International Mathematical Forum, 4, 2009, no. 15, 733 - 746

Annihilator Ideals in Almost Distributive Lattices

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Abstract

The concept of annihilator ideals is introduced in an Almost Distributive Lattice (ADL)R. It is proved that the set of all annihilator ideals of R forms a complete Boolean algebra. The sufficient condition for R to become a relatively complemented ADL is derived. The concept of annihilator preserving homomorphism is introduced in R. A sufficient condition for a homomorphism to be annihilator preserving is derived. Finally, it is proved that the homomorphic images and the inverse images of an annihilator ideal are again annihilator ideals.

Mathematics Subject Classification: 06D99, 06D15

Keywords: Almost Distributive Lattice(ADL), Boolean algebra, Annihilator ideal, relatively complemented ADL, annihilator preserving homomorphism

1 Introduction

An Almost Distributed Lattice(ADL) was introduced by U.M. Swamy and Rao.G.C [8] as an algebra $(R, \vee, \wedge)$ of type $(2, 2)$ which satisfies almost all the properties of a distributive lattice except possibly the commutative of $\vee$, the commutative of $\wedge$ and the right distributivity of $\vee$ over $\wedge$. It was also observed that any one of these three properties converts an ADL into a distributive lattice. Later a more general class called $\star$-ADLs was introduced in paper [10]. In [6], Mandelker studied the properties of relative annihilators and characterized the distributive lattice in terms of relative annihilators. In this paper the concept of Annihilator ideals in an ADL is introduced with suitable examples and proved some basic properties of the annihilator ideals, analogous to that in a distributive lattice[4]. It is proved that the set $\mathcal{A}(R)$ of all annihilator ideals of an ADL R with 0 can be made into a complete boolean algebra. A sufficient condition for an ADL to be relatively complemented is derived.

The concept of annihilator preserving homomorphism from an ADL R into another ADL R' is introduced as a homomorphism f satisfying the condition $f(A^*) = \{f(A)\}^*$, for any non-empty subset A of R. A sufficient condition for a homomorphism to be annihilator preserving is derived . It is proved that the images and the inverse images under a homomorphism of an annihilator ideal are again annihilator ideals. Finally, it is proved that for any ideal I of R, there exists a homomorphism from R to the set $Hom_R(I)$ of all homomorphisms defined on I, whose kernel is the annihilator of I.

2 Preliminary Notes

An Almost Distributive Lattice (ADL) is an algebra $(R, \vee, \wedge)$ of type (2,2) satisfying

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$ for any $x, y, z \in R$.

If R has an element 0 and satisfies $0 \wedge x = 0$ and $x \vee 0 = x$ along with the above properties, then R is called an ADL with 0.

Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define two binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0 \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL with x_0 as zero element and is called a discrete ADL.

If $(R, \vee, \wedge, 0)$ is an ADL with 0, then for any $a, b \in R$, define $a \leq b$ iff

$a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on R. 2.1

Theorem 2.1. *For any $a, b, c \in R$, we have the following:*
1. $a \vee b = a \Leftrightarrow a \wedge b = b$
2. $a \vee b = b \Leftrightarrow a \wedge b = a$
3. $a \wedge b = b \wedge a$ *whenever* $a \leq b$
4. $\wedge$ *is associative in* R.
5. $a \wedge b \wedge c = b \wedge a \wedge c$
6. $(a \vee b) \wedge c = (b \vee a) \wedge c$
7. $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$
8. $a \vee b = b \vee a$ *whenever* $a \wedge b = 0$
9. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
10. $a \wedge (a \vee b) = a, (a \wedge b) \vee b = b,$ *and* $a \vee (b \wedge a) = a$
11. $a \leq a \vee b$ *and* $a \wedge b \leq b$
12. $a \wedge a = a$ *and* $a \vee a = a$
13. $0 \vee a = a$ *and* $a \wedge 0 = 0$
14. *If* $a \leq c$ *and* $b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$
15. $a \vee b = a \vee b \vee a$

A non-empty subset I of R is called an ideal(filter)of R if $a \vee b \in I (a \wedge b \in I)$ and $a \wedge x \in I (x \vee a \in I)$ whenever $a, b \in I$ and $x \in R$. If I is an ideal of R and $a, b \in R$, then $a \wedge b \in I \Leftrightarrow b \wedge a \in I$. The set $\mathcal{I}(R)$ of all ideals of R is a complete distributive lattice with least element $\{0\}$ and the greatest element R under set inclusion in which, for any $I, J \in \mathcal{I}(R)$, $I \cap J$ is the infimum of I, J and the suprimum is given by $I \vee J = \{i \vee j / i \in I, j \in J\}$. For any $a \in R$, $(a] = \{a \wedge x / x \in R\}$ is the ideal generated by a. Similarly, for any $a \in R, [a) = \{x \vee a / x \in R\}$ is the filter generated by a. 2.2

Theorem 2.2. *For any $a, b \in R$, we have the following:*
1. $(a] \vee (b] = (a \vee b] = (b \vee a]$
2. $(a] \cap (b] = (a \wedge b] = (b \wedge a]$
3. $[a) \vee [b) = [a \wedge b) = [b \wedge a)$
4. $[a) \cap [b) = [a \vee b) = [b \vee a)$

Thus the set $\mathcal{PI}(R)$ of all principal ideals of R is a sublattice of the distributive lattice $I(R)$ of all ideals of R. We can introduce many concepts in an ADL R through the lattice $\mathcal{PI}(R)$. A proper ideal P of R is said to be prime if for any $x, y \in R$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. It is clear that a subset P of R is a prime ideal *iff* $R - P$ is a prime filter. An ADL R with 0 is called relatively complemented if each interval $[a, b], a \leq b$, in R is a complemented lattice. 2.3

Theorem 2.3. *If I is an ideal and F a filter in an ADL R such that $I \cap F = \emptyset$, then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

3 Annihilator ideals

In this section we give the definition of an annihilator ideal in an ADL R with 0 and prove some basic properties of the annihilator ideals. Also we prove various propeties of an ADL with annihilator ideals including a sufficient condition for an ADL R to become relatively complemented. 3.1

Definition 3.1. *For any non-empty subset A of an ADL R with 0, define*

$$A^* = \{x \in R / a \wedge x = 0, \text{ for all } a \in A\}$$

A^* is called the annihilator of A. For any $a \in R$, we have $\{a\}^* = (a]^*$, where $(a]$ is the principal ideal generated by a.For any $\emptyset \neq A \subseteq R$, we have clearly $A \cap A^* = (0]$.

The following two results can be directly verified by using definition 3.1. 3.2

Lemma 3.2. *For any non-empty subset A of R, A^* is an ideal of R.*

3.3

Lemma 3.3. *For any non-empty subsets I and J of R, we have the following:*

1). $I^* = \bigcap_{a \in I} (a]^*$
2). *If* $I \subseteq J$, *then* $J^* \subseteq I^*$ *and* $I^{**} \subseteq J^{**}$
3). $I \subseteq I^{**}$
4). $I^{***} = I^*$
5). $I^* \cap I^{**} = (0]$
6). $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq I^{**}$
7). $I \cap J = (0] \Leftrightarrow I \subseteq J^*$

3.4

Lemma 3.4. *For any two ideals I and J of R, we have the following:*

1. $(I \vee J)^* = I^* \cap J^*$
2. $(I \cap J)^{**} = I^{**} \cap J^{**}$

Proof: (1). Suppose $x \in I^* \cap J^*$ and $t = a \vee b \in I \vee J$, where $a \in I$ and $b \in J$. Then $t \wedge x = (a \vee b) \wedge x = (a \wedge x) \vee (b \wedge x) = 0 \vee 0 = 0$.
Therefore $I^* \cap J^* \subseteq (I \vee J)^*$. Converse follows from Lemma 3.3(2).
(2). Let $x \in I^{**} \cap J^{**}$, $y \in (I \cap J)^*$, $i \in I$ and $j \in J$. Since $i \wedge j \in I \cap J$ and $y \in (I \cap J)^*$, we get that $(y \wedge i) \wedge j = 0$. Which implies that $y \wedge i \in (j]^*$ for all $j \in J$. Hence $y \wedge i \in J^*$. Since $x \in J^{**}$, we get $(x \wedge y) \wedge i = 0$ for all $i \in I$. Hence $x \wedge y \in I^*$. Since $x \in I^{**}$, we get $x \wedge y \in I^{**}$. Thus $x \wedge y \in I^* \cap I^{**} = (0]$. Hence $x \wedge y = 0$ for all $y \in (I \cap J)^*$. Therefore $x \in (I \cap J)^{**}$.
Thus $I^{**} \cap J^{**} \subseteq (I \cap J)^{**}$. Converse follows from Lemma 3.3(2). □

The result (2) of the above lemma can be generalized as given in the following. 3.5

Corollary 3.5. *If $\{I_i/i \in \Delta\}$ is a family of ideals of R, then*
$(\bigcap\limits_{i\in\Delta} I_i)^{**} = \bigcap\limits_{i\in\Delta} (I_i)^{**}$.

We now define the concept of annihilator ideal in an ADL R with 0. 3.6

Definition 3.6. *Let R be an ADL with 0. An ideal I of R is called an annihilator ideal if $I = S^* = \{y \in R/y \wedge s = 0$, for all $s \in S\}$ for some non-empty subset S of R, or equivalently, $I = I^{**}$. We denote the set of all annihilator ideals of R by $\mathcal{A}(R)$.*

3.7

Example 3.7. *Let X be a discrete ADL with 0 and with atleast two elements, other than 0. Then $(X^n, \vee, \wedge, 0')$ is an ADL with zero $0' = (0, 0, ..., 0)$, where $\vee, \wedge$ are defined coordinate-wise.*
Now, let $I = \{(0, a_1, a_2, ...a_{n-1})/a_i \in X\}$. Then it can be observed that I is an ideal of R. Now it is clear that $I^* = \{(x, 0, 0, ..., 0)/x \in X\}$ and $I^{**} = \{(0, a_1, a_2, ...a_{n-1})/a_i \in X\} = I$. Hence I is an annihilator ideal of R.

3.8

Example 3.8. *Let $(R, +, \cdot, 0)$ be a commutative regular ring with unity. For any $a \in R$, let a^0 be the unique idempotent element in R such that $aR = a^0R$. For any $x, y \in R$, define $\quad x \wedge y = x^0y \quad$ and $\quad x \vee y = x + (1 - x^0)y$*
Then clearly $(R, \vee, \wedge, 0)$ is an ADL with 0.
Now consider $I = (x^0]$ and $J = (1 - x^0]$. Since $x^0 \wedge (1 - x^0) = 0$, we get that $(x^0] \subseteq (1 - x^0]^*$ and $(1 - x^0] \subseteq (x^0]^*$. Now $a \in (x^0]^*$ implies $a \wedge x^0 = 0$. So $a^0x^0 = 0$. Now $a(1 - x^0) = a - ax^0 = a - 0 = a$. Hence $a \in (1 - x^0]$. Thus $(x^0]^* \subseteq (1 - x^0]$. Therefore $I^* = (x^0]^* = (1 - x^0] = J$. Similarly, we can obtain $J^* = (1 - x^0]^* = (x^0] = I$. Hence I, J are the annihilator ideals in R.

3.9

Example 3.9. *Let $R = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on R as follows:*

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	b
b	b	b	b	b
c	c	b	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	a	b	c
c	0	0	c	c

Then clearly $(R, \vee, \wedge, 0)$ is an ADL with 0. Consider the set $I = \{0, a\} \subset R$. Then clearly I is an ideal in R. Now $I^* = \{0, c\}$ and also $I^{**} = \{0, a\} = I$. Thus I is an annihilator ideal in R. Similarly, the ideal $J = \{0, c\}$ of R, is another annihilator ideal in R.

3.10

Example 3.10. *Let $R = \{0, a, b, c\}$ and define $\vee$ and $\wedge$ on R as follows:*

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	a	a
b	b	b	b	b
c	c	a	b	c

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly $(R, \vee, \wedge, 0)$ is an ADL with 0. Consider the ideal $I = \{0, c\}$. Then $I^{**} = (0]^* = R$. Therefore I is not an annihilator ideal in R.

3.11

Lemma 3.11. *For $I, J \in \mathcal{A}(R)$, we have $I \cap J = (I^* \vee J^*)^*$.*

Proof: Since $I^* \subseteq I^* \vee J^*$,we get $(I^* \vee J^*)^* \subseteq I^{**} = I$. Similarly $(I^* \vee J^*)^* \subseteq J$. Hence $(I^* \vee J^*)^* \subseteq I \cap J$. Let $x \in I \cap J$ and $y \in I^* \vee J^*$. Then $y = t \vee s$ for some $t \in I^*$ and $s \in J^*$. Now $x \wedge y = x \wedge (t \vee s) = (x \wedge t) \vee (x \wedge s) = 0 \vee 0 = 0$. Hence $x \in (I^* \vee J^*)^*$. Therefore $I \cap J = (I^* \vee J^*)^*$. $\square$

3.12

Definition 3.12. *: Let R be an ADL. Then for any ideal I in R, define*

$$\bar{I} = \{\ (a] /\ a \in I\}$$

Then clearly $\bar{I}$ is an ideal in the lattice $\mathcal{PI}(R)$ of all principal ideals in R.

3.13

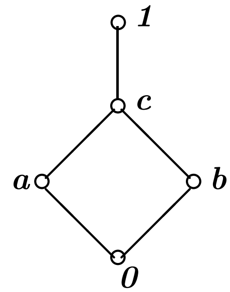
Theorem 3.13. *Let R be an ADL with 0. Then I is an annihilator ideal in R if and only if $\bar{I}$ is an annihilator ideal in $\mathcal{PI}(R)$.*

Proof: Assume that I is an annihilator ideal in R. Since $\bar{I}$ is an ideal, by the lemma 3.3(3) we have $\bar{I} \subseteq (\bar{I})^{**}$. Let $(a] \in (\bar{I})^{**}$ and $b \in I^*$. Then for any $c \in I$, $(b] \cap (c] = (b \wedge c] = (0]$. Hence $(b] \in (\bar{I})^*$. Since $(a] \in (\bar{I})^{**}$, we get $(a] \cap (b] = (0]$. Therefore $(a \wedge b] = (0]$, which implies that $a \wedge b = 0$. Hence $a \in I^{**} = I$. Thus $(a] \in \bar{I}$. Therefore $\bar{I}^{**} \subseteq \bar{I}$.
Conversly, let $\bar{I}$ be an annihilator ideal in $\mathcal{PI}(R)$. We have always $I \subseteq I^{**}$. Let $a \in I^{**}$ and $(b] \in (\bar{I})^*$. Now for any $c \in I, (c] \in \bar{I}$. Hence $(b] \cap (c] = (0]$. Thus $(b \wedge c] = (0]$, which implies that $b \wedge c = 0$. Therefore $b \in I^*$.

Now $a \in I^{**}$ and $b \in I^*$ imply that $a \wedge b = 0$. That is $(a] \cap (b] = (0]$.
Since $(b] \in (\bar{I})^*$ is arbitrary, $(a] \in (\bar{I})^{**} = \bar{I}$. Hence $(a] = (t]$ for some $t \in I$.
Thus $a \in I$. Hence $I^{**} \subseteq I$. Therefore I is an annihilator ideal in R. $\square$

If R is an ADL, then we know that $(\mathcal{I}(R), \vee, \wedge)$ is a distributive lattice. The set $\mathcal{A}(R)$ is not a sublattice of $\mathcal{I}(R)$ of all ideals of R. For, consider the following example. 3.14

Example 3.14. *Consider the distributive lattice $R = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure :*



Consider the ideals $I = \{0, a\}$ and $J = \{0, b\}$.
Now $I^* = \{0, b\} = J$ and $J^* = \{0, a\} = I$.
Hence $I^{**} = \{0, a\} = I$ and $J^{**} = \{0, b\} = J$.
Thus I and J are both annihilator ideals in R.
Now $I \vee J = \{0, a, b, c\}$. So $(I \vee J)^* = \{0\}$
Hence $(I \vee J)^{**} = R$
Therefore $I \vee J$ is not an annihilator ideal in R.

However, we have the following. 3.15

Theorem 3.15. *Let R be an ADL with 0. Then the set $\mathcal{A}(R)$ of all annihilator ideals of R forms a complete Boolean Algebra.*

Proof: For I, J in $\mathcal{A}(R)$, define $I \wedge J = I \cap J$ and $I \veebar J = (I^* \cap J^*)^*$.
Let $I, J \in \mathcal{A}(R)$. Then $I^{**} = I$ and $J^{**} = J$. Hence $(I \cap J)^{**} = I^{**} \cap J^{**} = I \cap J$.
Thus $I \cap J \in \mathcal{A}(R)$. We have also $I \veebar J \in \mathcal{A}(R)$.
It can be easily observed that $\langle \mathcal{A}(R), \wedge, \veebar \rangle$ is a lattice.
Since $(0]^* = R$ and $R^* = (0]$, we get that $(0], R \in \mathcal{A}(R)$ are the least and the greatest elements of $\mathcal{A}(R)$. Therefore, $\langle \mathcal{A}(R), \wedge, \veebar \rangle$ is a bounded lattice.
Let $I \in \mathcal{A}(R)$. Then clearly $I^* \in \mathcal{A}(R)$ and $I \wedge I^* = I \cap I^* = (0]$, $I \veebar I^* = (I^* \cap I^{**})^* = (I^* \cap I)^* = (0]^* = R$. Thus I^* is the complement of I for any $I \in \mathcal{A}(R)$. Therefore $\langle\ \mathcal{A}(R), \wedge, \veebar,\ ^*, (0], R\ \rangle$ is a complemented lattice.
Let $I, J, K \in \mathcal{A}(R)$. We prove that $I \veebar (J \wedge K) = (I \veebar J) \wedge (I \veebar K)$.
We first prove that $(I \veebar J) \wedge K \subseteq I \veebar (J \wedge K)$.
We have $I \cap K \cap [I^* \cap (J \cap K)^*] = (0]$, so that $K \cap I^* \cap (J \cap K)^* \subseteq I^*$.
Similarly $J \cap K \cap [I^* \cap (J \cap K)^*] = (0]$ implies that $K \cap I^* \cap (J \cap K)^* \subseteq J^*$.
Hence $K \cap I^* \cap (J \cap K)^* \subseteq I^* \cap J^*$. Thus by lemma 3.3(7), we get that $[K \cap I^* \cap (J \cap K)^*] \cap (I^* \cap J^*)^* = (0]$. That is $I^* \cap (J \cap K)^* \cap [K \cap (I^* \cap J^*)^*] = (0]$. Thus $K \cap (I^* \cap J^*)^* \subseteq [I^* \cap (J \cap K)^*]^*$. Hence $(I \veebar J) \wedge K \subseteq I \veebar (J \wedge K)$.
We now prove the distributivity.
Now, $(I \veebar J) \cap (I \veebar K) \subseteq I \veebar [J \cap (I \veebar K)] = I \veebar [(I \veebar K) \cap J] \subseteq I \veebar [I \veebar (K \cap J)]$.
Thus $\langle \mathcal{A}(R), \wedge, \veebar,\ ^*, (0], R \rangle$ is a boolean algebra.
We now prove the Completeness. Let $A_i \in \mathcal{A}(R)$ for $i \in \Delta$.

Then, by Corollary 3.5, $\left\{\bigcap_{i\in\Delta} A_i\right\}^{**} = \bigcap_{i\in\Delta} A_i^{**} = \bigcap_{i\in\Delta} A_i$, since each $A_i \in \mathcal{A}(R)$.

Thus $\langle \mathcal{A}(R), \wedge, \underline{\vee}, {}^*, (0], R\rangle$ is a complete boolean algebra. $\square$

Now we derive a sufficient condition for an ADL R to become relatively complemented. But first we need the following two important lemmas. 3.16

Lemma 3.16. *Let R be an ADL and F a filter in R. If $a \leq c$ and $a \in F \vee [c)$.*
Then $a = f \wedge c$ for some suitable $f \in F$.

Proof: Since $a \in F \vee [c)$, we can write $a = f \wedge x$ for some $f \in F$ and $x \in [c)$. Since $x \in [c)$, we get $x = x \vee c$, which implies $x \wedge c = c$.
Now $a \leq c \Rightarrow a = a\wedge c = f\wedge x\wedge c = f\wedge c$. $\square$
3.17

Lemma 3.17. *Let R be an ADL. If R is not relatively complemented then there exists two distinct prime ideals in R, one of which contains the other.*

Proof: Suppose R is not relatively complemented. Then there exists three elements $a, b, c \in R$ such that $b < c < a$ and c has no complement in the interval $[b, a]$. Write $F = \{x \in R/(c \vee x) \wedge a = a\}$. We first prove that F is a filter. Clearly $a \in F$. Let $x, y \in F$. Then $[c \vee (x \wedge y)] \wedge a = [(c \vee x) \wedge (c \vee y)] \wedge a = (c \vee x) \wedge [(c \vee y) \wedge a] = (c \vee x) \wedge a = a$. Hence $x \wedge y \in F$.
Let $x \in F$ and $r \in R$. Then $[c \vee r \vee x] \wedge a = [r \vee c \vee x] \wedge a = (r \wedge a) \vee [(c \vee x) \wedge a] = (r \wedge a) \vee a = a$. Hence $r \vee x \in F$. Therefore F is a filter.
Now consider the Filter $E = F \vee [c)$. Suppose $b \in E$. Since $b < c$ and $b \in F \vee [c)$, by the above lemma , we have $b = f \wedge c$ for some $f \in F$.
Now $b = b \wedge a = (f \wedge c) \wedge a = (c \wedge f) \wedge a = c \wedge (f \wedge a) \longrightarrow (1)$
Again $c \vee (f \wedge a) = (c \wedge a) \vee (f \wedge a) = (c \vee f) \wedge a = a$, because $f \in F \longrightarrow (2)$
From(1)and(2) we can obtain that $f \wedge a$ is a relative complement of c in $[b, a]$. Which is a controdiction. Hence $b \notin E$. Thus $(b] \cap E = \emptyset$. Therefore by theorem 2.3, there exists a prime ideal P of R such that $(b] \subseteq P$ and $P\cap E = \emptyset$. Now $P \cap E = \emptyset \Rightarrow P \cap \{F \vee [c)\} = \emptyset \Rightarrow (P \cap F) \vee (P \cap [c)) = \emptyset \Rightarrow P \cap F = \emptyset$ and $P \cap [c) = \emptyset$. Now consider the ideal $I = (c] \vee P$. Suppose $a \in I$. Since $a > c$ and $a \in I = (c] \vee P$, we can write $a = c \vee p$ for some $p \in P$. Thus $(c\vee p)\wedge a = a$, which implies $p \in F$. Hence $p \in P\cap F$. Which is a controdiction to that $P\cap F = \emptyset$. Hence $I\cap[a) = \emptyset$. Thus again by theorem 2.3, there exists a prime ideal Q such that $I \subseteq Q$ and $[a)\cap Q = \emptyset$. Hence $P \subset I \subseteq Q$. $\square$
3.18

Theorem 3.18. *Let R be an ADL with 0, in which every ideal is an annihilator ideal. Then R is relatively complemented.*

Proof: Assume that $I^{**} = I$ for all $I \in \mathcal{I}(R)$. Suppose R is not relatively complemented. Then by the above lemma, there exists two distinct prime ideals say P, Q in R such that $P \subset Q$. Choose $q \in Q - P$. Let $x \in Q^*$. Then $x \wedge q = 0 \in P$. So $x \in P$, because of P is a prime ideal and $q \notin P$. Hence $Q^* \subseteq P$. Thus $Q^* \subseteq P \subset Q$. Therefore $Q^* = Q \cap Q^* = (0]$. Hence $Q^{**} = (0]^* = R$. Thus $Q \subsetneq Q^{**}$, because Q is a proper ideal. Which is a controdiction. Hence R must be relatively complemented. $\square$

4 Annihilator Preserving homomorphisms

In this section we study some basic properties of annihilator preserving homomorphisms and derive a sufficient condition for a homomorphism to be annihilator preserving. Finally, we prove that the images and inverse images of annihilator ideals are againe annihilator ideals. 4.1

Definition 4.1. *(Rao. G.C.[7]) Let R and R' be two ADLs with zero elements 0 and $0'$ respectively. Then a mapping $f : R \to R'$ is called a homomorphism if it satisfies the following :*

(1). $f(a \vee b) = f(a) \vee f(b)$

(2). $f(a \wedge b) = f(a) \wedge f(b)$.

It is clear that $f(0) = 0'$. The kernel of the homomorphism f is defined by $Ker f = \{x \in R /\ f(x) = 0'\}$.

The following lemma is a direct verification. 4.2

Lemma 4.2. *Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and $f : R \to R'$ a homomorphism. Then we have the following :*

(1). *If f is onto, then for any ideal J of R, $f(J)$ is an ideal of R'.*

(2). *For any ideal J of R', $f^{-1}(J)$ is an ideal of R containing $Ker f$.*

4.3

Lemma 4.3. *Let $f : R \to R'$ be a homomorphism. Then for any non-empty subset A of R, we have $f(A^*) \subseteq (f(A))^*$.*

Proof: Let $a \in f(A^*)$ and $y \in f(A)$. Then there exists $b \in A^*$ and $x \in A$ such that $a = f(b)$ and $y = f(x)$. Now $a \wedge y = f(b) \wedge f(x) = f(b \wedge x) = f(0) = 0'$. That is $a \wedge y = 0'$ for all $y \in f(A)$. Hence $a \in (f(A))^*$. Thus $f(A^*) \subseteq (f(A))^*$. $\square$

If R is an ADL with 0, then for any $A \subseteq R$, $(f(A))^* \subseteq f(A^*)$ is not true ingeneral. For, consider the following example. 4.4

Example 4.4. *Let $R = \{0, a, b, c\}$ be a disctrete ADL. Define a mappping $f : R \longrightarrow R$ by $f(x) = 0$ for all $x \in R$. Then clearly f is a homomorphism on R. Take $A = \{a, b\}$. Then clearly $A^* = \{0\}$ and $f(A) = \{0\}$. Hence $f(A^*) = \{0\}$ and $\{f(A)\}^* = R$. Thus $(f(A))^* \nsubseteq f(A^*)$.*

We now define the concept of annihilator preserving homomorphism. 4.5

Definition 4.5. *Let $f : R \to R'$ be a homomorphism. Then f is called annihilator preserving if $f(A^*) = \{f(A)\}^*$, for any $(0] \subset A \subset R$.*

4.6

Example 4.6. *Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ADLs. Write $R = A \times B = \{(0,0), (0,b_1), (0,b_2), (a,0), (a,b_1), (a,b_2)\}$. Then $(R, \vee, \wedge, \overline{0})$ is an ADL under point-wise operations. Also the zero element in R is $(0,0)$. Let $R' = \{0', a', b', c'\}$ be another ADL in which the operations $\vee', \wedge'$ are defined as followes:*

$\vee'$	$0'$	a'	b'	c'
$0'$	$0'$	a'	b'	c'
a'	a'	a'	c'	c'
b'	b'	c'	b'	c'
c'	c'	c'	c'	c'

$\wedge'$	$0'$	a'	b'	c'
$0'$	$0'$	$0'$	$0'$	$0'$
a'	$0'$	a'	$0'$	a'
b'	$0'$	$0'$	b'	b'
c'	$0'$	a'	b'	c'

Now define the mapping $f : R \longrightarrow R'$ as follows:

$f((0,0)) = 0'$; $f((a,0) = a'$

$f((0,b_1)) = f((0,b_2)) = b'$; $f((a,b_1)) = f((a,b_2)) = c'$.

Then clearly f is a homomorphism from R onto R'. It can also be verified that f is annihilator preserving.

4.7

Example 4.7. *Let R and R' be two dense ADLs(i.e. an ADL R in which $(a]^* = (0]$, for all $0 \neq a \in R$) Then every homomorphism from R into R' is annihilator preserving.*

Unlike in rings, if f is a homomorphism of an ADL R with 0 into another ADL R' with $0'$ such that $Ker f = \{0\}$ and f is onto, then f need not be an isomorphism. It may be seen in the following example. 4.8

Example 4.8. *Let $R = \{0, a, b\}$ and $R' = \{0', c\}$ be two disctrete ADLs. Define a mappping $f : R \longrightarrow R'$ by $f(0) = 0'$ and $f(a) = f(b) = c$. Then clearly f is a homomorphism from R into R'. Clearly f is onto. Also $Ker f = \{0\}$. But f is not one-one.*

However, we have the following. 4.9

Theorem 4.9. *Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and $f : R \longrightarrow R'$ a homomorphism. If $Ker f = \{0\}$ and f is onto, then f is annihilator preserving.*

Proof: Let A be a subset of R such that $(0] \subset A \subset R$. We have always $f(A^*) \subseteq \{f(A)\}^*$. Now, let $x \in \{f(A)\}^* \subseteq R'$. Since f is onto, there exists $y \in R$ such that $f(y) = x \in \{f(A)\}^*$.

$\Rightarrow f(y) \wedge f(a) = 0'$ for all $a \in A$.
$\Rightarrow f(y \wedge a) = 0'$
$\Rightarrow y \wedge a \in Ker f = \{0\}$
$\Rightarrow y \wedge a = 0$ for all $a \in A$.
$\Rightarrow y \in A^*$
$\Rightarrow x = f(y) \in f(A^*)$

Hence $\{f(A)\}^* \subseteq f(A^*)$. Therefore $\{f(A)\}^* = f(A^*)$. □
4.10

Theorem 4.10. *Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and $f : R \longrightarrow R'$ a homomorphism. If $Ker f = \{0\}$, then f^{-1} preserves annihilators.*

Proof Let $(0] \subset B \subset R'$. It is enough to prove that $f^{-1}(B^*) = \{f^{-1}(B)\}^*$. Let $x \in \{f^{-1}(B)\}^*$. Then $x \wedge b = 0 \quad \forall\, b \in f^{-1}(A)$

$\Rightarrow x \wedge b = 0 \quad \forall\, f(b) \in B$
$\Rightarrow f(x) \wedge f(b) = 0' \quad \forall\, f(b) \in B$
$\Rightarrow f(x) \in B^*$
$\Rightarrow x \in f^{-1}(B^*)$

Hence $\{f^{-1}(B)\}^* \subseteq f^{-1}(B^*)$.

Conversly, let $x \in f^{-1}(B^*)$ and $b \in f^{-1}(B)$. Then $f(x) \in B^*$ and $f(b) \in B$. Hence $f(x \wedge b) = f(x) \wedge f(b) \in B^* \cap B = (0']$. Thus $x \wedge b \in Ker f = (0]$. Therefore $x \wedge b = 0$, for all $b \in f^{-1}(B)$. Hence $x \in \{f^{-1}(B)\}^*$. □
4.11

Theorem 4.11. *Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and $f : R \to R'$ an annihilator preserving epimorphism. If $Ker f = \{0\}$, then we have:*

$$A^* = B^* \Leftrightarrow \{f(A)\}^* = \{f(B)\}^*$$

for any two non-empty subsets A, B of R.

Proof : let A, B be two non-empty subsets of R. Assume that $A^* = B^*$. Then clearly $f(A^*) = f(B^*)$. Since f is annihilator preserving, $\{f(A)\}^* = \{f(B)\}^*$.

Conversly assume that $\{f(A)\}^* = \{f(B)\}^*$.
Let $t \in A^*$. Then $t \wedge a = 0 \;\forall\, a \in A$

$\Rightarrow f(t \wedge a) = f(0)$

$\Rightarrow f(t) \wedge f(a) = 0' \ \ \forall\, a \in A$
$\Rightarrow f(t) \in \{f(A)\}^*$
$\Rightarrow f(t) \in \{f(B)\}^*$ by hypothsis
$\Rightarrow f(t) \wedge f(b) = 0'$ $\forall\, b \in B$
$\Rightarrow f(t \wedge b) = 0'$
$\Rightarrow t \wedge b \in Kerf = \{0\}$
$\Rightarrow t \wedge b = 0 \ \ \forall\, b \in B$
$\Rightarrow t \in B^*$

Hence $A^* \subseteq B^*$. Similarly we can obtain $B^* \subseteq A^*$. Therefore $A^* = B^*$. □

4.12

Theorem 4.12. *Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and $f : R \to R'$ a homomorphism. Then we have the following:*

(1). *If f is annihilator preserving and onto, then $f(I)$ is an annihilator ideal of R' for every annihilator ideal I of R.*

(2). *If f^{-1} preserves annihilators , then $f^{-1}(I)$ is an annihilator ideal of R for every annihilator ideal I of R'.*

Proof: (1). Let I be an annihilator ideal of R. Then by lemma 4.2(1), $f(I)$ is an ideal of R'. Since f is annihilator preserving, $\{f(I)\}^{**} = f(I^{**}) = f(I)$. Therefore $f(I)$ is an annihilator ideal in R'.

(2). Let J be an annihilator ideal of R'. Then by lemma 4.2(2), $f^{-1}(J)$ is an ideal of R. Since f^{-1} preserves annihilators, we get $\{f^{-1}(J)\}^{**} = f^{-1}(J^{**}) = f^{-1}(J)$. □

4.13

Corollary 4.13. *Let R and R' be two ADLs with zero elements 0 and $0'$ respectively and $f : R \to R'$ an annihilator preserving homomorphism. Then $Kerf$ is an annihilator ideal of R.*

Proof: Since $Kerf = f^{-1}(\{0'\})$ and $\{0'\}$ an annihilator ideal in R', the result follows. □

We now prove that for every ideal I of R, there exists a homomorphism whose kernel is the annihilator of I. Let us denote the set of all homomorphisms defined on I by $Hom_R(I)$. Then it can be easily observed that the set $Hom_R(I)$ is an ADL with respect to the point-wise operations given by :

$(f \wedge g)(x) = f(x) \wedge g(x)$
$(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in I$ and $f, g \in Hom_R(I)$.

4.14

Theorem 4.14. *Let R be an ADL with 0. Then for any ideal I of R, there exists a homomorphism f from R to $Hom_R(I)$ such that $I^* = Ker\, f$.*

Proof: Let I be an ideal in R. Then for any $r \in R$, define

$\theta_r : I \longrightarrow I$ by $\theta_r(x) = x \wedge r \quad \forall\, x \in I$

$\underline{\theta_r \text{ is well-defined}}$: Let $r \in R$ be an arbitrary element. Let $x \in I$. Since I is an ideal of R, we get $\theta_r(x) = x \wedge r \in I$. Thus θ_r is well-defined for each $r \in R$.
$\underline{\theta_r \text{ is a homomorphism}}$: Fix $r \in R$. Let $x, y \in I$. Then

$$\begin{aligned} \theta_r(x \wedge y) &= (x \wedge y) \wedge r \\ &= (x \wedge y) \wedge r \wedge r \\ &= (x \wedge r) \wedge (y \wedge r) \\ &= \theta_r(x) \wedge \theta_r(y) \\ \text{Again} \qquad \theta_r(x \vee y) &= (x \vee y) \wedge r \\ &= (x \wedge r) \vee (y \wedge r) \\ &= \theta_r(x) \vee \theta_r(y) \end{aligned}$$

Thus θ_r is a homomorphism on I. Hence $\theta_r \in Hom_R(I)$.
Now define $f : R \longrightarrow Hom_R(I)$ by $f(r) = \theta_r \quad \forall\, r \in R$.
Clearly f is well-defined.
Now, for any $r, s \in R$, $f(r \wedge s) = \theta_{r\wedge s}$. For any $x \in I$, $\theta_{r\wedge s}(x) = x \wedge (r \wedge s) = (x \wedge r) \wedge (x \wedge s) = \theta_r(x) \wedge \theta_s(x)$. Thus $f(r \wedge s) = \theta_{r\wedge s} = \theta_r \wedge \theta_s = f(r) \wedge f(s)$.
Again, $f(r \vee s) = \theta_{r\vee s}$. Now for any $x \in I$, $\theta_{r\vee s}(x) = x \wedge (r \vee s) = (x \wedge r) \vee (x \wedge s) = \theta_r(x) \vee \theta_s(x)$. Thus $f(r \vee s) = \theta_{r\vee s} = \theta_r \vee \theta_s = f(r) \vee f(s)$.
Therefore f is a homomorphism. Hence $Kerf$ is an ideal in R.
We now prove that $I^* = Kerf$. We have

$$\begin{aligned} r \in Kerf &\Leftrightarrow f(r) = \theta_0, \text{ which is the zero element of } Hom_R(I). \\ &\Leftrightarrow \theta_r = \theta_0 \\ &\Leftrightarrow \theta_r(x) = \theta_0(x) \text{ for all } x \in I \\ &\Leftrightarrow x \wedge r = \theta_0(x) = 0 \text{ for all } x \in I \\ &\Leftrightarrow r \in I^* \end{aligned}$$

References

[1] Birkhoff. G.: *Lattice Theory*, Amer.Math.Soc.Colloq. XXV, Providence, (1967), U.S.A.

[2] Burris. S., Sankappanavar. H.P.: *A Course in Universal Algebra*, Springer Verlag, (1981).

[3] Cornish. W.H.: *Normal Lattices*, J.Austral.Math.Soc., **14** (1972), 200-215.

[4] Cornish. W.H.: *Annulets and α- ideals in Distributive Lattices*, J.Austral.Math.Soc., **15** (1973), 70-77.

[5] Kist. J.E.: *Minimal Prime ideals in Commutative Semigroups*, Proc. London Math. Soc. (3) **13** (1963), 31-50.

[6] Mandelker. M : *Relative annihilators in lattices*, Duke Math. J, **37** (1970), 377-386.

[7] Rao. G.C.: *Almost Distributive Lattices*, Doctoral Thesis, Andhra University, Waltair, (1980).

[8] Swamy. U.M., Rao.G.C.: *Almost Distributive Lattices*, J.Austral.Math.Soc. (Series A), **31** (1981), 77-91.

[9] Swamy. U.M., Rao. G.C., Nanaji Rao. G.: *Pseudo-Complementation on Almost Distributive Lattices*, Southeast Asian Bulletin of mathematics, **24** (2000), 95-104.

[10] Swamy. U.M., Rao. G.C., Nanaji Rao. G.: *Stone Almost Distributive Lattices*, Southeast Asian Bulletin of mathematics, **27** (2003), 513-526.

Received: August 11, 2008