Interval Valued \((\in, \in \vee q)\)-Fuzzy Ideal in Rings

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Abstract
The notion of an interval-valued \((\in, \in \vee q)\)-fuzzy subring(ideal,prime) in ring is introduced and their characterizations are investigated.

Mathematics Subject Classification: 03E72,16D25

Keywords: quasi-coincidence,interval-valued \((\in, \in \vee q)\)-fuzzy subring(ideal, prime)

1 Introduction
Fuzzy set was initiated by Zadeh[10] and so many researchers were conducted on the generalizations of the notion of fuzzy sets. In [11], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set. Liu applied the concept of fuzzy sets to the theory of rings and introduced the notions of fuzzy subring and fuzzy ideal of a ring[8]. This concept discussed further by many researchers[1, 3, 4, 6, 7]. In [9], Ming and Ming introduced the concept of quasi-coincidence of a fuzzy point with a fuzzy subset. Based on quasi-coincidence, S.K. Bhakat and P.Das[2] introduced a new type fuzzy subring(ideal,prime) of ring called an \((\in, \in \vee q)\)-fuzzy subring(ideal,prime).

In this paper, we concentrate on the quasi-coincidence of a fuzzy interval value with an interval valued fuzzy set and introduced the notions of \((\in, \in \vee q)\)-fuzzy subring(ideal,prime) which is an extended notion of \((\in, \in \vee q)\)-fuzzy subring(ideal,prime). And we give some interesting properties are investigated.

2 Preliminaries
Let \(R\) be a ring. By a subring of \(R\) we mean a nonempty subset \(S\) of \(R\) such that \(S\) is closed under the operations of addition and multiplication in \(R\). A subring \(I\) of a ring \(R\) is called an ideal of \(R\) if for all \(x \in R, r \in I, rx, xr \in I\).
Note that a nonempty subset \( I \) of a ring \( R \) is an ideal if and only if it satisfies: (i) for all \( a, b \in I \), \( a - b \in I \), (ii) for all \( a \in I \), \( ra, ar \in I \) implies \( a \in I \) or \( b \in I \).

An ideal \( I \) of a ring \( R \) is called a prime if for all \( a, b \in R \), \( ab \in I \) implies \( a \in I \) or \( b \in I \).

We now review some fuzzy logic concepts. Let \( X \) be a set. A fuzzy set in \( X \) is a function \( \mu : X \to [0,1] \).

**Definition 2.1.** [8] A fuzzy set \( \mu \) of a ring \( R \) is called a fuzzy subring(ideal) of \( R \) if it satisfies:

(i) \( \forall x, y \in R, \mu(x-y) \geq \min\{\mu(x), \mu(y)\} \),

(ii) \( \forall x, y \in R, \mu(xy) \geq \min\{\mu(x), \mu(y)\}\{\mu(xy) \geq \mu(x)\} \).

**Definition 2.2.** [5] A fuzzy ideal \( \mu \) of a ring \( R \) is called a fuzzy prime ideal of \( R \) if it satisfies: \( \forall x, y \in R, \mu(xy) = \mu(x) \) or \( \mu(xy) = \mu(y) \).

A fuzzy set \( \mu \) in a set \( X \) of the form

\[
\mu(y) := \begin{cases} 
  t \in (0,1] & \text{if } y = x, \\
  0 & \text{if } y \neq x,
\end{cases}
\]

is called a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \).

For a fuzzy point \( x_t \) and a fuzzy set \( \mu \) in a set \( X \), To say that \( x_t \in \mu \) (resp. \( x_t \in q_\mu \)) means that \( \mu(x) \geq t \) (resp. \( \mu(x) + t > 1 \)), and in this case, \( x_t \) is said to belong to (resp. be quasi-coincident with) a fuzzy set \( \mu \). To say that \( x_t \in \vee q_\mu \) (resp. \( x_t \in \wedge q_\mu \)) means that \( x_t \in \mu \) or \( x_t \in q_\mu \) (resp. \( x_t \in \mu \) and \( x_t \in q_\mu \)) [9].

Based on membership and quasi-coincidence, S.K. Bhakat and P. Das introduced the notion of \((\in, \in \vee q)\)-fuzzy subrings and ideals of a ring [2].

The notion of interval valued fuzzy set was introduced by Zadeh [10, 11]. To consider the notion of interval valued fuzzy set, we need following notations.

By an interval number \( \hat{a} \), we mean an interval \([\underline{a}, \overline{a}]\), where \( 0 \leq \underline{a} \leq \overline{a} \leq 1 \). The interval \([a, a]\) can be simply identified with the number \( a \in [0,1] \). Let \( D[0,1] \) denotes the set of all interval numbers. Consider the interval numbers

\[
\hat{a}_i = [\underline{a}_i, \overline{a}_i], \hat{b}_i = [\underline{b}_i, \overline{b}_i] \in D[0,1], i \in I,
\]

we define

\[
\begin{align*}
\text{rmin}(\hat{a}_i, \hat{b}_i) &= [\min\{\underline{a}_i, \underline{b}_i\}, \min\{\overline{a}_i, \overline{b}_i\}], \\
\text{rmax}(\hat{a}_i, \hat{b}_i) &= [\max\{\underline{a}_i, \underline{b}_i\}, \max\{\overline{a}_i, \overline{b}_i\}], \\
\text{rinfs}_i &= \left[\bigwedge_{i \in I} \underline{a}_i, \bigwedge_{i \in I} \overline{a}_i\right], \text{rsups}_i = \left[\bigvee_{i \in I} \underline{a}_i, \bigvee_{i \in I} \overline{a}_i\right].
\end{align*}
\]

We also define the symbols “\( \leq \)”, “\( = \)”, “\( < \)” in case of two interval numbers in \( D[0,1] \).

1. \( \hat{a}_1 \leq \hat{a}_2 \) if and only if \( \underline{a}_1 \leq \underline{a}_2 \) and \( \overline{b}_1 \leq \overline{b}_2 \).

2. \( \hat{a}_1 = \hat{a}_2 \) if and only if \( \underline{a}_1 = \underline{a}_2 \) and \( \overline{b}_1 = \overline{b}_2 \).

3. \( \hat{a}_1 < \hat{a}_2 \) if and only if \( \underline{a}_1 < \underline{a}_2 \) and \( \overline{b}_1 < \overline{b}_2 \).
Under these notation, the concept of an interval-valued fuzzy set defined on a non-empty set $X$ as objects having the form

$$A = \{(x, [\mu_A(x), \overline{\mu}_A(x)]) : x \in X\},$$

where $\mu_A$ and $\overline{\mu}_A$ are two fuzzy sets in $X$ such that $\mu_A(x) \leq \overline{\mu}_A(x)$ for all $x \in X$. Let $\widehat{\mu}_A(x) = [\mu_A(x), \overline{\mu}_A(x)]$, $\forall x \in X$. Then $\widehat{\mu}_A(x) \in D[0,1]$, $\forall x \in X$, and therefore the interval-valued fuzzy set $A$ is given by

$$A = \{(x, \widehat{\mu}_A(x)) : x \in X\}, \text{ where } \widehat{\mu}_A : X \to D[0,1].$$

For a given interval valued fuzzy sets $A$ and $B$ in a set $X$, we define

- $A \subseteq B \iff (\forall x \in X) (\mu_A(x) \leq \mu_B(x), \overline{\mu}_A(x) \leq \overline{\mu}_B(x)).$
- $A = B \iff A \subseteq B$ and $B \subseteq A.$
- $A \cap B = \{(x, [\min\{\mu_A(x), \mu_B(x)\}, \min\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}]) : x \in X\}$
- $A \cup B = \{(x, [\max\{\mu_A(x), \mu_B(x)\}, \max\{\overline{\mu}_A(x), \overline{\mu}_B(x)\}]) : x \in X\}.$

### 3 Interval valued $(\in, \in \vee q)$-fuzzy subrings and fuzzy ideals

In what follows, $R$ is a ring unless otherwise specified. An interval valued fuzzy set $A$ in $R$ of the form

$$\widehat{\mu}_A(y) = \begin{cases} \widehat{a} \neq [0,0] & \text{if } y = x, \\ \text{if } y \neq x \end{cases}$$

is called an fuzzy interval value with support $x$ and interval value $\widehat{a}$ and is denoted by $\mathcal{U}(x; \widehat{a})$

Throughout this paper, we assume that $\widehat{\mu}_A(x) = [\mu_A(x), \overline{\mu}_A(x)]$ must satisfy the following two properties:

1. Any two interval numbers of $D[0,1]$ are comparable;
2. $r\min\{\mu_A(x), \overline{\mu}_A(x)\} < [0.5, 0.5]$ or $r\min\{\mu_A(x), \overline{\mu}_A(y)\} \geq [0.5, 0.5]$ for all $x \in R.$

We say that a fuzzy interval value $\mathcal{U}(x; \widehat{a})$ belong to (resp. is a quasi-coincident with) an interval valued fuzzy set $A$, written by $\mathcal{U}(x; \widehat{a}) \in A(\text{resp. } \mathcal{U}(x; \widehat{a}) \in qA)$, if $\widehat{\mu}_A(x) \geq \widehat{a}$ (resp. $\widehat{\mu}_A(x) + \widehat{a} > [1,1]$). If $\widehat{\mu}_A(x) < \widehat{a}$ (resp. $\widehat{\mu}_A(x) + \widehat{a} \leq [1,1]$), then we write $\mathcal{U}(x; \widehat{a}) \notin A(\text{resp. } \mathcal{U}(x; \widehat{a}) \notin qA)$. If $\mathcal{U}(x; \widehat{a}) \in A$ or $\mathcal{U}(x; \widehat{a}) \in qA$, then we write $\mathcal{U}(x; \widehat{a}) \in \in qA$. The symbol $\in \in \vee q$ means $\in \vee q$ does not hold.
Definition 3.1. An interval valued fuzzy set $A$ in $R$ is called a $(\in, \in \Delta)$-fuzzy subring of $R$ if for all $x, y \in R$ and $a, b \in (0, 1]$, 

(S1) $U(x; \hat{a})$ and $U(y; \hat{b}) \in A$ imply $U(x + y; \min\{\hat{a}, \hat{b}\}) \in \in \Delta \setminus qA$, 

(S2) $U(x; \hat{a}) \in A$ imply $U(-x; \hat{a}) \in \in \Delta \setminus qA$, 

(S3) $U(x; \hat{a})$ and $U(y; \hat{b}) \in A$ imply $U(xy; \min\{\hat{a}, \hat{b}\}) \in \in \Delta \setminus qA$.

Theorem 3.2. $A$ is an interval valued $(\in, \in \Delta)$-fuzzy subring of $R$ if and only if for all $x, y \in R$ the following three conditions are satisfied:

(S4) $\hat{\mu}_A(x + y) \geq \min\{\hat{\mu}_A(x), \hat{\mu}_A(y), [0.5, 0.5]\}$,

(S5) $\hat{\mu}_A(-x) \geq \min\{\hat{\mu}_A(x), [0.5, 0.5]\}$,

(S6) $\hat{\mu}_A(xy) \geq \min\{\hat{\mu}_A(x), \hat{\mu}_A(y), [0.5, 0.5]\}$.

Proof. (S1) $\Rightarrow$ (S4) Assume that (S4) is not valid, then there exists $x, y \in R$ such that $\hat{\mu}_A(x + y) < \min\{\hat{\mu}_A(x), \hat{\mu}_A(y), [0.5, 0.5]\}$. We consider the following two cases:

(i) $\min\{\hat{\mu}_A(x), \hat{\mu}_A(y)\} < [0.5, 0.5]$ and (ii) $\min\{\hat{\mu}_A(x), \hat{\mu}_A(y)\} \geq [0.5, 0.5]$.

For the case (i) we have $\hat{\mu}_A(x + y) < \min\{\hat{\mu}_A(x), \hat{\mu}_A(y)\}$. Choose $\hat{a}$ such that $\hat{\mu}_A(x + y) < \hat{a} < \min\{\hat{\mu}_A(x), \hat{\mu}_A(y)\}$. Then $U(x; \hat{a}), U(y; \hat{a}) \in A$. Since $\hat{\mu}_A(x + y) < \hat{a}$ and $\hat{\mu}_A(x + y + \hat{a}) < [1, 1]$, we have $U(x + y; \hat{a}) \notin A$ and $U(x + y; \hat{a}) \notin qA$. From this implies that $U(x + y; \hat{a}) \notin \in \Delta \setminus qA$, which contradictis (S1). For the (ii) case we have $\hat{\mu}_A(x + y) < [0.5, 0.5] \leq \min\{\hat{\mu}_A(x), \hat{\mu}_A(y)\}$. Then $U(x; [0.5, 0.5]), U(x; [0.5, 0.5]) \in A$. But $U(x + y; [0.5, 0.5]) \in \in \Delta \setminus qA$ which contradictis (S1). Therefore (S4) holds.

(S4) $\Rightarrow$ (S1) Let $x, y \in R$ and $a, b \in (0, 1]$ be such that $U(x; \hat{a}) \in A$ and $U(y; \hat{b}) \in A$. Then $\hat{\mu}_A(x) \geq \hat{a}$ and $\hat{\mu}_A(y) \geq \hat{b}$. Now we have $\hat{\mu}_A(x + y) \geq \min\{\hat{\mu}_A(x), \hat{\mu}_A(y), [0.5, 0.5]\} \geq \min\{\hat{a}, \hat{b}, [0.5, 0.5]\}$. If $\min\{\hat{a}, \hat{b}\} < [0.5, 0.5]$ we have $\hat{\mu}_A(x + y) \geq \min\{\hat{a}, \hat{b}\}$. If $\min\{\hat{a}, \hat{b}\} \geq [0.5, 0.5]$, which implies that $\hat{\mu}_A(x + y) \geq [0.5, 0.5]$, according to $\hat{\mu}_A(x + y) + \min\{\hat{a}, \hat{b}\} \geq [1, 1]$. Hence $U(x + y; \min\{\hat{a}, \hat{b}\}) \in \in \Delta \setminus qA$. Therefore (S3) holds.

(S2) $\Rightarrow$ (S5) Assume that (S5) is not valid, then there exists $x \in R$ such that $\hat{\mu}_A(-x) < \min\{\hat{\mu}_A(x), [0.5, 0.5]\}$. We consider the following two caesas:

(i) $\hat{\mu}_A(x) < [0.5, 0.5]$ and (ii) $\hat{\mu}_A(x) \geq [0.5, 0.5]$.

For the case (i) we have $\hat{\mu}_A(-x) = \hat{r} < \hat{\mu}_A(x) = \hat{t}$. Choose $\hat{a}$ such that $\hat{r} < \hat{a} < \hat{t}$ and $\hat{r} + \hat{t} < [1, 1]$. Then $U(x; [0.5, 0.5]) \in A$, but $U(-x; [0.5, 0.5]) \in \in \Delta \setminus qA$ which contradictis (S1). For the (ii) case we have $\hat{\mu}_A(-x) < \min\{\hat{\mu}_A(x), [0.5, 0.5]\}$.
Proof. Suppose that interval valued \((U(S^5))\) holds.

(S5) \implies (S2) Let \(x \in R\) and \(a \in (0, 1)\) be such that \(U(x; a) = 0\). Then \(\mu^a(x) \geq a\). Now we have \(\hat{\mu}_A(x) \geq \hat{a}\). Therefore \((S4)\) holds.

(S3) \implies (S6) This proof is similar to \((S2) \iff (S5)\)

\(\square\)

**Definition 3.3.** An interval valued fuzzy set \(A\) in \(R\) is called an \((\xi, \eta)\)-fuzzy ideal of \(R\) if for all \(x, y \in R\) and \(t \in (0, 1)\),

1. \(A\) is an interval valued \((\xi, \eta)\)-fuzzy subring of \(R\),
2. \(U(x; \hat{t}) \in A\) and \(y \in R\) imply \(U(xy; \hat{t}), U(yx; \hat{t}) \in \eta A\).

**Theorem 3.4.** \(A\) is an interval valued \((\xi, \eta)\)-fuzzy ideal of \(R\) if and only if for all \(x, y \in R\) the following three conditions are satisfied:

1. \(\hat{\mu}_A(x - y) \geq \text{rmin}\{\hat{\mu}_A(x), \hat{\mu}_A(y), [0.5, 0.5]\}\),
2. \(\hat{\mu}_A(xy), \hat{\mu}_A(yx) \geq \text{rmin}\{\hat{\mu}_A(x), [0.5, 0.5]\}\).

Proof. Straightforward.

**Definition 3.5.** An interval valued fuzzy set \(A\) in \(R\) is called an \((\xi, \eta)\)-fuzzy prime ideal of \(R\) if it satisfies: \(\forall x, y \in R, t \in (0, 1)\),

\[U(xy; \hat{t}) \in A \implies U(x; \hat{t}) \in \eta A\text{ or } U(y; \hat{t}) \in \eta A\]

**Lemma 3.6.** Let \(A\) be a subset of \(R\). A characteristic function \(\chi_A\) is an interval valued \((\xi, \eta)\)-fuzzy ideal of \(R\) if and only if \(A\) is an ideal of \(R\).

Proof. Suppose that \(\chi_A\) is an interval valued \((\xi, \eta)\)-fuzzy ideal of \(R\). Let \(x, y \in A\). Then \(\chi_A(x) = [1, 1] = \chi_A(y)\), and so

\[\chi_A(x - y) \geq \text{rmin}\{\chi_A(x), \chi_A(y), [0.5, 0.5]\} = [0.5, 0.5]\]

It follows that \(\chi_A(x - y) = [1, 1]\) so that \(x - y \in A\). On the other hand, if \(a \in A\) and \(r \in R\), then \(\chi_A(ar) \geq \text{rmax}\{\chi_A(a), \chi_A(r)\} = \text{rmax}\{[1, 1], \chi_A(r)\} = [1, 1]\) and \(\chi_A(ra) \geq \text{rmax}\{\chi_A(r), \chi_A(a)\} = \text{rmax}\{\chi_A(r), [1, 1]\} = [1, 1]\) so that \(ar \in A\) and \(ra \in A\). Therefore \(A\) is an ideal of \(R\).

Conversely, assume that \(A\) be an ideal of ring \(R\). It is clear that \(U(\chi_A; [1, 1]) = A\). We first show that \(\chi_A(x - y) \geq \text{rmin}\{\chi_A(x), \chi_A(y), [0.5, 0.5]\}\) for all \(x, y \in R\). Let \(x, y \in R\). If \(x, y \in A\), then \(x - y \in A\) and so

\[\chi_A(x - y) = [1, 1] \geq \text{rmin}\{\chi_A(x), \chi_A(y), [0.5, 0.5]\} = [0.5, 0.5]\]
If \( x, y \notin A \), then \( \chi_A(x) = [0, 0] = \chi_A(y) \) and thus
\[
\chi_A(x - y) \geq \operatorname{rmin}\{\chi_A(x), \chi_A(y), [0.5, 0.5]\} = [0.0].
\]

If \( x \in A \) and \( y \notin A \), then \( \chi_A(x) = [1, 1] \) and \( \chi_A(y) = [0, 0] \). It follows that
\[
\chi_A(x - y) \geq \operatorname{rmin}\{\chi_A(x), \chi_A(y), [0.5, 0.5]\} = [0.0].
\]

Similarly for the case \( x \notin A \) and \( y \in A \), we get
\[
\chi_A(x - y) \geq \operatorname{rmin}\{\chi_A(x), \chi_A(y), [0.5, 0.5]\}
\]
. Similary, it can be shown that \( \chi_A(xy), \chi_A(yx) \geq \operatorname{rmin}\{\chi_A(x), [0.5, 0.5]\} \) for all \( x, y \in R \). Therefore \( A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \), and the proof is complete.

**Theorem 3.7.** Let \( A \) be a subset of \( R \). A function \( \chi_A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy prime ideal if and only if \( A \) is a prime ideal of \( R \).

**Proof.** Let \( \chi_A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy prime ideal of \( R \). By Lemma 3.6, we know that \( \chi_A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \). Now we assume that \( xy \in A \) for all \( x, y \in R \). Then \( \chi_A(xy) = [1, 1] \), which implies \( U(xy; \hat{t}) \in \chi_A \) for all \( t \in (0, 1] \). Because of primality of \( \chi_A \), we have \( U(x; \hat{t}) \in \vee q\chi_A \) or \( U(y; \hat{t}) \in \vee q\chi_A \). Note that \( \chi_A(x) > [0, 0] \) or \( \chi_A(y) > [0, 0] \). This implies that \( \chi_A(x) = [1, 1] \) or \( \chi_A(y) = [1, 1] \) and hence \( x \in A \) or \( y \in A \). Therefore \( A \) is prime ideal of \( R \). Conversely, \( A \) is a prime ideal of \( R \). By Lemma 3.6, we know that \( \chi_A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \). Let \( U(xy; \hat{t}) \in \chi_A \) for all \( x, y \in R \) and \( t \in (0, 1] \). Then \( \chi_A(xy) = [1, 1] \), which implies \( xy \in A \). Because of primality of \( A \), we have \( x \in A \) or \( y \in A \). This implies that \( U(x; \hat{t}) \in \chi_A \) or \( U(y; \hat{t}) \in \vee q\chi_A \). Therefore \( \chi_A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy prime ideal of \( R \).

Let \( R \) be a ring. Then, for an interval valued fuzzy set \( A \) of \( R \), the **level subset** of \( A \) in \( R \) is defined to be the following subset of \( R \),
\[
U(A; \hat{t}) = \{ x \in R \mid \mu_A(x) \geq \hat{t} \} \text{ for } t \in (0, 1].
\]

**Theorem 3.8.** Let \( A \) be an interval valued fuzzy subset in \( R \). Then \( A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \) if and only if \( U(A; \hat{t}) \) is an ideal of \( R \) for every \( t \in (0, 0.5] \).

**Proof.** Assume that \( A \) is an interval valued \((\varepsilon, \in \vee q)\)-fuzzy ideal of \( R \) and let \( t \in (0, 0.5] \) be such that \( x, y \in U(A; \hat{t}) \) Then
\[
\mu_A(x - y) \geq \operatorname{rmin}\{\mu_A(x), \mu_A(y), [0.5, 0.5]\} \geq \operatorname{rmin}\{\hat{t}, [0.5, 0.5]\} = \hat{t}
\]
and so \( x - y \in \mathcal{U}(A; \hat{t}) \). Let \( x \in \mathcal{U}(A; \hat{t}) \) and \( r \in R \). Then we have
\[
\mu(xr), \mu(rx) \geq \text{rmin}\{\mu(x), [0.5, 0.5]\} \geq \text{rmin}\{\hat{t}, [0.5, 0.5]\} = \hat{t}.
\]
Hence \( xr, rx \in \mathcal{U}(A; \hat{t}) \). Thus \( \mathcal{U}(A; \hat{t}) \) is an ideal of \( R \) for every \( t \in (0, 0.5] \).

Conversely, let \( A \) be an interval valued fuzzy subset in ring \( R \) such that \( \mathcal{U}(A; \hat{t}) \) is an ideal of \( R \) for every \( t \in (0, 0.5] \). Assume that \( A \) is not an interval valued \((\in, \in \vee q)\)-fuzzy ideal of \( R \), then there exists \( x, y \in R \) such that \( \mu(x - y) < \text{rmin}\{\mu(x), \mu(y), [0.5, 0.5]\} \). Then we can choose \( \hat{t} \) such that \( \mu(x - y) < \hat{t} < \text{rmin}\{\mu(x), \mu(y), [0.5, 0.5]\} \). Thus \( x, y \in \mathcal{U}(A; \hat{t}) \). Since \( \mathcal{U}(A; \hat{t}) \) is an ideal of \( R \), we have \( x - y \in \mathcal{U}(A; \hat{t}) \). Thus \( \mu(x - y) \geq \hat{t} \), a contradiction. Hence \( \mu(x - y) \geq \text{rmin}\{\mu(x), \mu(y), [0.5, 0.5]\} \) for all \( x, y \in R \). Similarly, we get \( \mu(xy), \mu(yx) \geq \text{rmin}\{\mu(x), [0.5, 0.5]\} \) for all \( x, y \in R \). Therefore \( A \) is an interval valued \((\in, \in \vee q)\)-fuzzy ideal of \( R \).

**Theorem 3.9.** Let \( A \) be an interval valued fuzzy subset of \( R \). Then \( A \) is an interval valued \((\in, \in \vee q)\)-fuzzy prime ideal of \( R \) if and only if \( \mu(x) \) satisfies the following assertions:
\[
(\forall x, y \in R) \,(\text{rmax}\{x, y\}) \geq \text{rmin}\{x, y, [0.5, 0.5]\}).
\]

**Proof.** Let \( A \) be an interval valued \((\in, \in \vee q)\)-fuzzy prime ideal of \( R \). If there exist \( x, y \in R \) such that \( \text{rmax}\{\mu(x), \mu(y)\} < \text{rmin}\{\mu(x), \mu(y), [0.5, 0.5]\} \). Then we can choose \( \hat{t} \) such that \( \text{rmax}\{\mu(x), \mu(y)\} < \hat{t} < \text{rmin}\{\mu(x), \mu(y), [0.5, 0.5]\} \). Thus \( \mu(x) < \hat{t} < \mu(y) \), which implies \( \mu(x) + \mu(y) < 1 \), a contradiction. Conversely, let \( \mu\) satisfy \( \mu(x) \geq \text{rmin}\{\mu(x), [0.5, 0.5]\} \) for all \( x \in R \). Then \( \mu(x) \geq \text{rmin}\{\mu(x), [0.5, 0.5]\} \) for all \( x \in R \). This imply \( \mathcal{U}(x; \hat{t}) \in \mathcal{U}(y; \hat{t}) \in \mathcal{U}(A; \hat{t}) \). Therefore \( A \) is an interval valued \((\in, \in \vee q)\)-fuzzy prime ideal of \( R \).

**Theorem 3.10.** Let \( A \) be an interval valued fuzzy subset of \( R \). An interval valued \((\in, \in \vee q)\)-fuzzy ideal is interval valued \((\in, \in \vee q)\)-fuzzy prime ideal if and only if \( \mathcal{U}(A; \hat{t}) \) is a prime ideal of \( R \) for all \( t \in (0, 0.5] \).

**Proof.** Assume that \( A \) is an interval valued \((\in, \in \vee q)\)-fuzzy prime ideal of \( R \). By Theorem 3.9, we know that \( \mathcal{U}(A; \hat{t}) \) is an ideal of \( R \) for every \( t \in (0, 0.5] \). Let \( xy \in \mathcal{U}(A; \hat{t}) \). Since \( A \) is an interval valued \((\in, \in \vee q)\)-fuzzy prime ideal of \( R \),
\[
\text{rmax}\{\mu(x), \mu(y)\} \geq \text{rmin}\{\mu(x), [0.5, 0.5]\} \geq \text{rmin}\{\hat{t}, [0.5, 0.5]\} = \hat{t}
\]
and so \( \mathcal{U}(x; \hat{t}) \in A \) or \( \mathcal{U}(y; \hat{t}) \in A \). Therefore \( \mathcal{U}(A; \hat{t}) \) is a prime ideal of \( R \) for every \( t \in (0, 0.5] \).

Conversely, let \( \mathcal{U}(A; \hat{t}) \) is a prime ideal of \( R \) for every
$t \in (0,0.5]$. By Theorem 3.9, we know that $A$ is an interval valued $(\in, \in \lor q)$-fuzzy ideal of $R$. Let $\hat{t} \leq [0.5,0.5]$. Assume that $\mathcal{U}(xy;\hat{t}) \in A$. Since $\mathcal{U}(A;\hat{t})$ is a prime ideal of $R$, $x \in \mathcal{U}(A;\hat{t})$ or $y \in \mathcal{U}(A;\hat{t})$, and so $\mathcal{U}(x;\hat{t}) \in A$ or $\mathcal{U}(y;\hat{t}) \in A$. If $t > [0.5,0.5]$, since $\mathcal{U}(A;[0.5,0.5])$ is prime, we have $x \in \mathcal{U}(A;[0.5,0.5])$ or $y \in \mathcal{U}(A;[0.5,0.5])$. Hence $\mathcal{U}(x;\hat{t}) \in \lor qA$ or $\mathcal{U}(y;\hat{t}) \in \lor qA$. Therefore $A$ is an interval valued $(\in, \in \lor q)$-fuzzy ideal of $R$.

References


Received: August, 2008