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Filters of BCH-Algebras Based on Bipolar-Valued Fuzzy Sets

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Abstract

The notions of a bipolar fuzzy regularity, a bipolar fuzzy regular subalgebra, a bipolar fuzzy filter, and a bipolar fuzzy closed quasi filter in BCH-algebras are introduced, and several properties are investigated. A characterization of a bipolar fuzzy filter is provided.

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1 Introduction

In the traditional fuzzy sets, the membership degrees of elements range over the interval $[0, 1]$. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set, and the membership degree 0 indicates that an element does not belong to the fuzzy set. The membership degrees on the interval $(0, 1)$ indicate the partial membership to

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the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set (see [3, 9]). In the viewpoint of satisfaction degree, the membership degree 0 is assigned to elements which do not satisfy some property. The elements with membership degree 0 are usually regarded as having the same characteristics in the fuzzy set representation. By the way, among such elements, some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property. The traditional fuzzy set representation cannot tell apart contrary elements from irrelevant elements. Consider a fuzzy set "*young*" defined on the *age* domain $[0, 100]$ (see Fig. 1 in [6]). Now consider two ages 50 and 95 with membership degree 0. Although both of them do not satisfy the property "*young*", we may say that age 95 is more apart from the property rather than age 50 (see [6]).

Only with the membership degrees ranged on the interval [0, 1], it is difficult to express the difference of the irrelevant elements from the contrary elements in fuzzy sets. If a set representation could express this kind of difference, it would be more informative than the traditional fuzzy set representation. Based on these observations, Lee [6] introduced an extension of fuzzy sets named bipolar-valued fuzzy sets. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets.

In [5], Jun and Song applied the notion of bipolar-valued fuzzy set to BCH-algebras. They introduced the concept of bipolar fuzzy subalgebras/ideals of a BCH-algebra, and investigated several properties. They gave relations between a bipolar fuzzy subalgebra and a bipolar fuzzy closed ideal, and provided conditions for a bipolar fuzzy subalgebra to be a bipolar fuzzy closed ideal. They also gave characterizations of a bipolar fuzzy closed ideal.

In this paper, we introduce the notion of a bipolar fuzzy regularity, a bipolar fuzzy regular subalgebra, a bipolar fuzzy filter, and a bipolar fuzzy closed quasi filter in BCH-algebras, and then we investigate several properties. We give a characterization of a bipolar fuzzy filter.

2 Preliminaries

2.1 Basic results on BCH-algebras

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BCH-algebra* we mean a system $(X; *, 0) \in K(\tau)$ in which the following axioms hold:

$$(\forall x \in X)(x * x = 0), \tag{2.1}$$

$$(\forall x, y \in X)(x * y = 0 \ \& \ y * x = 0 \ \Rightarrow \ x = y), \tag{2.2}$$

$$(\forall x, y, z \in X)((x * y) * z = (x * z) * y). \tag{2.3}$$

Any BCH-algebra X satisfies the following axioms:

$$(\forall x \in X)(x * 0 = x), \tag{2.4}$$
$$(\forall x \in X)(x * 0 = 0 \;\Rightarrow\; x = 0), \tag{2.5}$$
$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)), \tag{2.6}$$
$$(\forall x \in X)(0 * (0 * (0 * x)) = 0 * x). \tag{2.7}$$

A BCH-algebra X is said to be *medial* if it satisfies:

$$(\forall x, y, a, b \in X)((x * y) * (a * b) = (x * a) * (y * b)). \tag{2.8}$$

A subset S of a BCH-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset R of a BCH-algebra X is said to be *regular* if it satisfies:

$$(\forall x \in R)(\forall y \in X)(x * y \in R \;\Rightarrow\; y \in R). \tag{2.9}$$

For any $x, y \in X$, we denote $x \overleftarrow{\wedge} y = y*(y*x)$. A subset F of a BCH-algebra X is called a *filter* of X (see [2]) if it satisfies:

$$(\forall x, y \in F)(x \overleftarrow{\wedge} y \in F,\; y \overleftarrow{\wedge} x \in F), \tag{2.10}$$
$$(\forall x \in F)(\forall y \in X)(x \leq y \;\Rightarrow\; y \in F). \tag{2.11}$$

2.2 Basic results on bipolar-valued fuzzy set

As an extension of fuzzy sets, Lee [6] introduced the notion of bipolar-valued fuzzy sets. So, this subsection is based on his paper (see [6] [7]). Fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets etc. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. Bipolar-valued fuzzy sets have membership degrees that represent the degree of satisfaction to the property corresponding to a fuzzy set and its counter-property. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degrees on $(0, 1]$ indicate that elements somewhat satisfy the property, and the membership degrees on $[-1, 0)$ indicate that elements somewhat satisfy the implicit counter-property (see [6]). In[6], Figure 2 shows a bipolar-valued fuzzy set redefined for the fuzzy set "*young*" of Figure 1. The negative membership degrees indicate the satisfaction extent of elements to an implicit counter-property (e.g., old against the property young). This kind of bipolar-valued fuzzy set representation enables the elements with membership degree 0 in traditional fuzzy sets, to be expressed into the elements with membership degree

0 (irrelevant elements) and the elements with negative membership degrees (contrary elements). The age elements 50 and 95, with membership degree 0 in the fuzzy sets of Figure 1, have 0 and a negative membership degree in the bipolar-valued fuzzy set of Figure 2, respectively. Now it is manifested that 50 is an irrelevant age to the property young and 95 is more apart from the property young than 50, i.e., 95 is a contrary age to the property young (see [6]).

In the definition of bipolar-valued fuzzy sets, there are two kinds of representations, so called canonical representation and reduced representation. In this paper, we use the canonical representation of a bipolar-valued fuzzy sets. Let X be the universe of discourse. A *bipolar-valued fuzzy set* Φ in X is an object having the form $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$ where $\mu_\Phi^P : X \to [0,1]$ and $\mu_\Phi^N : X \to [-1,0]$ are mappings. The positive membership degree $\mu_\Phi^P(x)$ denoted the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$, and the negative membership degree $\mu_\Phi^N(x)$ denotes the satisfaction degree of x to some implicit counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$. If $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$. If $\mu_\Phi^P(x) = 0$ and $\mu_\Phi^N(x) \neq 0$, it is the situation that x does not satisfy the property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$ but somewhat satisfies the counter-property of $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$. It is possible for an element x to be $\mu_\Phi^P(x) \neq 0$ and $\mu_\Phi^N(x) \neq 0$ when the membership function of the property overlaps that of its counter-property over some portion of the domain (see [7]). For the sake of simplicity, we shall use the symbol $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ for the bipolar-valued fuzzy set $\Phi = \{(x, \mu_\Phi^P(x), \mu_\Phi^N(x) \mid x \in X\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

3 Bipolar fuzzy filters

In what follows, let X be a BCH-algebra unless otherwise specified.

Definition 3.1. [5] A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy subalgebra* of X if it satisfies the following assertions:

$$\begin{aligned} &\mu_\Phi^P(x * y) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}, \\ &\mu_\Phi^N(x * y) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\} \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

For a bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X and $(\beta, \alpha) \in [-1, 0] \times [0, 1]$, the *positive* α*-cut* and *negative* β*-cut* are denoted by $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$, and are defined as follows:

$$P(\mu_\Phi^P;\alpha) := \{x \in H \mid \mu_\Phi^P(x) \geq \alpha\},$$

$$N(\mu_\Phi^N;\beta) := \{x \in H \mid \mu_\Phi^N(x) \leq \beta\},$$

respectively. The *bipolar* (α,β)*-cut* of $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N)$, denoted by $B(\Phi;(\alpha,\beta))$, is defined to be the set

$$B(\Phi;(\alpha,\beta)) := \{x \in X \mid \mu_\Phi^P(x) \geq \alpha,\ \mu_\Phi^N(x) \leq \beta\}.$$

Note that $B(\Phi;(\alpha,\beta)) = P(\mu_\Phi^P;\alpha) \cap N(\mu_\Phi^N;\beta)$.

Definition 3.2. A bipolar fuzzy set $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N)$ in X is said to satisfy the *bipolar fuzzy regularity* if the following assertions hold:

$$\begin{aligned} &\mu_\Phi^P(y) \geq \min\{\mu_\Phi^P(x*y),\mu_\Phi^P(x)\}, \\ &\mu_\Phi^N(y) \leq \max\{\mu_\Phi^N(x*y),\mu_\Phi^N(x)\} \end{aligned} \tag{3.2}$$

for all $x, y \in X$.

A bipolar fuzzy subalgebra satisfying the bipolar fuzzy regularity is called a *bipolar fuzzy regular subalgebra.*

Example 3.3. Consider a BCH-algerba $X = \{0, 1, 2, 3\}$ with Cayley table as follows:

$*$	0	1	2	3
0	0	3	0	3
1	1	0	3	2
2	2	3	0	1
3	3	0	3	0

Let $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	1	2	3
μ_Φ^P	0.7	0	0.7	0
μ_Φ^N	-0.8	-0.1	-0.8	-0.1

Then $\Phi = (X;\mu_\Phi^P,\mu_\Phi^N)$ satisfies the bipolar fuzzy regularity, and moreover it is a bipolar fuzzy regular subalgebra of X.

Example 3.4. Consider a BCH-algerba $X = \{0, 1, 2, 3, 4\}$ with Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	1	2	3	4
μ_Φ^P	α_1	α_2	α_3	α_2	α_3
μ_Φ^N	β_1	β_2	β_3	β_2	β_3

where $\alpha_1 > \alpha_2 > \alpha_3$ in $[0, 1]$ and $\beta_1 < \beta_2 < \beta_3$ in $[-1, 0]$. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X which does not satisfy the bipolar fuzzy regularity since

$$\mu_\Phi^P(1) = \alpha_2 < \alpha_1 = \min\{\mu_\Phi^P(0 * 1), \mu_\Phi^P(0)\}$$

and/or

$$\mu_\Phi^N(1) = \beta_2 > \beta_1 = \max\{\mu_\Phi^N(0 * 1), \mu_\Phi^N(0)\}.$$

Definition 3.5. [5] A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy closed ideal* of X if it satisfies the following assertions:

$$\mu_\Phi^P(0 * x) \geq \mu_\Phi^P(x) \;\;\&\;\; \mu_\Phi^N(0 * x) \leq \mu_\Phi^N(x), \tag{3.3}$$

$$\begin{aligned} &\mu_\Phi^P(y) \geq \min\{\mu_\Phi^P(y * x), \mu_\Phi^P(x)\}, \\ &\mu_\Phi^N(y) \leq \max\{\mu_\Phi^N(y * x), \mu_\Phi^N(x)\} \end{aligned} \tag{3.4}$$

for all $x, y \in X$.

Definition 3.6. A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy filter* of X if it satisfies the following assertions:

$$(\forall x, y \in X)(x * y = 0 \;\Rightarrow\; \mu_\Phi^P(x) \leq \mu_\Phi^P(y),\; \mu_\Phi^N(x) \geq \mu_\Phi^N(y)). \tag{3.5}$$

and

$$\begin{aligned} &\min\{\mu_\Phi^P(x \overleftarrow{\wedge} y), \mu_\Phi^P(y \overleftarrow{\wedge} x)\} \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}, \\ &\max\{\mu_\Phi^N(x \overleftarrow{\wedge} y), \mu_\Phi^N(y \overleftarrow{\wedge} x)\} \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\} \end{aligned} \tag{3.6}$$

for all $x, y \in X$. If, moreover, a bipolar fuzzy filter $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ satisfies (3.3), we say that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a *bipolar fuzzy closed filter* of X.

Example 3.7. Consider a BCH-algerba $X = \{0, 1, 2, 3\}$ with Cayley table as follows:

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	1	2	3
μ_Φ^P	0.8	0.8	0.5	0
μ_Φ^N	-0.7	-0.7	-0.6	-0.1

Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy filter of X.

Definition 3.8. A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is called a *bipolar fuzzy closed quasi-filter* of X if it satisfies conditions (3.3) and (3.5).

Proposition 3.9. *Every bipolar fuzzy closed quasi-filter* $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ *of* X *satisfies the following assertions:*

(1) $(\forall x, y \in X)\ (\mu_\Phi^P(x * y) = \mu_\Phi^P(y * x),\ \mu_\Phi^N(x * y) = \mu_\Phi^N(y * x))$.

(2) $(\forall x, y \in X)\ (\mu_\Phi^P(x \overleftarrow{\wedge} y) = \mu_\Phi^P(x),\ \mu_\Phi^N(x \overleftarrow{\wedge} y) = \mu_\Phi^N(x))$.

(3) $(\forall x, y \in X)\ (x * y = 0 \ \Rightarrow\ \mu_\Phi^P(x) = \mu_\Phi^P(y),\ \mu_\Phi^N(x) = \mu_\Phi^N(y))$.

Proof. (1) Let $x, y \in X$. Using (2.6), (2.3) and (2.1), we can easily verify that $(0 * (x * y)) * (y * x) = 0$. It follows from (3.3) and (3.5) that

$$\mu_\Phi^P(y * x) \geq \mu_\Phi^P(0 * (x * y)) \geq \mu_\Phi^P(x * y)$$

and

$$\mu_\Phi^N(y * x) \leq \mu_\Phi^N(0 * (x * y)) \leq \mu_\Phi^N(x * y).$$

Similarly, we have $\mu_\Phi^P(y * x) \leq \mu_\Phi^P(x * y)$ and $\mu_\Phi^N(y * x) \geq \mu_\Phi^N(x * y)$. Hence (1) is valid.

(2) Using (1), (2.1), (2.3) and (2.4), we have

$$\begin{aligned} \mu_\Phi^P(x \overleftarrow{\wedge} y) &= \mu_\Phi^P(y * (y * x)) = \mu_\Phi^P((y * x) * y) \\ &= \mu_\Phi^P((y * y) * x) = \mu_\Phi^P(0 * x) \\ &= \mu_\Phi^P(x * 0) = \mu_\Phi^P(x) \end{aligned}$$

and

$$\begin{aligned}\mu_\Phi^N(x \overleftarrow{\wedge} y) &= \mu_\Phi^N(y*(y*x)) \;=\; \mu_\Phi^N((y*x)*y)\\ &= \mu_\Phi^N((y*y)*x) \;=\; \mu_\Phi^N(0*x)\\ &= \mu_\Phi^N(x*0) \;=\; \mu_\Phi^N(x).\end{aligned}$$

(3) Let $x, y \in X$ be such that $x*y=0$. Using (2.4) and (2), we get

$$\mu_\Phi^P(x) = \mu_\Phi^P(x*0) = \mu_\Phi^P(x*(x*y)) = \mu_\Phi^P(y \overleftarrow{\wedge} x) = \mu_\Phi^P(y)$$

and

$$\mu_\Phi^N(x) = \mu_\Phi^N(x*0) = \mu_\Phi^N(x*(x*y)) = \mu_\Phi^N(y \overleftarrow{\wedge} x) = \mu_\Phi^N(y).$$

This completes the proof. □

Theorem 3.10. *Every bipolar fuzzy closed quasi-filter* $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ *of* X *which is also a bipolar fuzzy subalgebra is a bipolar fuzzy closed ideal of* X.

Proof. It is sufficient to show that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ satisfies (3.4). Let $x, y \in X$. Using Proposition 3.9(1), (2.1), (2.3) and (3.1), we have

$$\begin{aligned}\mu_\Phi^P(y) &= \mu_\Phi^P(y*0) \;=\; \mu_\Phi^P(0*y)\\ &= \mu_\Phi^P((x*x)*y) \;=\; \mu_\Phi^P((x*y)*x)\\ &\geq \min\{\mu_\Phi^P(x*y), \mu_\Phi^P(x)\}\\ &= \min\{\mu_\Phi^P(y*x), \mu_\Phi^P(x)\}\end{aligned}$$

and

$$\begin{aligned}\mu_\Phi^N(y) &= \mu_\Phi^N(y*0) \;=\; \mu_\Phi^N(0*y)\\ &= \mu_\Phi^N((x*x)*y) \;=\; \mu_\Phi^N((x*y)*x)\\ &\leq \max\{\mu_\Phi^N(x*y), \mu_\Phi^N(x)\}\\ &= \max\{\mu_\Phi^N(y*x), \mu_\Phi^N(x)\}.\end{aligned}$$

Hence $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. □

Lemma 3.11. [5] *Every bipolar fuzzy closed ideal is a bipolar fuzzy subalgebra.*

Theorem 3.12. *Let* $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ *be a bipolar fuzzy closed ideal of* X. *If* $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ *satisfies the following assertion:*

$$(\forall x, y \in X)(\mu_\Phi^P(y*x) \geq \mu_\Phi^P(x*y),\ \mu_\Phi^N(y*x) \leq \mu_\Phi^N(x*y)), \tag{3.7}$$

then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ *is a bipolar fuzzy closed filter of* X.

Proof. By Lemma 3.11, $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy subalgebra of X. Hence

$$\begin{aligned}
\mu_\Phi^P(x \overleftarrow{\wedge} y) &= \mu_\Phi^P(y * (y * x)) \ \geq \ \min\{\mu_\Phi^P(y), \mu_\Phi^P(y * x)\} \\
&\geq \min\{\mu_\Phi^P(y), \min\{\mu_\Phi^P(y), \mu_\Phi^P(x)\}\} \\
&= \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\mu_\Phi^N(x \overleftarrow{\wedge} y) &= \mu_\Phi^N(y * (y * x)) \ \leq \ \max\{\mu_\Phi^N(y), \mu_\Phi^N(y * x)\} \\
&\leq \max\{\mu_\Phi^N(y), \max\{\mu_\Phi^N(y), \mu_\Phi^N(x)\}\} \\
&= \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}.
\end{aligned}$$

Similarly, we have

$$\mu_\Phi^P(y \overleftarrow{\wedge} x) \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}$$

and

$$\mu_\Phi^N(y \overleftarrow{\wedge} x) \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}.$$

Therefore

$$\min\{\mu_\Phi^P(x \overleftarrow{\wedge} y), \mu_\Phi^P(y \overleftarrow{\wedge} x)\} \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\}$$

and

$$\max\{\mu_\Phi^N(x \overleftarrow{\wedge} y), \mu_\Phi^N(y \overleftarrow{\wedge} x)\} \leq \max\{\mu_\Phi^N(x), \mu_\Phi^N(y)\}.$$

Let $x, y \in X$ satisfy $x * y = 0$. Then $0 * x = 0 * y$. It follows from (2.4) and the assumption that

$$\mu_\Phi^P(y) = \mu_\Phi^P(y * 0) \geq \mu_\Phi^P(0 * y) = \mu_\Phi^P(0 * x) \geq \mu_\Phi^P(x * 0) = \mu_\Phi^P(x)$$

and

$$\mu_\Phi^N(y) = \mu_\Phi^N(y * 0) \leq \mu_\Phi^N(0 * y) = \mu_\Phi^N(0 * x) \leq \mu_\Phi^N(x * 0) = \mu_\Phi^N(x).$$

Thus $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed filter of X. □

Lemma 3.13. *If a bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X satisfies the bipolar fuzzy regularity, then*

$$(\forall x \in X)(\mu_\Phi^P(0) \geq \mu_\Phi^P(x),\ \mu_\Phi^N(0) \leq \mu_\Phi^N(x)). \tag{3.8}$$

Proof. Taking $y = 0$ in (3.2) and using (2.4) induce the desired results. □

Theorem 3.14. *If a bipolar fuzzy set* $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ *in* X *satisfies the bipolar fuzzy regularity and the condition* (3.4), *then* $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ *is a bipolar fuzzy closed ideal of* X.

Proof. It is sufficient to show that $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ satisfies the condition (3.3). For any $x, y \in X$, we have

$$\begin{aligned} \mu_\Phi^P(0 * x) &\geq \min\{\mu_\Phi^P(0 * (0 * x)), \mu_\Phi^P(0)\} = \mu_\Phi^P(0 * (0 * x)) \\ &\geq \min\{\mu_\Phi^P((0 * (0 * x)) * x), \mu_\Phi^P(x)\} \\ &= \min\{\mu_\Phi^P((0 * x) * (0 * x)), \mu_\Phi^P(x)\} \\ &= \min\{\mu_\Phi^P(0), \mu_\Phi^P(x)\} = \mu_\Phi^P(x) \end{aligned}$$

and

$$\begin{aligned} \mu_\Phi^N(0 * x) &\leq \max\{\mu_\Phi^N(0 * (0 * x)), \mu_\Phi^N(0)\} = \mu_\Phi^N(0 * (0 * x)) \\ &\leq \max\{\mu_\Phi^N((0 * (0 * x)) * x), \mu_\Phi^N(x)\} \\ &= \max\{\mu_\Phi^N((0 * x) * (0 * x)), \mu_\Phi^N(x)\} \\ &= \max\{\mu_\Phi^N(0), \mu_\Phi^N(x)\} = \mu_\Phi^N(x). \end{aligned}$$

Theorefore $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X. □

The following example shows that the converse of Theorem 3.14 may not be true.

Example 3.15. Let $X = \{0, 1, 2, 3, 4\}$ be a BCH-algebra which is given in Example 3.4. Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy set in X defined by

	0	1	2	3	4
μ_Φ^P	α_1	α_2	α_2	α_2	α_2
μ_Φ^N	β_1	β_2	β_2	β_2	β_2

where $\alpha_1 > \alpha_2$ in $[0, 1]$ and $\beta_1 < \beta_2$ in $[-1, 0]$. Then $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ is a bipolar fuzzy closed ideal of X which does not satisfy the bipolar fuzzy regularity.

Lemma 3.16. [4] *In a medial BCH-algebra* X, *we have* $x * y = 0 * (y * x)$ *for all* $x, y \in X$.

Theorem 3.17. *If* X *is medial, then every bipolar fuzzy closed ideal of* X *satisfies the bipolar fuzzy regularity.*

Proof. Assume that X is medial. Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy closed ideal of X and let $x, y \in X$. Then $\mu_\Phi^P(0 * (x * y)) \geq \mu_\Phi^P(x * y)$ and $\mu_\Phi^N(0 * (x * y)) \leq \mu_\Phi^N(x * y)$ by (3.3). It follows from (3.4) and Lemma 3.16 that

$$\begin{aligned} \mu_\Phi^P(y) &\geq \min\{\mu_\Phi^P(y * x), \mu_\Phi^P(x)\} \\ &= \min\{\mu_\Phi^P(0 * (x * y)), \mu_\Phi^P(x)\} \\ &\geq \min\{\mu_\Phi^P(x * y), \mu_\Phi^P(x)\} \end{aligned}$$

and

$$\begin{aligned} \mu_\Phi^N(y) &\leq \max\{\mu_\Phi^N(y * x), \mu_\Phi^N(x)\} \\ &= \max\{\mu_\Phi^N(0 * (x * y)), \mu_\Phi^N(x)\} \\ &\leq \max\{\mu_\Phi^N(x * y), \mu_\Phi^N(x)\}. \end{aligned}$$

Therefore $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ satisfies the bipolar fuzzy regularity. □

Using the notion of positive α-cut and negative β-cut, we state a characterization of bipolar fuzzy filters.

Theorem 3.18. *A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is a bipolar fuzzy filter of X if and only if for all $(\beta, \alpha) \in [-1, 0] \times [0, 1]$, the nonempty positive α-cut $P(\mu_\Phi^P; \alpha)$ and the nonempty negative β-cut $N(\mu_\Phi^N; \beta)$ are filters of X.*

Proof. Let $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ be a bipolar fuzzy filter of X and assume that $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$ are nonempty for $(\beta, \alpha) \in [-1, 0] \times [0, 1]$. If $x, y \in P(\mu_\Phi^P; \alpha)$ and $a, b \in N(\mu_\Phi^N; \beta)$, then $\mu_\Phi^P(x) \geq \alpha$, $\mu_\Phi^P(y) \geq \alpha$, $\mu_\Phi^N(a) \leq \beta$ and $\mu_\Phi^N(b) \leq \beta$. It follows from (3.6) that

$$\min\{\mu_\Phi^P(x \overleftarrow{\wedge} y), \mu_\Phi^P(y \overleftarrow{\wedge} x)\} \geq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\} \geq \alpha,$$
$$\max\{\mu_\Phi^N(a \overleftarrow{\wedge} b), \mu_\Phi^N(b \overleftarrow{\wedge} a)\} \leq \max\{\mu_\Phi^N(a), \mu_\Phi^N(b)\} \leq \beta$$

so that $\mu_\Phi^P(x \overleftarrow{\wedge} y) \geq \alpha$, $\mu_\Phi^P(y \overleftarrow{\wedge} x) \geq \alpha$, $\mu_\Phi^N(a \overleftarrow{\wedge} b) \leq \beta$ and $\mu_\Phi^N(b \overleftarrow{\wedge} a) \leq \beta$, that is, $x \overleftarrow{\wedge} y \in P(\mu_\Phi^P; \alpha)$, $y \overleftarrow{\wedge} x \in P(\mu_\Phi^P; \alpha)$, $a \overleftarrow{\wedge} b \in N(\mu_\Phi^N; \beta)$ and $b \overleftarrow{\wedge} a \in N(\mu_\Phi^N; \beta)$. Let $x, y \in X$ be such that $x \in P(\mu_\Phi^P; \alpha)$ and $x \leq y$. Then $\mu_\Phi^P(x) \geq \alpha$ and $x * y = 0$. Using (3.5), we have $\mu_\Phi^P(y) \geq \mu_\Phi^P(x) \geq \alpha$, and so $y \in P(\mu_\Phi^P; \alpha)$. Now let $a, b \in X$ be such that $a \in N(\mu_\Phi^N; \beta)$ and $a \leq b$. Then $\mu_\Phi^N(a) \leq \beta$ and $a * b = 0$. Using (3.5), we have $\mu_\Phi^N(b) \leq \mu_\Phi^N(a) \leq \beta$, and so $b \in N(\mu_\Phi^N; \beta)$. Hence $P(\mu_\Phi^P; \alpha)$ and $N(\mu_\Phi^N; \beta)$ are filters of X.

Conversely, suppose that $P(\mu_\Phi^P; \alpha) \neq \emptyset$ and $N(\mu_\Phi^N; \beta) \neq \emptyset$ are filters of X for all $(\beta, \alpha) \in [-1, 0] \times [0, 1]$. Assume that $\mu_\Phi^P(x_0) > \mu_\Phi^P(y_0)$ for some $x_0, y_0 \in X$ with $x_0 * y_0 = 0$. Then $\mu_\Phi^P(y_0) < \alpha_0 \leq \mu_\Phi^P(x_0)$ for some $\alpha_0 \in (0, 1)$. It follows that $x_0 \in P(\mu_\Phi^P; \alpha_0)$ and $y_0 \notin P(\mu_\Phi^P; \alpha_0)$. This is a contradiction. If $a_0, b_0 \in X$ satisfies $a_0 * b_0 = 0$ and $\mu_\Phi^N(a_0) < \mu_\Phi^N(b_0)$, then $\mu_\Phi^N(a_0) \leq \beta_0 < \mu_\Phi^N(b_0)$ for some $\beta_0 \in (-1, 0)$. Hence $a_0 \in N(\mu_\Phi^N; \beta_0)$ and $b_0 \notin N(\mu_\Phi^N; \beta_0)$, which is a contradiction. Therefore (3.5) is valid. Now, if (3.6) is not valid, then

$$\min\{\mu_\Phi^P(x \overleftarrow{\wedge} y), \mu_\Phi^P(y \overleftarrow{\wedge} x)\} < \alpha_0 \leq \min\{\mu_\Phi^P(x), \mu_\Phi^P(y)\} \tag{3.9}$$

or

$$\max\{\mu_\Phi^N(a \overleftarrow{\wedge} b), \mu_\Phi^N(b \overleftarrow{\wedge} a)\} > \beta_0 \geq \max\{\mu_\Phi^N(a), \mu_\Phi^N(b)\} \tag{3.10}$$

for some $x, y, a, b \in X$, $\alpha_0 \in (0, 1)$ and $\beta_0 \in (-1, 0)$. Thus (3.9) implies that $x, y \in P(\mu_\Phi^P; \alpha_0)$, but $x \overleftarrow{\wedge} y \notin P(\mu_\Phi^P; \alpha_0)$ or $y \overleftarrow{\wedge} x \notin P(\mu_\Phi^P; \alpha_0)$. If the case (3.10) is occur, then $a, b \in N(\mu_\Phi^N; \beta_0)$, but $a \overleftarrow{\wedge} b \notin N(\mu_\Phi^N; \beta_0)$ or $b \overleftarrow{\wedge} a \notin N(\mu_\Phi^N; \beta_0)$. This is a contradiction. Therefore (3.6) is valid. This completes the proof. □

Corollary 3.19. *A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is a bipolar fuzzy filter of X, then the following assertion is valid:*

$(\forall(\beta, \alpha) \in [-1, 0] \times [0, 1])(B(\Phi; (\alpha, \beta)) \neq \emptyset \ \Rightarrow \ B(\Phi; (\alpha, \beta))$ *is a filter of* $X)$.

Corollary 3.20. *A bipolar fuzzy set $\Phi = (X; \mu_\Phi^P, \mu_\Phi^N)$ in X is a bipolar fuzzy filter of X, then the following assertion is valid:*

$(\forall\alpha \in [0, 1])(B(\Phi; (\alpha, -\alpha)) \neq \emptyset \ \Rightarrow \ B(\Phi; (\alpha, -\alpha))$ *is a filter of* $X)$.

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