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On Lower Radical Constructions in Nonassociative Semirings

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Abstract

The notion of Radical classes is introduced in [4]. We prove here some useful equivalent conditions for a subclass of a fixed universal class $\mathbb{U}$ to be a radical class. We introduce the notion of lower radical construction and prove $\mathfrak{L}\mathbb{V} = \mathfrak{R}(\mathbb{V})$, where $\mathbb{V}$ is any arbitrary class of additively cancellative and semisubtractive semirings.

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1 Introduction

The paper is concerned with generalizing some results in ring theory. In correspondence to the Kurosh-Amitsur radical theory for associative rings, an abstract concept of radical classes and radicals for semirings has been introduced and investigated in a series of publications [5]-[8] by D. M. Olson and several coauthors. In this paper we discuss about lower radical construction and prove $\mathfrak{L}\mathbb{V} = \mathfrak{R}(\mathbb{V})$, where $\mathbb{V}$ is any arbitrary class of additively cancellative and semisubtractive semirings. Some of the statements in this paper are more or less known from [4], but we organize and prove them in a somewhat different way more appropriate for our purposes. There are many different definitions of a semiring appearing in the literature. Throughout this paper, a semiring will be defined as follows: A semiring is a set S together with two binary operations called addition $(+)$ and multiplication $(.)$ such that $(S, +)$ is a commutative monoid with identity element 0_S; $(S, .)$ is a monoid with identity element 1; multiplication distributes over addition from either side and 0 is multiplicative absorbing, that is, $a.0 = 0.a = 0$ for each $a \in S$. A semiring is commutative if $(S, .)$ is a commutative semigroup. A subset I of a semiring S will be called an ideal of S if I is an additive subsemigroup of $(S, +)$, $IS \subseteq I$ and $SI \subseteq I$. An ideal I of a semiring S is called proper if and only if $I \neq S$ and a proper ideal I of S is called maximal if and only if there is no ideal J of S satisfying $I \subset J \subset S$. An ideal I of a semiring S will be called subtractive (k-ideal) if for $a \in I, a+b \in I, b \in S$ imply $b \in I$. A semiring S is said to be Semisubtractive if for any arbitrary $a \neq b$ in S there is always some x in S satisfying $b + x = a$ or some y in S satisfying $a + y = b$. The k-closure $\overline{I}$ of an ideal I, defined by $\overline{I} = \{s \in S/s + a \in I \ forsome \ a \in I\}$ is an ideal of S as well. An ideal I of a semiring S is called a k-ideal if $\overline{I} = I$ is true. We denote the set of all ideals of S by $\mathcal{I}(S)$ and the set of all k-ideals by $\mathcal{K}(S)$. Note that the ideals defined in this way should be called more precisely "semiring ideals". This is of importance if (associative) rings occur in semiring-theoretical investigations, since a ring R, considered as a semiring, may have semiring ideals which are not "ring ideals" in the usual meaning. More precisely, a semiring ideal I of a ring R is a ring ideal of R if and only if $\overline{I} = I$ holds, i.e. if and only if I is a k-ideal of R.

Each homomorphism $\phi\colon S \to T$of semirings corresponds to a congruence k of S and the homomorphic image $\phi(S)$ is isomorphic to the semiring S/k of congruence classes. In this paper we mainly use congruences that are determined by an ideal I of S according to $sk_1s' \Leftrightarrow$ there are

$$a_i \in I \text{ satisfying } s + a_1 = s' + a_2.$$

In this case one usually denotes S/k_1 by S/I. Moreover, $k_I = k_{\overline{I}}$ and thus $S/I = S/\overline{I}$ hold for all ideals I of S with the same k-closure $\overline{I}, S/I$ has always an absorbing zero, namely the congruence class $\overline{I} = [a]_I = [a]_{\overline{I}}$ determined by each $a \in I$. We also mention that a semiring has in general much more congruences than those determined by its ideals. For a last concept of this kind, let $\phi\colon S \to T$ be a surjective homomorphism for semirings which have a zero. Then ϕ is called a semi-isomorphism and denoted by $\phi\colon S \tilde{\to} T$ if $\phi(0_S) = 0_T$ and $\phi^{-1}(0_T) = 0_S$ are satisfied. We emphasize here that such a semi-isomorphism, despite of misleading name, has in general very little in common with an isomorphism. Here we denote a surjective semiring homomorphism from S to T by $S \longmapsto T$ and a semiring ideal I of S by $I \rhd S$.

Theorem 1.1. [4] *Let S be a semiring, T a semiring with an absorbing zero 0_T, and $\phi\colon S \to T$ a surjective homomorphism. Then $K = \phi^{-1}(0_T)$ is a k-ideal of S (also called the kernel of ϕ) and $\phi([s]_K) = \phi(s)$ for all $s \in S$ defines a semi-isomorphism $\phi\colon S/K \tilde{\to} T$ which satisfies $\phi \circ k_K^{\#} = \phi$, where $k_K^{\#}$ denotes the natural homomorphism of S onto $S/K = S/k_K$.*

Theorem 1.2. [4] *For a semiring S with an absorbing zero 0 let A be a subsemiring which contains 0 and B an ideal of S. Then $\phi([a]_{A\cap\overline{B}}) = [a]_B$ for all $a \in A \subseteq A + B$ defines a semi-isomorphism*

$$\phi\colon A/A\cap\overline{B} \tilde{\to} A+B/B.$$

Theorem 1.3. [4] *Let A, B be ideals of a semiring S with the additional condition $A \subseteq B$. Then $\overline{\phi}([s]_B) = [[s]_A]_{\overline{B}/A}$ for all $s \in S$ defines an isomorphism*

$$\overline{\phi}\colon S/B \to (S/A)/(\overline{B}/A).$$

In [9], we have introduced $R^e = \{[a,b]/a,b \in R\}$, the ring of differences of any cancellative semiring R with respect to the following well defined operations

$$[a,b]+[c,d] = [a+c, b+d] \text{ and}$$
$$[a,b][c,d] = [ac+bd, ad+bc].$$

Let R be any cancellative semiring. If I is a [left, right] ideal of R, then $I^e = \{[a,b]/a,b \in I\}$ is a [left, right] ideal of R^e. Conversely, if J is a [left, right] ideal of R^e, then $J^c = \{a \in R/[a,0] \in J\}$ is a [left, right] ideal of R.

Proposition 1.4. [9]
(a) Let R be a cancellative semiring. Then for any k-ideal I of R, $I = (I^e)^c$
(b) Let R be a cancellative semiring and I be a proper k-ideal of R then I^e is a proper ideal of R^e.
(c) If I, I' are any two k-ideals of a cancellative semiring R and $I \subset I'$, then $I^e \subset I'^e$.
(d) Let R be a cancellative semiring. Then for any two ideals J and J' of R^e, $J \subset J' \Rightarrow J^c \subset J'^c$

Note 1. *It should be noted that J^c is a k-ideal.*

Proposition 1.5. [1] *A homomorphism f from any cancellative and semisubtractive semiring S to any cancellative semiring T is always a steady homomorphism.*

Remark 1.6. [1] *If S is cancellative and semisubtractive, then S/I is cancellative and semisubtractive for any k-ideal I of S. From Proposition 1.5 [1], the semi-isomorphism theorems for semirings (semimodules) are the isomorphism theorems for cancellative and semisubtractive semirings (semimodules).*

2 Radical Classes

Definition 2.1. *A non-empty sub-class $\mathfrak{R}$ of a fixed Universal class $\mathbb{U}$ is called a radical class of $\mathbb{U}$ if $\mathfrak{R}$ satisfy the following two conditions;*
(R_1) $\mathfrak{R}$ is homomorphically closed.
(R_2) for each $S \in \mathbb{U} - \mathfrak{R}$ there is a k-ideal $K \in \mathcal{K}(S) - \{S\}$ such that $\mathcal{I}(S/K) \cap \mathfrak{R} = \{(0)\}$.

Theorem 2.2. *Each radical class $\mathfrak{R}$ of $\mathbb{U}$ is closed under extension i.e. for all $S \in \mathbb{U}$ and $I \in \mathcal{I}(S)$ the following condition holds true*
(R_e) $I \in \mathfrak{R}$ and $S/I \in \mathfrak{R} \Rightarrow S \in \mathfrak{R}$.

Theorem 2.3. *A subclass (which is non-empty) $\mathfrak{R}$ of $\mathbb{U}$ is a radical class if and only if $\mathfrak{R}$ satisfies (R_1), (R_e) and the following inductive property (R_i) i.e. each ascending chain $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \ldots$ of $\mathfrak{R}$-ideals (k-ideals) I_n of any semiring $S \in \mathbb{U}$ satisfies $B = \cup I_n \in \mathfrak{R}$.*

Theorem 2.4. *A class $\mathfrak{R}$ of semirings is radical class if and only if*
(a) $\mathfrak{R}$ is homomorphically closed.
(b) $\mathfrak{R}$ has the inductive property.
(c) $\mathfrak{R}$ is closed under extensions.

Theorem 2.5. *For any class $\mathfrak{R}$ of semirings, the following conditions are equivalent.*

I. $\mathfrak{R}$ is a radical class.

II. (Ri) If $A \in \mathfrak{R}$, then for any $A \longmapsto B \neq 0$ there is a $C \rhd A$ such that $0 \neq C \in \mathfrak{R}$.

(Rii) If A is a semiring of the universal class $\mathbb{U}$ and for any $A \longmapsto B \neq 0$ there is a $C \rhd B$ such that $0 \neq C \in \mathfrak{R}$, then $A \in \mathfrak{R}$

III. $\mathfrak{R}$ satisfies condition (Ri), has the inductive property and is closed under extensions.

Proof. (I $\Rightarrow$ III) Since every homomorphically closed class clearly satisfies (Ri). Moreover, by Theorem 2.4, $\mathfrak{R}$ has the inductive property and $\mathfrak{R}$ is closed under extension.
(III $\Rightarrow$ II) Assume that $\mathfrak{R}$ satisfies (Ri), has the inductive property and $\mathfrak{R}$ is closed under extension. We must prove (Rii). Assume that A is a semiring such that for every $A \longmapsto B \neq 0$, there exists a $C \rhd A$ such that $0 \neq C \in \mathfrak{R}$ and that $A \notin \mathfrak{R}$. Since $\mathfrak{R}$ has inductive property, by Zorn's lemma obtain a k-ideal I of A, which is maximal with respect to being k-ideals in $\mathfrak{R}$. Since $A \notin \mathfrak{R}$, $A/I \neq 0$ holds. Now $A \notin \mathfrak{R}$ and $A \to A/I$ is a surjective homomorphism . Therefore $A^e \notin \mathfrak{R}^e$ with $A^e \to A^e/I^e$ is a surjective homomorphism. So there exists an ideal J/I^e in A^e/I^e such that $J/I^e \in \mathfrak{R}^e$. Thus $I^e \in \mathfrak{R}^e$, $J/I^e \in \mathfrak{R}^e$ implies that $I^{ec} \in \mathfrak{R}^{ec}$ and $J^c/I^{ec} \in \mathfrak{R}^{ec}$

$$\Rightarrow I \in \mathfrak{R} \text{ and } J^c/I \in \mathfrak{R},$$

a contraction to the fact that I is a maximal k-ideal. Therefore $A \in \mathfrak{R}$, and hence (II). Finally (III $\Leftrightarrow$ I) by [4]. $\square$

Definition 2.6. *A mapping* $\rho\colon \mathbb{U} \to \mathbb{U}$ *is called a radical operator in* $\mathbb{U}$ *if each semiring* $S \in \mathbb{U}$ *has an image* $\rho(S) \in \mathcal{K}(S) \subseteq \mathbb{U}$ *such that the following conditions are satisfied for all* $S, T \in \mathbb{U}$.
(ρa): $\phi(\rho(S)) \subseteq \rho(\phi(S))$, *for each* $\phi\colon S \longmapsto T$.
(ρb): $\rho(S/\rho(S)) = 0$.
(ρc): $\rho(T) = T \triangleright S \Rightarrow T \subseteq \rho(S)$.
(ρd): $\rho(\rho(S)) = \rho(S)$.

Theorem 2.7. *Let* $\rho\colon \mathbb{U} \to \mathbb{U}$ *be a radical operator. Then*

$$\triangle = \{A \in \mathbb{U}/\ \rho(A) = A\}$$

is a radical class of $\mathbb{U}$.

Proof. Claim that $\triangle$ is homomorphically closed. Let $A \in \triangle$ and $\phi\colon A \to B$ be a surjective homomorphism. Then $B = \phi(A) = \phi(\rho(A)) \subseteq \rho(\phi(A)) = \rho(B)$. Shows that $\rho(B) = B$. Therefore $B \in \rho$. Hence ρ is homomorphically closed. Now for any semiring $A \in \mathbb{U} - \rho$, we have $\rho(A) \subseteq A$. Then clearly $\rho(A)$ is a k-ideal which satisfies (R_2). $\square$

3 Radical Construction

Let $\mathbb{U}$ be an universal class of (not necessarily associative) semirings and let $\mathbb{V} \subseteq \mathbb{U}$. We give here a construction for $\mathfrak{L}\mathbb{V}$, the lower radical class determined by $\mathbb{V}$ in $\mathbb{U}$. Using this construction, we establish that if $\mathbb{V}$ is hereditary class (if $A \in \mathbb{V}$ and I is an ideal of A, then $I \in \mathbb{V}$), then $\mathfrak{L}\mathbb{V}$ is also hereditary.

Let $\mathbb{V}$ be any arbitrary class of semirings. However there is always the smallest radical class containing $\mathbb{V}$. Using Theorem 2.4 and take the intersection of all radical classes that contains $\mathbb{V}$. This intersection $\mathfrak{L}\mathbb{V}$ is a radical class and is clearly the smallest radical class containing $\mathbb{V}$. $\mathfrak{L}\mathbb{V}$ is called lower radical determined by the class $\mathbb{V}$.

Starting with $\mathbb{V}$, an arbitrary class of semisubtractive and additively cancellative semirings. $\mathbb{V}_1 = \{A/A \textit{ is a homomrophic image of a semiring } B$ in $\mathbb{V}\}$ called homomorphic closure of $\mathbb{V}$. Proceeding inductively, if $\mathbb{V}_\mu$ has been defined for all ordinals $\mu < \lambda$, we define $\mathbb{V}_\lambda = \{A/\ \textit{there exists a } k-\textit{ideal } I \textit{ of } A \textit{ such that } I \in \mathbb{V}_{\lambda-1} \textit{ and } A/I \in \mathbb{V}_{\lambda-1}\}$ where $\lambda - 1$ exists. When λ is a limit ordinal, we define
$\mathbb{V}_\lambda = \{A/A \textit{ is the union of an ascending chain of ideals, each one is in } \mathbb{V}_\mu$ $\textit{for } \mu < \lambda\}$. Finally we define $\mathfrak{R}(\mathbb{V}) = \bigcup \mathbb{V}_\lambda$, where the union is taken over

all ordinals λ. Using this characterization, analogous to rings, we prove that $\mathfrak{R}(\mathbb{V}) = \bigcup \mathbb{V}_\lambda$.

Lemma 3.1. *If λ and μ are ordinals with $\mu < \lambda$, then $\mathbb{V}_\mu \subseteq \mathbb{V}_\lambda$.*

Proof. First we want to show that if $\mu < \lambda$, then $\mathbb{V}_\mu \subseteq \mathbb{V}_\lambda$. Clearly $\mathbb{V} \subset \mathbb{V}_1$. Also it is obvious that $\mathbb{V}_\mu \subseteq \mathbb{V}_\lambda$, for any limit ordinal λ. To see $\mathbb{V}_\mu \subseteq \mathbb{V}_{\mu+1}$. As $\phi(0) = 0$, the semiring 0 is in $\mathbb{V}_1$ and by induction also in every $\mathbb{V}_\lambda$ for $\lambda \geq 1$. Thus for $A \in \mathbb{V}_\mu$ we take $I = 0 \in \mathbb{V}_\mu$ and $A = A/I \in \mathbb{V}_\mu$. So $A \in \mathbb{V}_{\mu+1}$ by definition and the construction is monotonically increasing. □

Lemma 3.2. *For every ordinal $\lambda \geq 1$, $\mathbb{V}_\lambda$ is homomorphically closed. Hence $\mathfrak{R}(\mathbb{V})$ is homomorphically closed.*

Proof. To prove $\mathfrak{R}(\mathbb{V})$ is homomorphically closed. Of course $\mathbb{V}_1$ is homomorphically closed. Assume that the result is true for $\mathbb{V}_\mu$, for $\mu < \lambda$. We take $A \in \mathbb{V}_\lambda$ and for some k-ideal I consider the image A/I of A. When λ is a limit ordinal, there is a chain $\{K_i\}$ of ideals of A such that $\bigcup K_i = A$ and each $K_i \in \mathbb{V}_\mu$, for $\mu < \lambda$. Then $\{I + K_i/I\}$ is a chain of ideals of A/I such that A/I is the union of this chain. Since $I + K_i/I \cong K_i/I \bigcap K_i$ by Remark 1.6 [1]. So we have $K_i/I \bigcap K_i \in \mathbb{V}_\mu$ for $\mu < \lambda$. This means that $K_i/I \bigcap K_i \triangleright A/I$ with $\bigcup K_i/I \bigcap K_i = A/I$ and $K_i/I \bigcap K_i \in \mathbb{V}_\mu$, $\mu < \lambda$. Therefore $A/I \in \mathbb{V}_\mu$. If $\lambda - 1$ exists, then there exists a k-ideal J of A such that J and A/J are in $\mathbb{V}_{\lambda-1}$. Again by hypothesis on $\mathbb{V}_{\lambda-1}$ and by Remark 1.6 [1], we have $J + I/I \cong J/J \bigcap I \in \mathbb{V}_{\lambda-1}$ and by Theorem-1.3 [4] $(A/I)/\overline{(J + I)/I} \cong A/I + J \in \mathbb{V}_{\lambda-1}$, implies that $A/I \in \mathbb{V}_\lambda$. Thus by transfinite induction any $\mathbb{V}_\lambda$ is homomorphically closed and so is $\mathfrak{R}(\mathbb{V})$. □

We now show that $\mathfrak{R}(\mathbb{V})$ satisfies condition (b) and (c) of Theorem-2.4.

Lemma 3.3. *Let $A \in \mathbb{U}$ and let $\{I_\lambda\}$ be a chain of $\mathfrak{R}(\mathbb{V})$-ideals of A. Then $\bigcup I_\lambda$ is an $\mathfrak{R}(\mathbb{V})$-ideals of A.*

Proof. Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_\rho \subseteq \ldots$ be a strictly ascending chain of k-ideals of A such that $I_\rho \in \bigcup \mathbb{V}_\lambda$. Since A is a set and the construction is monotonic, there is an ordinal μ such that $I_\rho \in \mathbb{V}_\mu$, for each ρ. If we take any limit ordinal γ bigger than μ, we have $\bigcup I_\rho \in \mathbb{V}_\gamma \subseteq \bigcup \mathbb{V}_\lambda$ this gives us inductive property. □

Lemma 3.4. *If a k-ideal I and A/I are in $\mathfrak{R}(\mathbb{V})$, then $A \in \mathfrak{R}(\mathbb{V})$.*

Proof. To prove $\mathfrak{R}(\mathbb{V})$ is closed under extension. Let I be a k-ideal and assume that I, $A/I \in \mathfrak{R}(\mathbb{V})$. Then there exists an ordinal λ which is not a limit ordinal such that both I and A/I are in $\mathbb{V}_\lambda$. Therefore $A \in \mathbb{V}_{\lambda+1}$ by definition and thus $A \in \mathfrak{R}(\mathbb{V})$. Hence the extension property. □

Theorem 3.5. $\mathfrak{R}(\mathbb{V}) = \mathfrak{L}\mathbb{V}$.

Proof. By Theorem-2.4 Lemma 3.1, 3.2, 3.3 and 3.4 $\mathfrak{R}(\mathbb{V})$ is a radical class in $\mathbb{U}$. By the minimality of $\mathfrak{L}\mathbb{V}$ among radical classes in $\mathbb{U}$ which contains $\mathbb{V}$, $\mathfrak{L}\mathbb{V} \subseteq \mathfrak{R}(\mathbb{V})$. Now it is enough to prove that $\mathfrak{R}(\mathbb{V}) \subseteq \mathfrak{L}\mathbb{V}$. This accomplished by proving $\mathbb{V}_\lambda \subseteq \mathfrak{L}\mathbb{V}$ for any ordinal λ. Clearly $\mathbb{V}_1 \subseteq \mathfrak{L}\mathbb{V}$. Let λ be an ordinal exceeding one, and assume that $\mathbb{V}_\mu \subseteq \mathfrak{L}\mathbb{V}$ for ordinals $\mu < \lambda$. Let $A \in \mathbb{V}_\lambda$. If λ is limit ordinal, A is the union of a chain of ideals from the classes $\mathbb{V}_\mu$, $\mu < \lambda$. Thus by the induction hypothesis A is the union of $\mathfrak{L}\mathbb{V}$-ideals, so $A \in \mathfrak{L}\mathbb{V}$ by Theorem-2.4. If λ is not a limit ordinal, then there exists a k-ideal I and A/I of A both belongs to $\mathbb{V}_{\lambda-1} \subseteq \mathfrak{L}\mathbb{V}$. Again $A \in \mathfrak{L}\mathbb{V}$ by Theorem-2.4. Thus $\mathbb{V}_\lambda \subseteq \mathfrak{L}\mathbb{V}$ for all λ. Hence $\mathfrak{R}(\mathbb{V}) \subseteq \mathfrak{L}\mathbb{V}$. □

Theorem 3.6. *Let* $\mathbb{V} \subseteq \mathbb{U}$, *where* $\mathbb{U}$ *is some universal class. If* $\mathbb{V}$ *is hereditary, so is* $\mathfrak{L}\mathbb{V}$.

Proof. We prove $\mathbb{V}_\lambda$ is hereditary for each $\lambda \geq 1$, then clearly $\mathbb{V}_1$ is homomorphic closure and it is hereditary. Assume $\lambda > 1$ and $\mathbb{V}_\mu$ is hereditary class for each $\mu < \lambda$. Let $A \in \mathbb{V}_\lambda$ and suppose I is an k-ideal of A. If λ is a limit ordinal, $A = \bigcup I_\alpha$, where $\{I_\alpha\}$ is a chain of ideals belonging to one of the (hereditary) classes $\mathbb{V}_\mu$, $\mu < \lambda$. But then $I = (\bigcup I_\alpha) \bigcap I = \bigcup (I_\alpha \bigcap I)$ so $I \in \mathbb{V}_\mu$. If λ is not a limit ordinal, there is an k-ideal J of A so that $J,\ A/J \in V_{\lambda-1}$. Since $\mathbb{V}_{\lambda-1}$ is hereditary, $I \bigcap J$ and $I + J/J \cong I/I \bigcap J$ both belonging to $\mathbb{V}_{\lambda-1}$. This implies that $I \in \mathbb{V}_\lambda$. □

Lemma 3.7. *If* $\mathfrak{R}$ *is a radical class in* $\mathbb{U}$ *and for some* $K' \in \mathbb{U}$, *a subsemiring* $K \subseteq K'$ *is the set-theoretic union of* $\mathfrak{R}$*-ideals of* K', *then* $K \in \mathfrak{R}$.

Proof. If $K = \bigcup I_\alpha \notin \mathfrak{R}$, Then by definition of radical class $\mathfrak{R}$, there exists a k-ideal I of K such that $\mathcal{I}(K/I) \bigcap \mathfrak{R} = \{(0)\}$, for some $I \neq K$. Therefore there is some α such that $I_\alpha \nsubseteq I$, so $I_\alpha + I \cong I_\alpha / I \bigcap I_\alpha$ is a nonzero $\mathfrak{R}$-ideal of K/I. This contradiction proves that $K \in \mathfrak{R}$. □

Theorem 3.8. *For alternative semirings, if* $A_1,\ A_2$ *are homomorphically closed, hereditary classes of* $\mathbb{U}$*-semirings, then* $\mathfrak{L}(A_1 \bigcap A_2) = \mathfrak{L}A_1 \bigcap \mathfrak{L}A_2$.

References

[1] **R. P. Deore and K. B. Patil**, On Ideals In Semirings, *Southeast Asian Bulletin In Maths, Springer Verlag,* 32 (3)(2008), 417-425.

[2] **J. S. Golan**, *The Theory of Semirings with Applications in Mathematics and Theoretical Computer Sciences*, Pitman Monographs and surveys in Pure and Applied Mathematics 54, Longman Science Tech, Harlow-1992.

[3] **H. C. Hutchins; H. J. Weinert**, Homomorphisms and Kernels of Semifields, *Period. Math. Hungar.*, 21(1990), 113-152.

[4] **B. Morak**, On the Radical Theory for Semiring, *Beitrage Alg. und. Geom.*, 40(1999), 533-549.

[5] **D. M. Olson; G. A. P. Heyman; H. J. LeRoux**,Weakly Special Classes of Hemirings, *Quaestiones Math.*, 15(1992), 119-126.

[6] **D. M. Olson; T. L. Jenkins**, Radical theory for Hemirings, *J. Nature. Sci. Math.*, 23(1983), 23-32.

[7] **D. M. Olson; H. J. LeRouxG; A. P. Heyman**, Three Special Rradicals for Hemirings, *Quaestiones Math.*, 17(1994), 205-215.

[8] **D. M. Olson; A. C. Nance**, A note on radicals for hemirings, *Quaestiones Math.*, 12(1989), 307-314.

[9] **K.B. Patil and R. P. Deore**, Some Results on Semirings and Semimodules, *Bull. Cal. Math. Soc.*, 98(1)(2006), 49-56.

[10] **R. Tangeman and D. Kreiling**, Lower Radicals In Non-associative Rings, *J. Austral. Math. Soc.*, 14(1972), 419-423.

[11] **H. J. Weinert; R. Wiegandt**, A Kurosh-Amitsur radical theory for proper semifields, *Comm. Algebra.*, 20(1992), 2419-2458.

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