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Asymptotic Stability of Time Varying Delay-Difference System of Cellular Neural Networks via Matrix Inequalities

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Abstract. In this paper, we obtain some criteria for determining the asymptotic stability of the zero solution of time varying delay-difference system of cellular neural networks in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method. The result is applied to obtain new stability conditions for some classes of time varying delay-difference system such as time varying delay-difference system of cellular neural networks with multiple delays in terms of certain matrix inequalities. Our results can be well suited for computational purposes.

Keywords: Asymptotic stability; Cellular neural networks; Lyapunov function; Time varying delay-difference system; Matrix inequalities.

1 Introduction

In recent decades, cellular neural networks have been extensively studied in many aspects and successfully applied to many fields such as pattern identifying, voice recognizing, system controlling, signal processing systems, static image treatment, and solving nonlinear algebraic equations, etc. Such applications are based on the existence of equilibrium points, and qualitative properties of systems. In electronic implementation, time delays occur due to some reasons such as circuit integration, switching delays of the amplifiers and communication delays, etc. Therefore, the study of the asymptotic stability of cellular neural networks with delays is of particular importance to manufacturing high quality microelectronic cellular neural networks.

While stability analysis of continuous-time neural networks can employ the stability theory of differential equations by Wei et al. [12], it is much harder to study the stability of discrete-time neural networks by Gubta and Jin [8] with time delays by Arik [3] or impulses by Liu et al. [11]. The techniques currently available in the literature for discrete-time systems are mostly based on the construction Lyapunov second method by Infante [10]. For Lyapunov second method, it is well known that no general rule exists to guide the construction of a proper Lyapunov function for a given system. In fact, the construction of the Lyapunov function becomes a very difficult task.

In this paper, we consider time varying delay-difference system of cellular neural networks of the form

$$x(k+1) = -C(k)x(k) + A(k)S(x(k)) + B(k)S(x(k-h)), \tag{1.1}$$

where $x(k) \in \mathbf{R}^n$ is neuron state vector, $h > 0$, $C(k) = \text{diag}\{c_1(k), \dots, c_n(k)\}$, $c_i(k) \ge 0$, $i = 1, 2, ..., n$ is the $n \times n$ relaxation matrix function, $A(k)$ and $B(k)$ are the $n \times n$ weight matrices functions, $S(x) = [s_1(x_1), \dots, s_n(x_n)]^T$ where $s_i(x_i)$, $i = 1, 2, ..., n$ are the neuron activations.

The asymptotic stability of the zero solution of the delay-differential system of cellular neural networks has been developed during the past several years. We refer to monographs by Arik [2] and Chua and Yang [6] and the references cited therein. Much less is known regarding the asymptotic stability of the zero solution of the time varying delay-difference system of cellular neural networks. Therefore, the purpose of this paper is to establish sufficient conditions for the asymptotic stability of the zero solution of (1.1) in terms of certain matrix inequalities.

2 Preliminaries

The following notations will be used throughout the paper. $\mathbf{R}^+$ denotes the set of all non-negative real numbers; $\mathbf{Z}^+$ denotes the set of all non-negative integers; $\mathbf{R}^n$ denotes the n-finite-dimensional Euclidean space with the Euclidean norm $\|.\|$ and the scalar product between x and y is defined by $x^T y$; $M^{n\times m}$ denotes the space of all $(n \times m)$-matrices; and A^T denotes the transpose of the matrix A; A is the symmetric matrix if $A = A^T$.

Throughout this paper, we assume the neuron activations $s_i(x_i)$, $i = 1, 2, \dots, n$ are bounded on $\mathbf{R}$, and $s_i(x_i)$, $i = 1, 2, \dots, n$ are Lipschitz continuous, that is, there exists the constants $l_i > 0$, $i = 1, 2, \dots, n$ such that

$$|s_i(x_1) - s_i(x_2)| \le l_i |x_1 - x_2|, \ \forall x_1, x_2 \in \mathbf{R}. \tag{2.1}$$

By condition (2.1), $s_i(x_i)$ satisfy

$$|s_i(x_i)| \le l_i |x_i|, \; i = 1, 2, ..., n. \tag{2.2}$$

Matrix $Q \in \mathbf{R}^{n\times n}$ is positive semidefinite $(Q \ge 0)$ if $x^T Q x \ge 0$, for all $x \in \mathbf{R}^n$. If $x^T Q x > 0 (x^T Q x < 0$, resp.) for any $x \ne 0$, then Q is positive (negative, resp.) definite and denoted by $Q > 0, (Q < 0,$ resp.). It is easy to verify that $Q > 0$, $(Q < 0,$ resp.) Iff

$$\exists \beta > 0: \; x^T Q x \ge \beta \|x\|^2, \forall x \in \mathbf{R}^n,$$

$$(\; \exists \beta > 0: \; x^T Q x \le -\beta \|x\|^2, \forall x \in \mathbf{R}^n, \text{ resp.});$$

Matrix function $Q(t) \in M^{n\times n}$ is positive definite if

$$\exists \beta > 0: \; x^T Q(t) x \ge \beta \|x\|^2, \; \forall t \in \mathbf{R}^+, \; x \in \mathbf{R}^n.$$

Corollary 2.1 For any positive scalar ε and vectors x and y, the following inequality holds:

$$x^T y + y^T x \le \varepsilon x^T x + \varepsilon^{-1} y^T y.$$

Let us denote $V_\delta = \{x \in \mathbf{R}^n : \|x\| < \delta\}$.

Lemma 2.1 (Hale [9]) The zero solution of difference system is asymptotic stability if there exists a positive definite function $V(k, x(k)) : \mathbf{R}^n \to \mathbf{R}^+$ such that

$$\exists \beta > 0 : \Delta V(k, x(k)) = V(k, x(k+1)) - V(k, x(k)) \le -\beta \|x(k)\|^2,$$

along the solution of the system. In case the above condition holds for all $x(k) \in V_\delta$, we say that the zero solution is locally asymptotically stable.

We present the following technical lemmas, which will be used in the proof of our main result.

Lemma 2.2 (Chua and Yang [6]) For any constant symmetric matrix $M \in \mathbf{R}^{n\times n}, M = M^T > 0$, scalar $s \in \mathbf{Z}^+/\{0\}$, vector function $W : [0, s] \to \mathbf{R}^n$, we have

$$s \sum_{i=0}^{s-1} (w^T(i) M w(i)) \ge \left(\sum_{i=0}^{s-1} w(i) \right)^T M \left(\sum_{i=0}^{s-1} w(i) \right).$$

3 Main result

In this section, we will study asymptotic stability of zero solution of time-varying delay difference system of Cellular neural networks in terms of certain matrix inequalities by using the second Lyapunov method described by the following system of the form

$$x(k+1) = -C(k)x(k) + A(k)S(x(k)) + B(k)S(x(k-h)), \tag{3.1}$$

where $x(k) \in \mathbf{R}^n$ is neuron state vector, $h > 0,\ C(k) = \text{diag}\{c_1(k), \dots, c_n(k)\}$, $c_i(k) \geq 0$, $i = 1, 2, ..., n$ is the $n \times n$ relaxation matrix function, $A(k)$ and $B(k)$ are the $n \times n$ weight matrices functions, $S(x) = [s_1(x_1), \dots, s_n(x_n)]^T$ where $s_i(x_i)$, $i = 1, 2\ , ..., n$ are the neuron activations.

Theorem 3.1 T*he zero solution of (3.1) is asymptotically stable if there exists the symmetric positive definite matrices* P, G, W, $L = diag[l_1, \dots, l_n] > 0$ *and* $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ *satisfying the following matrix inequalities of the form*

$$\psi = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} < 0\,, \qquad (3.2)$$

where

$$\begin{aligned} (1,1) &= C^T(k)PC(k) - P + hG + W + \varepsilon A^T(k)PB(k)B^T(k)PA(k) \\ &\quad + \varepsilon_1 C^T(k)PB(k)B^T(k)PC(k) + \varepsilon_2 LA^T(k)PB(k)B^T(k)PA(k)L \\ &\quad + LA^T(k)PA(k)L + \varepsilon^{-1}LL, \\ (2,2) &= LB^T(k)PB(k)L + \varepsilon_1^{-1}LL + \varepsilon_2^{-1}LL - W, \\ (3,3) &= -hG. \end{aligned}$$

Proof. Consider the Lyapunov function $V = V_1 + V_2 + V_3$,
Where

$$\begin{aligned} V_1 &= x^T(k)Px(k), \\ V_2 &= \sum_{i=k-h}^{k-1} (h - k + i)x^T(i)Gx(i), \\ V_3 &= \sum_{i=k-h}^{k-1} x^T(i)Wx(i), \end{aligned}$$

P, G and W being symmetric positive definite solutions of (3.2).
The Lyapunov difference of the system is defined as $\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$,
where

$$\begin{aligned}
\Delta V_1 &= V_1(x(k+1)) - V_1(x(k)) \\
&= [-C(k)x(k) + A(k)S(x(k)) + B(k)S(x(k-h))]^T P \\
&\quad [-C(k)x(k) + A(k)S(x(k)) + B(k)S(x(k-h))] \\
&\quad -x^T(k)Px(k) \\
&= x^T(k)[C(k)PC(k) - P]x(k) \\
&\quad -x^T(k)C(k)PA(k)S(x(k)) - S^T(x(k))A^T(k)PC(k)x(k) \\
&\quad -x^T(k)C(k)PB(k)S(x(k-h)) - S^T(x(k-h)B^T(k)PC(k)x(k) \\
&\quad +S^T(x(k))A^T(k)PB(k)S(x(k-h)) + S^T(x(k-h))B^T(k)PA(k)S(x(k)) \\
&\quad +S^T(x(k))A^T PAS(x(k)) + S^T(x(k-h))B^T PBS(x(k-h)), \\
\Delta V_2 &= \Delta\left(\sum_{i=k-h}^{k-1}(h-k+i)x^T(i)Gx(i)\right) = hx^T(k)Gx(k) - \sum_{i=k-h}^{k-1} x^T(i)Gx(i), \\
\Delta V_3 &= \Delta\left(\sum_{i=k-h}^{k-1} x^T(i)Wx(i)\right) = x^T(k)Wx(k) - x^T(k-h)Wx(k-h).
\end{aligned} \tag{3.3}$$

By (2.2) and Corollary 2.1 are utilized in (3.3), respectively.
Note that

$$\begin{aligned}
S^T(x(k))A^T(k)PB(k)S(x(k-h)) + S^T(x(k-h))B^T(k)PA(k)S(x(k)) &\le \varepsilon_2 S^T(k)A^T(k)P \\
&\quad B(k)B^T(k)PA(k)S(k) \\
&\quad +\varepsilon_2^{-1}S^T(x(k-h)) \\
&\quad S(x(k-h)), \\
-x^T(k)C(k)PA(k)S(x(k)) - S^T(x(k))A^T(k)PC(k)x(k) &\le \varepsilon x^T(k)C(k)P \\
&\quad A(k)A^T(k)PC(k)x(k) \\
&\quad +\varepsilon^{-1}S^T(x(k))S(x(k)), \\
-x^T(k)C(k)PB(k)S(x(k-h)) - S^T(x(k-h))B^T(k)PC(k)x(k) &\le \varepsilon_1 x^T(k)C(k)P \\
&\quad B(k)B^T(k)PC(k)x(k) \\
&\quad +\varepsilon_1^{-1}S^T(x(k-h)) \\
&\quad S(x(k-h)),
\end{aligned}$$

$$\begin{aligned}
S^T(x(k-h))B^T(k)PB(k)S(x(k-h)) &\le x^T(k-h)LB^T(k)PB(k)Lx(k-h), \\
S^T(x(k))A^T(k)PA(k)S(x(k)) &\le x^T(k)LA^T(k)PA(k)Lx(k), \\
\varepsilon_2 S^T(k)A^T(k)PB(k)B^T(k)PA(k)S(k) &\le \varepsilon_2 x^T(k)LA^T(k)PB(k)B^T(k)PA(k)Lx(k), \\
\varepsilon_1^{-1}S^T(x(k-h))S(x(k-h)) &\le \varepsilon_1^{-1}x^T(k-h)LLx(k-h), \\
\varepsilon_2^{-1}S^T(x(k-h))S(x(k-h)) &\le \varepsilon_2^{-1}x^T(k-h)LLx(k-h), \\
\varepsilon^{-1}S^T(x(k))S(x(k)) &\le \varepsilon^{-1}x^T(k)LLx(k),
\end{aligned}$$

hence

$$\begin{aligned}\Delta V_1 \le\, & x^T(k)[C(k)PC(k)-P]x(k)\\ & +\varepsilon x^T(k)A^T(k)PB(k)B^T(k)PA(k)x(k)\\ & +\varepsilon_1 x^T(k)C(k)PB(k)B^T(k)PC(k)x(k)\\ & +x^T(k-h)LB^T(k)PB(k)Lx(k-h)\\ & +x^T(k)LA^T(k)PA(k)Lx(k)\\ & +\varepsilon_2 x^T(k)LA^T(k)PB(k)B^T(k)PA(k)Lx(k)\\ & +\varepsilon_1^{-1}x^T(k-h)LLx(k-h)\\ & +\varepsilon_2^{-1}x^T(k-h)LLx(k-h)\\ & +\varepsilon^{-1}x^T(k)LLx(k).\end{aligned}$$

Then we have

$$\begin{aligned}\Delta V \le\, & x^T(k)[C(k)PC(k)-P+hG+W+\varepsilon A^T(k)PB(k)B^T(k)PA(k)\\ & +\varepsilon_1 C(k)PB(k)B^T(k)PC(k)+\varepsilon_2 LA^T(k)PB(k)B^T(k)PA(k)L\\ & +LA^T(k)PA(k)L+\varepsilon^{-1}LL]x(k)\\ & +x^T(k-h)[LB^T(k)PB(k)L+\varepsilon_1^{-1}LL+\varepsilon_2^{-1}LL-W]x(k-h)\\ & -\sum_{i=k-h}^{k-1} x^T(i)Gx(i).\end{aligned}$$

Using Lemma 2.2, we obtain

$$\sum_{i=k-h}^{k-1} x^T(i)Gx(i) \ge \left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)^T (hG)\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right).$$

From the above inequality it follows that:

$$\begin{aligned}\Delta V \le\ & x^T(k)[C(k)PC(k) - P + hG + W + \varepsilon A^T(k)PB(k)B^T(k)PA(k)\\ & +\varepsilon_1 C(k)PB(k)B^T(k)PC(k) + \varepsilon_2 LA^T(k)PB(k)B^T(k)PA(k)L\\ & +LA^T(k)PA(k)L + \varepsilon^{-1}LL]x(k)\\ & +x^T(k-h)[LB^T(k)PB(k)L + \varepsilon_1^{-1}LL + \varepsilon_2^{-1}LL - W]x(k-h)\\ & -\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)^T (hG)\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)\\ & = \left(x^T(k), x^T(k-h), (\frac{1}{h}\sum_{i=k-h}^{k-1} x(i))^T \right)\begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix}\\ & \begin{pmatrix} x(k) \\ x(k-h) \\ (\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)) \end{pmatrix}\\ & = y^T(k)\begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} y(k)\\ & = y^T(k)\psi y(k),\end{aligned}$$

where

$$\begin{aligned}(1,1) =\ & C^T(k)PC(k) - P + hG + W + \varepsilon A^T(k)PB(k)B^T(k)PA(k)\\ & +\varepsilon_1 C^T(k)PB(k)B^T(k)PC(k) + \varepsilon_2 LA^T(k)PB(k)B^T(k)PA(k)L\\ & +LA^T(k)PA(k)L + \varepsilon^{-1}LL,\\ (2,2) =\ & LB^T(k)PB(k)L + \varepsilon_1^{-1}LL + \varepsilon_2^{-1}LL - W,\\ (3,3) =\ & -hG,\\ y(k) =\ & \begin{pmatrix} x(k) \\ x(k-h) \\ \left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right) \end{pmatrix}.\end{aligned}$$

By the condition (3.2), ΔV is the negative definite, there is a number $\beta > 0$ such that $\Delta V(y(k)) \le -\beta \|y(k)\|^2$, and hence, asymptotic stability of the delay difference system of (3.1) immediately follows from Lemma 2.1. This completes the proof. □

Example 3.1 Let us consider a time-varying delay-difference system of the Cellular neural networks (3.1), given by the system

$$x(k+1) = -C(k)x(k) + A(k)S(x(k)) + B(k)S(x(k-h)),$$

where the matrices are $A(k) = \begin{pmatrix} e^{-k} & 0 \\ 0 & e^{-k} \end{pmatrix}$, $B(k) = \begin{pmatrix} e^{-k} & 0 \\ 0 & e^{-k} \end{pmatrix}$,

$C(k) = \begin{pmatrix} e^{-k} & 0 \\ 0 & e^{-k} \end{pmatrix}$, $s_i(x_i) = \frac{2}{\pi}\tan^{-1}(x_i)$, $i = 1, 2$, $h = 1$ and $\varepsilon, \varepsilon_1, \varepsilon_2 = 0.5$.

Using the LMI Toolbox in MATLAB, we found that the LMIs in Theorem 3.1 are feasible and $P = \begin{pmatrix} 0.7714 & 0.2667 \\ 0.2667 & 0.3691 \end{pmatrix}$, $G = \begin{pmatrix} 0.7135 & 0.2556 \\ 0.2556 & 0.8923 \end{pmatrix}$,

$W = \begin{pmatrix} 0.5190 & 0.1132 \\ 0.1132 & 0.8518 \end{pmatrix}$, $L = \begin{pmatrix} 0.3376 & 0 \\ 0 & 0.1580 \end{pmatrix}$ are set of solutions to the LMIs (3.2).

By a straightforward, we have

$$\psi = \begin{pmatrix} -0.0086 & 0 \\ 0 & -0.0146 \end{pmatrix}.$$

The eigenvalues are -0.0086 and -0.0146, respectively. This implies the matrix $\psi < 0$. It follows from Lemma 2.1 that zero solution of time-varying delay difference system of Cellular neural networks is asymptotically stable.

5 Conclusions

In this paper, based on a discrete analog of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of time varying delay-difference system of cellular neural networks in terms of certain matrix inequalities.

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