(x) = g(x) + \int_a^b K(x, t) [S(t)]^p dt, \]

and so, \( S(x) \) must be the exact solution of Eq. (12).

5 Numerical Results and Discussion

The HAM provides an analytical solution in terms of an infinite power series. However, there is a practical need to evaluate this solution. The consequent series truncation, and the practical procedure conducted to accomplish this task, together transforms the analytical results into an exact solution, which is evaluated to a finite degree of accuracy. In order to investigate the accuracy of the HAM solution with a finite number of terms, two examples were solved. To show the efficiency of the present method for our problem in comparison with the exact solution we report absolute error which is defined by

\[ |Ey_{HAM}^m| = |y_{exact} - y_{HAM}^m| \]

where \( y_{HAM}^m = \sum_{i=0}^{m} y_i(x) \). MATLAB 7 is used to carry out the computations.

Example 1. Consider the nonlinear Fredholm integral equation

\[ y(x) = \ln(x + 1) + 2\ln(2(1 - x \ln 2 + x) - 2x - \frac{5}{4}) + \int_0^1 (x - t) y^2(t) dt. \]
For which the exact solution is \( y(x) = \ln(x + 1) \). We begin with 
\( y_0(x) = \ln(x + 1) + 2 \ln 2(1 - x \ln 2 + x) - 2x - \frac{5}{4} \). Its iteration formulation reads

\[
y_m(x) = \int_0^1 [(x - t) \sum_{j=0}^{m-1} y_j(t) y_{m-j-1}(t)] dt, \quad m = 1, 2, \ldots.
\]

Some numerical results of these solutions are presented in Table 1.

**Example 2.** The presented HAM iterative scheme is applied for solving the nonlinear integral equation

\[
y(x) = \sin(\pi x) + \frac{1}{5} \int_0^1 \cos(\pi x) \sin(\pi t) y^3(t) \, dt.
\]

The exact solution to this equation is \( y(x) = \sin(\pi x) + \frac{20 - \sqrt{391}}{3} \cos(\pi x) \). The formulas corresponding to this problem are

\[
y_0(x) = \sin(\pi x)
\]

\[
y_m(x) = \frac{1}{5} \int_0^1 \cos(\pi x) \sin(\pi t) \sum_{i=0}^{m-1} y_{m-i-1} \sum_{j=0}^{i} y_j y_{i-j} \, dt, \quad m = 1, 2, \ldots.
\]

Table 2. shows absolute errors of numerical results calculated according the presented method.
Table2. Numerical results of Example 2

| $x_i$ | $y_{exa}$   | $y_{15}^{HAM}$ | $|y_{exa} - y_{15}^{HAM}|$ |
|------|-------------|-----------------|-----------------------------|
| 0    | 0.0754266889 | 0.0754266889    | 5.53723733531$E - 15$      |
| 0.1  | 0.3807520383 | 0.3807520383    | 5.21804821573$E - 15$      |
| 0.2  | 0.6488067254 | 0.6488067254    | 4.55191440096$E - 15$      |
| 0.3  | 0.8533516897 | 0.8533516897    | 3.21964677141$E - 15$      |
| 0.4  | 0.9743646449 | 0.9743646449    | 1.77635683940$E - 15$      |
| 0.5  | 1.0000000000 | 1.0000000000    | 0                           |
| 0.6  | 0.9277483875 | 0.9277483875    | 1.77635683940$E - 15$      |
| 0.7  | 0.7646822990 | 0.7646822990    | 3.21964677141$E - 15$      |
| 0.8  | 0.5267637791 | 0.5267637791    | 4.55191440096$E - 15$      |
| 0.9  | 0.2372819503 | 0.2372819503    | 0.52735593669$E - 15$      |
| 1    | -0.0754266889| -0.0754266889   | 0.55372373353$E - 15$      |

6 Conclusion

The proposed method is a powerful procedure for solving integral equations. The examples analyzed illustrate the ability and reliability of the method presented in this paper and reveals that this one is very simple and effective. The obtained solutions, in comparison with exact solutions admit a remarkable accuracy. Results indicate that the convergence rate is very fast, and lower approximations can achieve high accuracy.

References


Received: August, 2008