A Subclass of Harmonic Univalent Functions
with Negative Coefficients Using
Fractional Calculus Operator

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Abstract

In this paper we introduce a new family \( \mathcal{A}(\alpha, \beta, \gamma) \) of harmonic univalent functions with negative coefficients using fractional calculus operator \( \Omega^\lambda \) which contains various well known classes of harmonic univalent functions as well as many new ones. We obtain coefficient estimate, distortion theorem and various other properties for functions belonging to the class \( \mathcal{A}(\alpha, \beta, \gamma) \) in the unit disk.

Mathematics Subject Classification: 30C45

Keywords: Harmonic Univalent Function, Fractional Derivative, Fractional Integral, Owa-Srivastava Fractional Calculus Operator

1 Introduction

A continuous complex valued function \( f = u + iv \) defined in a simply connected complex domain \( D \) is said to be harmonic in \( D \) if both \( u \) and \( v \) are real harmonic in \( D \). In any simply connected domain we can write \( f = h + \bar{g} \), where \( h \) and \( g \) are analytic in \( D \). A necessary and sufficient condition for \( f \) to be locally univalent and sense preserving in \( D \) is that \( |h'(z)| > |g'(z)| \) in \( D \).

Let \( \mathcal{A} \) be a class of functions \( f = h + \bar{g} \) that are univalent and sense preserving in the open unit disk \( U = \{z : |z| < 1\} \) where

\[
    h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U)
\]
and \[ g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_n| > 1 \] (2)

normalized by \( f(0) = 0 = f_z(0) - 1 \) with \( f_z(0) \) denotes partial derivative of \( f(z) \) at \( z = 0 \). We call \( h \) and \( g \) analytic part and co-analytic part of \( f \) respectively.

The harmonic function \( f = h + \bar{g} \) for \( g \equiv 0 \) reduces to an analytic function \( f = h \).

Let \( S^* (\alpha) \) and \( K(\alpha) \) denote the subclasses of \( A \) consisting of functions which are starlike of order \( \alpha \) and convex of order \( \alpha \) respectively in \( U \). Further let \( \tilde{A} \) be the subfamily of \( A \) consisting of harmonic functions of the form \( f = h + \bar{g} \), where

\[ h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \text{ and } g(z) = -\sum_{n=1}^{\infty} b_n z^n, \quad b_n \geq 0 . \] (3)

Several essentially equivalent definitions of fractional calculus have been studied in the literature \[9\], \[10\]. We state the following definitions due to Owa \[2\] which have been used rather frequently in the theory of analytic functions \[4\].

The fractional derivative for a function \( f(z) \) of order \( \lambda (0 \leq \lambda < 1) \) is defined by

\[ D_\lambda^z f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\eta)}{(z-\eta)^\lambda} d\zeta \] (4)

and the fractional integral for a function \( f(z) \) of order \( \lambda (\lambda > 0) \) is defined by

\[ D_{-\lambda}^z f(z) := \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\eta)}{(z-\eta)^{1-\lambda}} d\eta \] (5)

where \( f(z) \) is an analytic function in a simply connected region of the \( z \)-plane containing the origin and the multiplicity of \( (z-\eta)^\lambda \) in (1.5) (and that of \( (z-\eta)^{1-\lambda} \) in (1.6)) is removed by requiring \( \log(z-\eta) \) to be real when \( z-\eta > 0 \).

Using definition (1.5), the fractional derivative of order \( n + \lambda (0 \leq \lambda < 1; n \in N_0 := N \cup \{0\}) \) is defined by

\[ D_{z}^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_\lambda^z f(z) \] (6)

with the help of above definitions Owa and Srivastava \[4\] defined the fractional calculus operator \( \Omega^\lambda \), \( (\lambda \in \mathbb{R}; \lambda \neq 2, 3, 4, \ldots) \) by

\[ \Omega^\lambda f(z) = \Gamma(2-\lambda) z^\lambda D_{z}^{1-\lambda} f(z) \] (7)
A subclass of harmonic univalent functions

for functions belonging to the class \( \mathcal{A} \).

Choi at al. [6] investigated the subclass \( \mathcal{A}(\alpha, \beta, \gamma) \) of \( \mathcal{A} \) for \( \alpha < 2, \beta < 2 \) and \( \gamma < 1 \) which was defined by using the Owa-Srivastava fractional calculus operator \( \Omega^\lambda \) as follows:

\[
\mathcal{A}(\alpha, \beta, \gamma) := \left\{ f \in \mathcal{A} : \Re \left( \frac{\Omega^\alpha f(z)}{\Omega^\beta f(z)} \right) > \gamma \in U \right\} \tag{8}
\]

we note that \( \mathcal{A}(1, 0, \gamma) = S^*(\gamma) \) and \( \mathcal{A}(\alpha + 1, 0, \gamma) = S^*(\gamma, \alpha) \) which was studied by Owa.

We introduce the class of functions \( \bar{\mathcal{A}}(\alpha, \beta, \gamma) \) as follows:

\[
\bar{\mathcal{A}}(\alpha, \beta, \gamma) = \mathcal{A}(\alpha, \beta, \gamma) \cap \bar{\mathcal{A}}.
\]

We also see that

\[
\bar{\mathcal{A}}(\alpha + 1, 0, \gamma) = \bar{\mathcal{A}}^*(\gamma, \alpha)
\]

and

\[
\bar{\mathcal{A}}(1, 0, \gamma) = \bar{\mathcal{A}}^*(\gamma, 0).
\]

The special classes \( \bar{\mathcal{A}}^*(\gamma, \alpha) \) and \( \bar{\mathcal{A}}^*(\gamma, 0) \) were studied by Owa [1] and Silverman [7] respectively.

In this paper we obtain coefficient bounds, distortion theorem, inclusion properties of functions belonging to the class \( \bar{\mathcal{A}}(\alpha, \beta, \gamma) \) are also given. Further, we investigate radii of starlikeness and convexity for harmonic functions belonging to this class.

2 Coefficient Bounds

We begin with a necessary and sufficient condition for functions in \( \bar{\mathcal{A}}(\alpha, \beta, \gamma) \).

**Theorem 2.1** Let \( \beta \leq \alpha < 2 \) and \( \gamma < 1 \) also let the function \( f(z) \) be defined by \( f = h + \bar{g} \), where \( g \) and \( h \) are given by (3) then \( f(z) \in \bar{\mathcal{A}}(\alpha, \beta, \gamma) \) if and only if

\[
\sum_{n=2}^{\infty} \left( \frac{\Gamma(2 - \alpha)}{\Gamma(n + 1 - \alpha)} - \gamma \frac{\Gamma(2 - \beta)}{\Gamma(n + 1 - \beta)} \right) n!(a_n + b_n) \leq 1 - \gamma. \tag{9}
\]

The result (9) is sharp.

**Proof:** We have \( f = h + \bar{g} \). Using the definitions of fractional derivative and fractional calculus operator with this \( f \) we obtain

\[
\Omega^\lambda f(z) = z - \sum_{n=2}^{\infty} \psi(n, \lambda)(a_n + b_n)z^n, (a_n \geq 0, \lambda < 2), \tag{10}
\]
Thus we can rewrite inequality (12) as
\[ \beta \]
which is assertion (9) of Theorem 2. Hence
\[ (10) \]
and (10) is bounded by (11). Finally we note that the result is sharp the extremal function being
\[ f(z) = z - \frac{(1 - \lambda)\Gamma(n + 1 - \alpha)\Gamma(n + 1 - \beta)}{n\{\Gamma(2 - \alpha)\Gamma(n + 1 - \beta) - \gamma\Gamma(2 - \beta)\Gamma(n + 1 - \alpha)\}} z^n \quad (n \geq 2). \]

Corollary 2.2 Let \( f(z) \in \mathcal{A}(\alpha, \beta, \gamma) \), \( \beta \leq \alpha < 2 \) and \( \gamma < 1 \) then
\[ b_n + a_n \leq \frac{(1 - \gamma)\Gamma(n + 1 - \alpha)\Gamma(n + 1 - \beta)}{n\{\Gamma(2 - \alpha)\Gamma(n + 1 - \beta) - \gamma\Gamma(2 - \beta)\Gamma(n + 1 - \alpha)\}} \quad (n \geq 2). \]
3 Distortion Theorem

Our next result is on the distortion bounds for functions in $\overline{A}(\alpha, \beta, \gamma)$.

**Theorem 3.1** Let $0 < \alpha < 2, \beta \leq \alpha$ and $0 \leq \gamma < 1$ if $f(z) \in \overline{A}(\alpha, \beta, \gamma)$ then

$$|z| - \frac{(1 - \gamma)\{(2 - \alpha)^2 + (2 - \beta)^2\}}{4\{(2 - \beta) - \gamma(2 - \alpha)\}} \leq |f(z)| \leq 1 + \frac{(1 - \gamma)\{(2 - \alpha)^2 + (2 - \beta)^2\}}{4\{(2 - \beta) - \gamma(2 - \alpha)\}}$$

(15)

for $z \in U$.

**Proof:** Let $0 < \alpha < 2, \beta \leq \alpha$ and $0 \leq \gamma < 1$ if $f(z) \in \overline{A}(\alpha, \beta, \gamma)$. Then in view of Theorem 2.1 we obtain

$$(\psi(2, \alpha) - \gamma\psi(2, \beta)) \sum_{n=2}^{\infty} (a_n + b_n) \leq \sum_{n=2}^{\infty} (\psi(n, \alpha) - \gamma\psi(n, \beta)) (a_n + b_n)$$

$$\leq (1 - \gamma)$$

where $\psi(n, \alpha)$ and $\psi(n, \beta)$ are given by (11). This gives

$$\sum_{n=2}^{\infty} (a_n + b_n) \leq \frac{1 - \gamma}{\psi(2, \alpha) - \gamma\psi(2, \beta)}$$

$$= \frac{(1 - \gamma)(2 - \alpha)(2 - \beta)}{2\{(2 - \beta) - \gamma(2 - \alpha)\}}$$

$$\leq \frac{(1 - \gamma)\{(2 - \alpha)^2 + (2 - \beta)^2\}}{4\{(2 - \beta) - \gamma(2 - \alpha)\}}$$

(16)

by Young’s inequality. Consequently, we get

$$|f(z)| \geq |z| - |z|^2 \sum_{n=2}^{\infty} (a_n + b_n)$$

$$\geq |z| - \frac{(1 - \gamma)\{(2 - \alpha)^2 + (2 - \beta)^2\}}{4\{(2 - \beta) - \gamma(2 - \alpha)\}}$$

and

$$|f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} (a_n + b_n)$$

$$\leq 1 + \frac{(1 - \gamma)\{(2 - \alpha)^2 + (2 - \beta)^2\}}{4\{(2 - \beta) - \gamma(2 - \alpha)\}}$$

which completes proof of the Theorem 3.1.
Theorem 3.2 Let $\beta \leq 0, \alpha < 2$ and $0 \leq \gamma < 1$ if $f(z) \in \bar{A}(\alpha, \beta, \gamma)$ then
\[
|D_\alpha^z f(z)| \leq \frac{|z|^{-\alpha}}{\Gamma(2-\alpha)} \left( 1 + \frac{(1-\alpha)(1-\beta)}{(2-\beta) - \gamma(2-\alpha)} \right) \tag{17}
\]
and
\[
|D_\alpha^z f(z)| \geq \frac{|z|^{-\alpha}}{\Gamma(2-\alpha)} \left| |z| - \frac{(1-\alpha)(1-\beta)}{(2-\beta) - \gamma(2-\alpha)} \right| \tag{18}
\]
for $z \in U$.

Proof: Let $f(z) \in \bar{A}(\alpha, \beta, \gamma)$. Then using Theorem 2.1 we obtain
\[
\sum_{n=2}^{\infty} \psi(n, \alpha)(a_n + b_n) \leq (1-\gamma) + \gamma \sum_{n=2}^{\infty} \psi(n, \beta)(a_n + b_n) \tag{19}
\]
where $\psi(n, \alpha)$ and $\psi(n, \beta)$ are as given in (11). Since $\beta \leq 0$, it is easily seen that
\[
0 < \psi(n, \beta) \leq \psi(2, \beta) = \frac{2}{2-\beta}. \tag{20}
\]
By applying (16), (19) and (20) we obtain
\[
\sum_{n=2}^{\infty} \psi(n, \alpha)(a_n + b_n) \leq (1-\gamma) + \gamma \sum_{n=2}^{\infty} \psi(n, \beta)(a_n + b_n)
\leq \frac{(1-\gamma)(2-\beta)}{(2-\beta) - \gamma(2-\alpha)} \tag{21}
\]
Hence by virtue of (7) and (21) we observe that
\[
|\Gamma(2-\alpha)z^\alpha D_\alpha^z f(z)| \leq |z| + |z|^2 \sum_{n=2}^{\infty} \psi(n, \alpha)(a_n + b_n)
\leq 1 + \frac{(1-\gamma)(2-\beta)}{(2-\beta) - \gamma(2-\alpha)}
\]
and
\[
|\Gamma(2-\alpha)z^\alpha D_\alpha^z f(z)| \geq |z| - \frac{(1-\gamma)(2-\beta)}{(2-\beta) - \gamma(2-\alpha)}.
\]
This completes the proof of the assertions (17) and (18) of Theorem 3.2.
4 Properties of The Class $\tilde{A}(\alpha, \beta, \gamma)$

We note that,

**Theorem 4.1** Let $\beta_1 \leq \beta_2 \leq \alpha < 2$ and $0 \leq \gamma < 1$. Then $\tilde{A}(\alpha, \beta_1, \gamma) \subset \tilde{A}(\alpha, \beta_2, \gamma)$.

**Proof:** The proof is straightforward using Theorem 2.1 and hence we omit the details.

**Theorem 4.2** Let $\beta \leq \alpha_1 \leq \alpha_2 < 2$ and $0 \leq \gamma < 1$. Then $\tilde{A}(\alpha_2, \beta, \gamma) \subset \tilde{A}(\alpha_1, \beta, \gamma)$.

**Corollary 4.3** Let $\beta_1 \leq \beta_2 \leq \alpha_1 \leq \alpha_2 < 2$ and $0 \leq \gamma < 1$. Then $\tilde{A}(\alpha_2, \beta_1, \gamma) \subset \tilde{A}(\alpha_1, \beta_2, \gamma)$.

Next, we find the radii of starlikness and convexity for functions belonging to the class $\tilde{A}(\alpha, \beta, \gamma)$ in the following:-

**Theorem 4.4** Let $\beta \leq \alpha < 2$ and $\gamma < 1$ and let $f(z) \in \tilde{A}(\alpha, \beta, \gamma)$. Then $f(z)$ is star like of order $\sigma (0 \leq \sigma < 1)$ in $|z| < r_1(\alpha, \beta, \gamma, \sigma)$, where

$$r_1(\alpha, \beta, \gamma, \sigma) = \inf \left[ \frac{(1-\sigma)n!\{\Gamma(2-\alpha)\Gamma(n+1-\beta) - \gamma\Gamma(n+1-\alpha)\}}{(n-\sigma)(1-\gamma)\Gamma(n+1-\alpha)\Gamma(n+1-\beta)} \right]^{1/(n-1)}$$

where the infimum is taken over all $n \geq 2$.

**Proof:** Let $f(z) \in \tilde{A}(\alpha, \beta, \gamma)$. Then it is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq (1-\sigma)$$

for $|z| < r_1(\alpha, \beta, \gamma, \sigma)$. The proof now follows.

**Corollary 4.5** Let $\beta \leq \alpha < 2$ and $\gamma < 1$ and let $f(z) \in \tilde{A}(\alpha, \beta, \gamma)$. Then $f(z)$ is convex of order $\sigma (0 \leq \sigma < 1)$ in $|z| < r_2(\alpha, \beta, \gamma, \sigma)$, where

$$r_2(\alpha, \beta, \gamma, \sigma) = \inf \left[ \frac{(1-\sigma)n!\{\Gamma(2-\alpha)\Gamma(n+1-\beta) - \gamma\Gamma(n+1-\alpha)\}}{n(n-\sigma)(1-\gamma)\Gamma(n+1-\alpha)\Gamma(n+1-\beta)} \right]^{1/(n-1)}$$

where the infimum is taken over all $n \geq 2$.

Let $\tilde{A}_0(\alpha, \beta, \gamma)$ denote the subclass of $\tilde{A}(\alpha, \beta, \gamma)$ if the co-analytic part of $f$ is zero i.e. $g = 0$. Then we have,
Theorem 4.6 Let \( \beta \leq \alpha < 2 \) and \( \gamma < 1 \). Also let

\[
f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad (a_{n,j} \geq 0; j = 1, 2, \ldots, m)
\]

be in the class \( \mathcal{A}_0(\alpha, \beta, \gamma) \). Then the function \( g(z) \) defined by

\[
g(z) = \sum_{j=1}^{m} f_j \left( \frac{z}{m} \right)
\]

also belongs to the class \( \mathcal{A}_0(\alpha, \beta, \gamma) \).

**Proof:** From (22) we can write

\[
g(z) = z - \sum_{n=2}^{\infty} \left( \frac{1}{m^n} \sum_{j=1}^{m} a_{n,j} \right) z^n.
\]

Since \( f_j(z) \in \mathcal{A}_0(\alpha, \beta, \gamma) \subseteq \mathcal{A}(\alpha, \beta, \gamma) \), \( j = 1, 2, \ldots, m \) by using Theorem 2.1 we obtain

\[
\sum_{n=2}^{\infty} \{ \psi(n, \alpha) - \gamma \psi(n, \beta) \} \left( \frac{1}{m^n} \sum_{j=1}^{m} a_{n,j} \right) \\
\leq \frac{1}{m^n} \{(1 - \gamma) + (1 - \gamma) + \cdots + (1 - \gamma)\} \\
\leq 1 - \gamma
\]

where \( \psi(n, \alpha) \) and \( \psi(n, \beta) \) are as given in Theorem 2.1. Hence \( g \in \mathcal{A}_0(\alpha, \beta, \gamma) \). This completes the proof of theorem.

Theorem 4.7 Let \( \beta \leq \alpha < 2, \gamma < 1 \) and \( \sigma > -1 \). If \( f(z) \in \mathcal{A}_0(\alpha, \beta, \gamma) \) then the function \( G(z) \) given by

\[
G(z) = \frac{\sigma + 1}{z^\sigma} \int_0^z t^{\sigma-1} \sum_{j=1}^{m} \frac{j}{m} f_j \left( \frac{t}{j} \right) \, dt, (\sigma > -1, z \in U),
\]

where \( f_j(z) \) are same as defined in Theorem 4.6, is also in the class \( \mathcal{A}_0(\alpha, \beta, \gamma) \).

**Proof:** Let \( f(z) \in \mathcal{A}_0(\alpha, \beta, \gamma) \) then we have

\[
f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0, z \in U).
\]

Now from (23) it is easy to show that

\[
G(z) = \sum_{j=1}^{m} g_j(z),
\]
where
\[ g_j(z) = \frac{1}{m} \left( z - \sum_{n=2}^{\infty} b_{n,j} z^n \right), \]
\[ b_{n,j} = \left( \frac{\sigma + 1}{\sigma + n} \right)^{j-1} a_n, \quad j = 1, 2, \ldots, m. \]

Once again using Theorem 2.1 we obtain
\[ \sum_{n=2}^{\infty} \left\{ \psi(n, \alpha) - \gamma \psi(n, \beta) \right\} b_{n,j} = \sum_{n=2}^{\infty} \left\{ \psi(n, \alpha) - \gamma \psi(n, \beta) \right\} \left( \frac{\sigma + 1}{\sigma + n} \right)^{j-1} a_n \]
\[ \leq \sum_{n=2}^{\infty} \left\{ \psi(n, \alpha) - \gamma \psi(n, \beta) \right\} a_n \]
\[ \leq (1 - \gamma) \]
where \( \psi(n, \alpha) \) and \( \psi(n, \beta) \) are as given in Theorem 2.1. Hence \( G(z) \in \mathcal{A}_0(\alpha, \beta, \gamma) \). This completes the proof of the theorem.

References


Received: September, 2008