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Certain Classes of Almost Contact Riemannian Manifolds

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Abstract. Certain classes of almost contact Riemannian Manifolds, viz., almost Kenmotsu, nearly Kenmotsu, Quasi-Kenmotsu and special contact metric Manifolds are defined and obtained some properties of these manifolds. Also, it has been shown that the structure vector field ξ of the almost contact metric structure (Φ, ξ, η, G) is not a Killing vector field on a nearly Kenmotsu vector field.

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Introduction:

The study of odd dimensional manifolds with contact and almost contact structures was initiated by Boothby and Wang in 1958 rather from topological point of view. Sasaki and Hatakeyama reinvestigated them using tensor calculus in 1961. Almost contact metric structures and Sasakian structures viz., almost Sasakian, nearly Sasakian etc., were proposed by Sasaki[5] in 1960 and 1965 respectively. Later, Kenmotsu [3] defined a class of almost contact Riemannian manifold, called Kenmotsu manifold, similar in parallel to Sasakian manifold in 1972. In this paper, we defined almost Kenmotsu, nearly Kenmotsu, Quasi-Kenmotsu and special contact metric Manifolds. The relation among these manifolds have been obtained and studied some properties of these manifolds.

1. Preliminaries:

Let M = M^{2m+1} be a (2m + 1) dimensional almost contact metric manifold with structure tensors (Φ, ξ , η, G) where Φ is a tensor of type (1,1), ξ is a vector field, η is a 1-form and G is the associated Riemannian metric on M. Then by definition, we have

$$\Phi^2 X = \overline{\overline{X}} = -X + \eta(X)\xi, \quad \Phi\xi = 0$$

$$G(\Phi X, \Phi Y) = G(X, Y) - \eta(X)\,\eta(Y) \tag{1.1}$$

The fundamental 2 – form Ω is defined by:

$$\Omega(X,Y) = G(\overline{X},\ Y) \text{ where we put } \overline{X} = \Phi X. \tag{1.2}$$

If M is an almost contact metric manifold, we have

$$(D_X\,\Omega)(Y,\xi) = -(D_X\,\eta)(\overline{Y}) \tag{1.3}$$

$$(D_X\,\Omega)(\overline{Y},Z) - (D_X\,\Omega)(Y,\overline{Z}) = (D_X\,\eta)(Y)\eta(Z) + (D_X\,\eta)(Z)\eta(Y) \tag{1.4}$$

where D is the Riemannian connection determined by the metric G.
On the almost contact metric manifold if further we have

$$(D_X\,\Phi)(Y) = -\eta(Y)(\overline{X}) - G(X,\overline{Y})\xi \tag{1.5}$$

it is called Kenmotsu Manifold[3].
From (1.1) and (1.5), we get

$$D_X\,\xi = -\overline{\overline{X}} = X - \eta(X)\xi \tag{1.6}$$

Then from equations (1.1) and (1.6), we get

$$(D_X\,\eta)(Y) = G(X, Y) - \eta(X)\,\eta(Y) = G(\overline{X},\overline{Y}) \tag{1.7}$$

Similarly from (1.5) we also have

$$\begin{aligned}(D_X\,\Omega)(Y,Z) + (D_Y\,\Omega)(Z,X) + (D_Z\,\Omega)(X,Y) \\ = -2[\eta(X)G(Y,\overline{Z}) + \eta(Y)G(Z,\overline{X}) + \eta(Z)G(X,\overline{Y})]\end{aligned} \tag{1.8}$$

An almost contact structure is said to be Normal if $N_0(X,Y)$ vanishes, where

$$N_0(X,Y) \overset{def}{=} N_\Phi(X,Y) + d\eta(X,Y)\xi \tag{1.9}$$

Here, N_ϕ (X,Y) is known as the Nijenhuis tensor of ϕ.

2. Almost Kenmotsu and S-contact metric manifolds:

Definition (2.1): An almost contact metric manifold M on which there exists a function f such that η = df if dη = 0. Then the manifold M is called an Almost Kenmotsu manifold (or) contact metric manifold.
From the above definition, if D is an affine connection on contact metric manifold, we have

$$(D_X\,\eta)(Y) - (D_Y\,\eta)(X) + \eta[T(X,Y)] = 0 \tag{2.1)a}$$

where T is the torsion tensor of D.
If D is symmetric, then from (2.1a), we have

$$(D_X\,\eta)(Y) - (D_Y\,\eta)(X) = 0 \tag{2.1)b}$$

Note that, in the sequel, we shall always take D as a Riemannian connection in this paper.

Definition (2.2): An almost Kenmotsu manifold (contact metric manifold) on which if the condition

$$(D_X \eta)(Y) + (D_Y \eta)(X) = 2G(\overline{X}, \overline{Y}) \tag{2.2}$$

is satisfied, then it is called a special contact metric manifold or in short S-contact metric manifold.

Therefore, from (2.1b) and (2.2), for an S-contact metric manifold, we get

$$(D_X \eta)(Y) = (D_Y \eta)(X) = G(\overline{X}, \overline{Y}) = -\Omega(\overline{X}, Y) \tag{2.3}$$

Theorem(2.1): In an almost contact metric manifold, if $D_X \xi = X - \eta(X)\xi$ is satisfied, it is an S-contact metric manifold.

Proof: On the almost contact metric manifold, we have

$$(D_X \eta)(Y) = G(D_X\xi, Y)$$

$$= G(X - \eta(X)\xi, Y) = G(\overline{X}, \overline{Y}).$$

Similarly, we see that

$$(D_Y \eta)(X) = G(\overline{Y}, \overline{X})$$

On adding and subtracting the above two values, we get both (2.2) and (2.3) respectively, which proves the theorem.

Proposition (2.1): In an S-contact metric manifold, we get

$$D_X \xi = X - \eta(X)\xi . \tag{2.4}$$

Proof: The equation (2.3) is equivalent to (2.4).

Corollary(2.1): The following holds on an S-contact metric manifold;

$$(D_X \Omega)(Y, \xi) = G(\overline{X}, Y). \tag{2.5}$$

Proof: Taking the covariant differentiation of $\Omega(Y, \xi) = 0$, we get

$$(D_X \Omega)(Y, \xi) = -\Omega(Y, D_X \xi) = \Omega(Y, \overline{\overline{X}}) = G(\overline{X}, Y).$$

Theorem(2.2): In an S-contact metric manifold, we have

(i) $(D_{\overline{Z}} \Omega)(X, Y) = -{}'K(X, Y, Z, \xi)$

(ii) $(D_{\overline{Z}} \Phi)(X) = -K(Z, \xi, X)$ (2.6)

Proof: The equation $-\Omega(\overline{Y}, Z) = (D_Y \eta)(Z)$ implies

$$-(D_X \Omega)(\overline{Y}, Z) + (D_X \Omega)(Y, \overline{Z}) + (D_Y \Omega)(\overline{X}, Z) - (D_Y \Omega)(X, \overline{Z})$$

$$= G(D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi, Z).$$

Using (1.5), the above equation implies

$$\eta(Y)G(X,Z) - \eta(X)G(Y,Z) = -{}'K(X, Y, Z, \xi)$$

which can also be written as (2.6)(i) and hence, also we have (2.6)(ii).

Theorem(2.3): On an S-contact metric manifold, the condition

$$(D_{\overline{Z}} \Omega)(X, Y) = \eta(X)G(Y, Z) - \eta(Y)G(X, Z) \tag{2.7}$$

is equivalent to the condition

$$(D_{\overline{Z}} \Omega)(\overline{X}, \overline{Y}) = -{}'K(\overline{X}, \overline{Y}, Z, \xi) = 0. \tag{2.8}$$

Proof: From (2.6) and (2.7) we get

$$(D_{\bar{Z}}\,\Omega)(X,Y) = -'K(X,Y,Z,\xi) = 0$$
$$= \eta(X)G(Y,Z) - \eta(Y)G(X,Z)$$

Barring X and Y in the above equation, we get (2.8). Again, barring X and Y in (2.8), we get

$$(D_{\bar{Z}}\,\Omega)(X,Y) = \eta(Y)(D_{\bar{Z}}\,\Omega)(X,\xi) - \eta(X)(D_{\bar{Z}}\,\Omega)(Y,\xi)$$

Using (2.5), the above equation gives (2.7).

3. Quasi Kenmotsu manifold:

Definition(3.1): An almost contact metric manifold is said to be Quasi Kenmotsu manifold if

$$(D_{\bar{X}}\,\Omega)(Y,Z) + (D_{\bar{Y}}\,\Omega)(Z,X) + (D_{\bar{Z}}\,\Omega)(X,Y) = 0\,. \tag{3.1}$$

Theorem(3.1): The necessary and sufficient condition for a Quasi-Kenmotsu manifold to be Normal is

$$(D_Z\,\Omega)(X,\bar{Y}) = -\,\eta(X)(D_Z\,\eta)(Y). \tag{3.2}$$

Proof: We know that the necessary and sufficient condition for a Quasi-Kenmotsu manifold which is an almost contact manifold to be normal is $N_0 = 0$. That is,

$$N_\Phi(X,Y) + d\eta(X,Y)\xi = 0,$$

which is represented as

$$(D_{\bar{X}}\,\Omega)(Y,Z) - (D_{\bar{Y}}\,\Omega)(X,Z) + (D_X\,\Omega)(Y,\bar{Z}) - (D_Y\,\Omega)(X,\bar{Z}) + d\eta(X,Y)\eta(Z) = 0\,.$$

By using the equations (1.8), (1.4) and (2.3) the above equation gives (3.2).

4. Kenmotsu manifold:

Definition(4.1): An S-contact metric manifold on which the equation (2.8) holds is called a Kenmotsu manifold.

Theorem(4.1): A normal S-contact metric manifold is Kenmotsu.

Proof: By $N_0(X, Y) = 0$, we have

$$(D_{\bar{X}}\,\Omega)(Y,Z) - (D_{\bar{Y}}\,\Omega)(X,Z) + (D_X\,\Omega)(Y,\bar{Z}) - (D_Y\,\Omega)(X,\bar{Z}) + d\eta(X,Y)\eta(Z) = 0\,.$$

Using the equations (2.1), (1.8), (1.4) and (2.3) respectively, the above equation gives

$$(D_Z\,\Omega)(X,\bar{Y}) = -\,\eta(X)G(Y,Z) + \eta(X)\,\eta(Y)\,\eta(Z)\,,$$

which also implies

$$(D_Z\,\Omega)(X,Y) = -\,\eta(X)G(\bar{Z},Y) + \eta(Y)\,G(Z,\bar{X})\,.$$

This shows that the manifold is of Kenmotsu type. Hence, an alternate definition of Kenmotsu manifold is given as: A Kenmotsu manifold is a Normal S-contact metric manifold.

5. Nearly Kenmotsu manifold:

Definition(5.1): An almost contact metric manifold on which

$$(D_X\,\Phi)(Y)+(D_Y\,\Phi)(X) = -\,\eta(Y)(\overline{X}) - \eta(X)(\overline{Y}) \tag{5.1}$$

is satisfied, is called a nearly Kenmotsu manifold.

Theorem(5.1): On nearly Kenmotsu manifold ξ is not a Killing vector field.
Proof: Operate (5.1) with G and put Y = ξ , we get

$$(D_X\;\Omega)(\xi,Z) + (D_\xi\;\Omega)(X,Z) = \Omega(Z,\,X)\,.$$

Using (1.3) and barring Z in this equation, we see that

$$-(D_X\,\eta)(Z) + (D_\xi\,\Omega)(X,\,\overline{Z}) = -\,\Omega(\overline{Z},\,X) \tag{5.2}$$

Consequently, we have

$$-[(D_X\,\eta)(Z) + (D_Z\,\eta)(X)] - [(D_\xi\,\Omega)(\overline{X},Z) - (D_\xi\,\Omega)(X,\,\overline{Z})] = \;2\Omega(\overline{X},\,Z).$$

Use of (1.4) in this equation yields

$$-[(D_X\,\eta)(Z) + (D_Z\,\eta)(X)] - (D_\xi\,\eta)(X)\eta(Z) - (D_\xi\,\eta)(Z)\eta(X) = \;2\Omega(\overline{X},\,Z). \tag{5.3}$$

Writing ξ for X in (5.2), we get $(D_\xi\;\eta)(Z)\;=0$ which proves the theorem.

Theorem(5.2): A normal nearly Kenmotsu manifold is Kenmotsu.
Proof: Since we have $N_0 = 0$, it holds that

$$[\overline{X},\overline{Y}] + \overline{\overline{[X,\,Y]}} - \overline{[\overline{X},Y]} - \overline{[X,\overline{Y}\,]} + \;d\eta(X,Y)\xi\; = 0.$$

Operating the above by η implies

$$\eta[(D_{\overline{X}}\,\phi)Y - (D_{\overline{Y}}\,\phi)\,X\,] + \;d\eta(X,Y)\;=0. \tag{5.4}$$

Barring X in (5.1), and on operation with η, we get

$$\eta\;(D_{\overline{X}}\,\phi)Y \;\;=\;\; -\eta\,(D_Y\,\phi)\,\overline{X}\;\;. \tag{5.5}$$

Similarly, we see that

$$\eta\;(D_{\overline{Y}}\,\phi)\,X \;\;=\;\; -\eta\,(D_X\,\phi)\overline{Y}\;\;. \tag{5.6}$$

Using (5.5) and (5.6), equation (5.4) assumes the form

$$(D_X\;\Omega)(\overline{Y},\,\xi) - (D_Y\;\Omega)(\overline{X},\,\xi) + d\eta(X,\,Y) = 0.$$

Use of (1.3) in this equation yields $d\eta(X,\,Y) = 0$**,** which shows that the manifold is of almost Kenmotsu. By equation (2.2) it is of S-contact metric manifold and therefore the result follows from theorem(4.1).

Theorem(5.3): A manifold which is of nearly Kenmotsu and Quasi Kenmotsu is of Kenmotsu.
Proof: On a Nearly Kenmotsu manifold, we have

$$(D_X\,\Omega)(Y,Z) + (D_Y\,\Omega)(X,Z) = -\,\eta(X)\,G(\overline{Y},Z) - \eta(Y)\,G(\overline{X}\,,Z)$$

Addding the above equation to (1.8), we get

$$\begin{aligned} &2(D_X\,\Omega)(Y,Z) + (D_Z\,\Omega)(X,Y) \\ &\qquad = -3\eta(Y)G(\overline{X}\,,Z)\; - \eta(X)G(\overline{Z},Y) - 2\eta(Z)G(\overline{Y},\,X). \end{aligned} \tag{5.7}$$

From (1.5) we have

$$(D_X\,\Omega)(Z,Y) + (D_Z\,\Omega)(X,Y) = -\,\eta(X)\,G(\overline{Z},\,Y) - \eta(Z)\,G(\overline{X}\,,Y). \tag{5.8}$$

Hence from (5.7) and (5.8) we have

$$(D_X \Omega)(Y,Z) = -\eta(Y)G(\overline{X}, Z) - \eta(Z)G(\overline{Y}, X)$$

which proves

$$(D_X \Phi)(Y) = -\eta(Y)(\overline{X}) - G(X, \overline{Y})\xi.$$

It shows that the manifold is Kenmotsu.

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