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Ideals of Subtraction Algebras Based on Interval Valued Fuzzy Set

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Abstract

The notion of an interval-valued fuzzy ideal in subtraction algebra is introduced and related properties are investigated.

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1 Introduction

The notion of a fuzzy set was introduced by Zadeh in 1965 [5]. In [6], Zadeh made an extension of the concept of a fuzzy set by an interval-valued fuzzy set. In [4], B. M. Schein considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). Since then, Y. B. Jun et al.[2] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. Lee and Park[3] introduced the notion of a fuzzy ideal in subtraction algebras. In this paper, we introduce the notion of a interval valued fuzzy ideal in subtraction algebras and investigate related properties.

2 Preliminaries

We review some deinitions and properties that are necessary for this paper.

A non-empty set X with a single binary operation "$-$' is called a *subtraction algebra* if it satisfies following axioms:

(S1) $x - (y - x) = x$,

(S2) $x-(x-y)=y-(y-x)$,

(S3) $(x-y)-z=(x-z)-y$.

for all $x,y,z\in X$,

Note that (S3) permits us to omit parentheses in expressions of the form $(x-y)-z$. A partial ordering " $\leq$ " on X can be defined by

$$a\leq b \Leftrightarrow a-b=0,$$

where $0=a-a$ is an element that does not depend on the choice of $a\in X$. Then, it is clear that $(X;\leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0,a]$ is a Boolean algebra with respect to the induced order. Here $a\wedge b=a-(a-b)$; the complement of an element $b\in[0,a]$ is $a-b$; and if $b,c\in[0,a]$, then

$$\begin{aligned} b\vee c &= (b'\wedge c')'=a-((a-b)\wedge(a-c)) \\ &= a-((a-b)-((a-b)-(a-c))). \end{aligned}$$

Any subtraction algebra X has the following properties(see[2]):

(a1) $(x-y)-y=x-y$.

(a2) $x-0=x$ and $0-x=0$.

(a3) $(x-y)-x=0$.

(a4) $x-(x-y)\leq y$.

(a5) $(x-y)-(y-x)=x-y$.

(a6) $x-(x-(x-y))=x-y$.

(a7) $(x-y)-(z-y)\leq x-z$.

(a8) $x\leq y$ if and only if $x=y-w$ for some $w\in X$.

(a9) $x\leq y$ implies $x-z\leq y-z$ and $z-y\leq z-x$ for all $z\in X$.

(a10) $x,y\leq z$ implies $x-y=x\wedge(z-y)$.

(a11) $(x\wedge y)-(x\wedge z)\leq x\wedge(y-z)$.

Definition 2.1. *[2] A subset A of subtraction algebra X is called an ideal of X if it satisfies:*

(b1) $(\forall a\in A)(\forall x\in A)(x-y\in A)$,

(b2) $(\forall a,b\in X)$ $(\exists a\vee b\Rightarrow a\vee b\in A)$.

Proposition 2.2. *[2] A subset A of subtraction algebra X is called an ideal of X if and only if it satisfies:*

(b3) $0 \in A$,

(b4) $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Proposition 2.3. *[2] Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

Now we review some fuzzy logic concepts. A *fuzzy set* in X is a function $\mu : X \to [0, 1]$. An *interval-valued fuzzy set* (briefly, *i-v fuzzy set*) A defined on X is given by

$$\mathrm{A} = \{(x, [\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^U(x)]) \mid x \in X\}$$

where μ_{A}^L and μ_{A}^U are two fuzzy sets in X such that $\mu_{\mathrm{A}}^L(x) \leq \mu_{\mathrm{A}}^U(x)$ for all $x \in X$. For the sake of simplicity, we shall use the notation $\mathrm{A} = [\mu_{\mathrm{A}}^L, \mu_{\mathrm{A}}^U])$ instead of $\mathrm{A} = \{(x, [\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^U(x)]) \mid x \in X\}$. Let $\bar{\mu}_{\mathrm{A}}(x) = \left[\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^U(x)\right], \forall x \in X$ and let $D[0, 1]$ denotes the family of all closed subintervals of $[0, 1]$. If $\mu_{\mathrm{A}}^L(x) = \mu_{\mathrm{A}}^U(x) = c$ (say) where $0 \leq c \leq 1$, then we have $\bar{\mu}_{\mathrm{A}}(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0, 1]$. Thus $\bar{\mu}_{\mathrm{A}}(x) \in D[0, 1]$, $\forall x \in X$, and therefore the i-v fuzzy set A is given by

$$A = \{(x, \bar{\mu}_{\mathrm{A}}(x))\}, \forall x \in X, \text{ where } \bar{\mu}_{\mathrm{A}} : X \to D[0, 1].$$

Now let us define what is known as *refined minimum* (briefly, rmin) of two elements in $D[0, 1]$. Let $D_1, D_2 \in D[0, 1]$, where $D_1 := [a_1, b_1]$, $D_2 := [a_2, b_2]$. Define $\mathrm{rmin}(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$.

We also define

$$D_1 \leq D_2 \text{ if and only if } a_1 \leq a_2 \text{ and } b_1 \leq b_2;$$
$$D_1 = D_2 \text{ if and only if } D_1 \leq D_2 \text{ and } D_1 \leq D_2.$$

Definition 2.4. *[3] A fuzzy set μ in X is called a fuzzy ideal of X if it satisfies:*

(c1) $(\forall x, y \in X)\ (\mu(x - y) \geq \mu(x))$,

(c2) $(\forall x, y \in X)\ (\exists x \vee y \Rightarrow \mu(x \vee y) \geq \min\{\mu(x), \mu(y)\})$.

3 Interval-valued fuzzy ideals

In what follows, X denotes a subtraction algebra unless otherwise specified. First, we can extend the concept of fuzzy ideal to the concept of interval valued fuzzy ideal of X as follows:

Definition 3.1. *An i-v fuzzy set* A *in* X *is called a interval-valued fuzzy ideal (briefly, i-v fuzzy ideal) of* X *if it satisfies:*

(d1) $(\forall x, y \in X)\ (\bar{\mu}_{\mathrm{A}}(x - y) \geq \bar{\mu}_{\mathrm{A}}(x))$,

(d2) $(\forall x, y \in X)\ (\exists x \vee y \Rightarrow \bar{\mu}_{\mathrm{A}}(x \vee y) \geq \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\})$.

Example 3.2. *Let* $X = \{0, 1, 2\}$ *be a set with the following Cayley table:*

$-$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X; -)$ *is a subtraction algebra (see [3]). Define an i-v fuzzy set* A *in* X *defined by*

$$\bar{\mu}_{\mathrm{A}}(x) = \begin{cases} [0.5, 0.7] & \text{if } x = 0 \\ \quad , 0.5 & \\ & \text{if } x = 1 \\ \quad , 0.6 & \\ & \text{if } x = 2. \end{cases}$$

By routine calculations we know that A *is an i-v fuzzy ideal of* X.

Proposition 3.3. *Let* A *be an i-v fuzzy set* A *of* X *such that*

(e1) $(\forall x, a, b \in X)\,(\bar{\mu}_{\mathrm{A}}(x - ((x - a) - b)) \geq \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}(a), \bar{\mu}_{\mathrm{A}}(b)\})$.

Then A *is an i-v fuzzy ideal of* X.

Proof. For any $x, y \in X$, we can derive

$$\begin{aligned} \bar{\mu}_{\mathrm{A}}(x - y) &= \bar{\mu}_{\mathrm{A}}((x - y) - (((x - y) - x) - x)) \\ &\geq \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(x)\} = \bar{\mu}_{\mathrm{A}}(x), \end{aligned}$$

by appling (e1). Now suppose $x \vee y$ exists for $x, y \in X$. Putting $w := x \vee y$, we have $x \vee y = w - ((w - x) - y)$ by Proposition 2.3. It follows from (e1) that

$$\bar{\mu}_{\mathrm{A}}(x \vee y) = \bar{\mu}_{\mathrm{A}}(w - ((w - x) - y)) \geq \min\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\}.$$

Therefore A is a i-v fuzzy ideal of X. □

Proposition 3.4. *Let* A *be a i-v fuzzy set of* X. *If* A *satisfies conditions*

(f1) $(\forall x \in X)\ (\bar{\mu}_{\mathrm{A}}(0) \geq \bar{\mu}_{\mathrm{A}}(x))$ *and*

(f2) $(\forall x, y, z \in X)\ (\bar{\mu}_{\mathrm{A}}(x - z) \geq \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}((x - y) - z), \bar{\mu}_{\mathrm{A}}(y)\})$,

then we have the following condition hold:

$$(\forall a, x \in X)(x \leq a \ \Rightarrow\ \bar{\mu}_{\mathrm{A}}(x) \geq \bar{\mu}_{\mathrm{A}}(a)).$$

Proof. Let $a, x \in X$ be such that $x \leq a$. Then $x - a = 0$, and hence by (a2),(f1) and (f2)

$$\begin{aligned} \bar{\mu}_{\mathrm{A}}(x) &= \bar{\mu}_{\mathrm{A}}(x - 0) \ \geq\ \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}((x - a) - 0), \bar{\mu}_{\mathrm{A}}(a)\} \\ &= \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}(0), \bar{\mu}_{\mathrm{A}}(a)\} \ =\ \bar{\mu}_{\mathrm{A}}(a), \end{aligned}$$

Hence $\bar{\mu}_{\mathrm{A}}(x) \geq \bar{\mu}_{\mathrm{A}}(a)$. □

Theorem 3.5. *An i-v fuzzy set* $A = [\mu_{\mathrm{A}}^L, \mu_{\mathrm{A}}^U]$ *in* X *is an i-v fuzzy ideal of* X *if and only if* μ_{A}^L *and* μ_{A}^U *are fuzzy ideals of* X.

Proof. Suppose that μ_{A}^L and μ_{A}^U are fuzzy ideals of X. Let $x, y \in X$. We have $\bar{\mu}_{\mathrm{A}}(x - y) = \left[\mu_{\mathrm{A}}^L(x - y), \mu_{\mathrm{A}}^U(x - y)\right] \geq \left[\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^U(x)\right] = \bar{\mu}_{\mathrm{A}}(x)$ by (d1). Next we assume that $x \vee y$ exists in X for all $x, y \in X$. By (d2), we have

$$\begin{aligned} \bar{\mu}_{\mathrm{A}}(x \vee y) &= \left[\mu_{\mathrm{A}}^L(x \vee y), \mu_{\mathrm{A}}^U(x \vee y)\right] \\ &\geq \left[\min\{\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^L(y)\}, \min\{\mu_{\mathrm{A}}^U(x), \mu_{\mathrm{A}}^U(y)\}\right] \\ &= \mathrm{rmin}\left\{\left[\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^U(x)\right], \left[\mu_{\mathrm{A}}^L(y), \mu_{\mathrm{A}}^U(y)\right]\right\} \\ &= \mathrm{rmin}\left\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\right\}. \end{aligned}$$

Therefore A is an i-v fuzzy ideal of X.

Conversely, assume that A is an i-v fuzzy ideal of X. For any $x, y \in X$, from (d3), we get $\left[\mu_{\mathrm{A}}^L(x - y), \mu_{\mathrm{A}}^U(x - y)\right] = \bar{\mu}_{\mathrm{A}}(x - y) \geq \bar{\mu}_{\mathrm{A}}(x) = \left[\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^U(x)\right]$. It follows that $\mu_{\mathrm{A}}^L(x - y) \geq \mu_{\mathrm{A}}^L(x)$ and $\mu_{\mathrm{A}}^U(x - y) \geq \mu_{\mathrm{A}}^U(x)$. Next we assume that $x \vee y$ exists in X for all $x, y \in X$. By (d3), we have

$$\begin{aligned} \left[\mu_{\mathrm{A}}^L(x \vee y), \mu_{\mathrm{A}}^U(x \vee y)\right] &= \bar{\mu}_{\mathrm{A}}(x \vee y) \geq \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\}) \\ &= \mathrm{rmin}\left\{\left[\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^U(x)\right], \left[\mu_{\mathrm{A}}^L(y), \mu_{\mathrm{A}}^U(y)\right]\right\} \\ &= \left[min\{\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^L(y)\}, \min\{\mu_{\mathrm{A}}^U(x), \mu_{\mathrm{A}}^U(y)\}\right]. \end{aligned}$$

It follows that

$$\mu_{\mathrm{A}}^L(x - y) \geq \min\{\mu_{\mathrm{A}}^L(x), \mu_{\mathrm{A}}^L(y)\}$$

and

$$\mu_{\mathrm{A}}^U(x - y) \geq \min\{\mu_{\mathrm{A}}^U(x), \mu_{\mathrm{A}}^U(y)\}.$$

Therefore μ_{A}^L and μ_{A}^U are fuzzy ideal of X. □

Theorem 3.6. *If A is an i-v fuzzy ideal of X, then the non-empty set*

$$\bar{U}(A;[\delta_1,\delta_2]) := \{x \in X \mid \bar{\mu}_{\mathrm{A}}(x) \geq [\delta_1,\delta_2]\}$$

is an ideal of X for every $[\delta_1,\delta_2] \in D[0,1]$.

Proof. Suppose that A is a i-v fuzzy ideal of X and let $[\delta_1,\delta_2] \in D[0,1]$ be such that $\bar{U}(A;[\delta_1,\delta_2]) \neq \emptyset$. Let $x \in X$ and $a \in \bar{U}(A;[\delta_1,\delta_2])$. Then $\bar{\mu}_{\mathrm{A}}(a) \geq [\delta_1,\delta_2]$ and so $\bar{\mu}_{\mathrm{A}}(a-x) \geq \bar{\mu}_{\mathrm{A}}(a) \geq [\delta_1,\delta_2]$ by (d1). Hence $a-x \in \bar{U}(A;[\delta_1,\delta_2])$. Assume that $a \vee b$ exists in X for all $a,b \in \bar{U}(A;[\delta_1,\delta_2])$. By (d2), we have

$$\bar{\mu}_{\mathrm{A}}(a \vee b) \geq \min\{\bar{\mu}_{\mathrm{A}}(a), \bar{\mu}_{\mathrm{A}}(b)\} \geq [\delta_1,\delta_2]\,,$$

and thus $a \vee b \in \bar{U}(A;[\delta_1,\delta_2])$. Therefore $\bar{U}(A;[\delta_1,\delta_2])$ is an ideal of X. □

We then call $\bar{U}(A;[\delta_1,\delta_2])$ the *i-v level ideal of A.*

Theorem 3.7. *Every ideal of X can be realized as an i-v level ideal of an i-v fuzzy ideal of X.*

Proof. Let S be a ideal of X. We consider the i-v fuzzy set A of X defined by

$$\bar{\mu}_{\mathrm{A}}(x) = \begin{cases} [\alpha_1, \alpha_2], & x \in S, > 0, 0 \text{[redacted]} \\ , & \text{otherwise} \end{cases}$$

where $\alpha_1, \alpha_2 \in (0, 1]$ with $\alpha_1 < \alpha_2$. It is easy check that $(\bar{\mu}_{\mathrm{A}}(x - y) \geq \bar{\mu}_{\mathrm{A}}(x))$ for all $x, y \in X$, and $\bar{\mu}_{\mathrm{A}}(x \vee y) \geq \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\}$ whenevr there exists $x \vee y$ for all $x, y \in X$. Using Theorem 3.6 and $\bar{U}(A; [\delta_1, \delta_2]) = S$, we can conclude that every ideal of X can be realized as an i-v level ideal of an i-v fuzzy ideal of X. □

Lemma 3.8. *If A is an i-v fuzzy ideal of X, then $\bar{\mu}_{\mathrm{A}}(0) \geq \bar{\mu}_{\mathrm{A}}(x)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 3.9. *If A is an i-v fuzzy ideal of X, then the set*

$$X_{\bar{\mu}_{\mathrm{A}}} := \{x \in X \mid \bar{\mu}_{\mathrm{A}}(x) = \bar{\mu}_{\mathrm{A}}(0)\}$$

is a ideal of X.

Proof. Let $a \in X_{\bar{\mu}_{\mathrm{A}}}$ and $x \in X$. Then $\bar{\mu}_{\mathrm{A}}(a) = \bar{\mu}_{\mathrm{A}}(0)$, and so $\bar{\mu}_{\mathrm{A}}(a - x) \geq \bar{\mu}_{\mathrm{A}}(a) = \bar{\mu}_{\mathrm{A}}(0)$ by (d1). Combining this and Lemma 3.8, we get $\bar{\mu}_{\mathrm{A}}(a - x) = \bar{\mu}_{\mathrm{A}}(0)$, i.e., $a - x \in X_{\bar{\mu}_{\mathrm{A}}}$. Let $x, y \in X_{\bar{\mu}_{\mathrm{A}}}$ and assume that there exists $x \vee y$. By means of (d2), we know that

$$\bar{\mu}_{\mathrm{A}}(x \vee y) \geq \mathrm{rmin}\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\} = \bar{\mu}_{\mathrm{A}}(0).$$

Thus $\bar{\mu}_{\mathrm{A}}(x \vee y) = \bar{\mu}_{\mathrm{A}}(0)$ by Lemma 3.8, and so $x \vee y \in X_{\bar{\mu}_{\mathrm{A}}}$. Therefore $X_{\bar{\mu}_{\mathrm{A}}}$ is an ideal of X. □

Theorem 3.10. *For a nonzero element w of X, let A be an i-v fuzzy suset of X and $\bar{\mu}_{\mathrm{A}}$ be a fuzzy set in X defined by*

$$\bar{\mu}_{\mathrm{A}}(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in (w], \\ {}_1, \beta_2 \text{otherwise}, \end{cases}$$

where $(w] := \{x \in X \mid x \leq w\}$ and $[\alpha_1, \alpha_2] > [\beta_1, \beta_2]$. Then $\bar{\mu}_{\mathrm{A}}$ is a fuzzy ideal of X.

Proof. Let $x, y \in X$. If $x \notin (w]$, then obviously $\bar{\mu}_{\mathrm{A}}(x) = [\beta_1, \beta_2] \leq \bar{\mu}_{\mathrm{A}}(x - y)$. Assume that $x \in (w]$. Then $x - y \leq x \leq w$. Hence $x - y \in (w]$. It follows that $\bar{\mu}_{\mathrm{A}}(x - y) = [\alpha_1, \alpha_2] = \bar{\mu}_{\mathrm{A}}(x)$. Therefore (d1) is valid. Next we show that (d2) is hold. If $x \notin (w]$ or $y \notin (w]$, then

$$\min\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\} = [\beta_1, \beta_2] \leq \bar{\mu}_{\mathrm{A}}(x \vee y)$$

whenever $x \vee y$ exists in X. Suppose that $x, y \in (w]$. Then $x \leq w$ and $y \leq w$, and so $x \vee y$ exists by Proposition 2.3. Since $x \vee y = w - ((w - y) - x)$, it follows from (a3) that $x \vee y \leq w$, i.e., $x \vee y \in (w]$. Hence $\bar{\mu}_{\mathrm{A}}(x \vee y) = [\alpha_1, \alpha_2] = \min\{\bar{\mu}_{\mathrm{A}}(x), \bar{\mu}_{\mathrm{A}}(y)\}$. Therefore $\bar{\mu}_{\mathrm{A}}$ is a fuzzy ideal of X. □

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