International Mathematical Forum, 4, 2009, no. 13, 645 - 651

On Fuzzy Dot Subalgebras of d-Algebras

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Abstract

In this paper, we introduce the notion of fuzzy dot subalgebras in d-algebras, and consider its various properties.

Mathematics Subject Classification: 06F35, 03G25, 03E72

Keywords: d-algebra, fuzzy dot algebra, fuzzy σ-product relation, left (resp, right) fuzzy relation

1 Introduction

Y. Imai and Iseki [1, 2] introduced two classes of abstract: BCK-algebras and BCI-algebras. J. Negges [3] introduced the class of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. L. A. Zadeh [5] introduced the notion of fuzzy sets. In this paper, we introduce the notion of a fuzzy dot subalgebra of a d-algebra as a generalization of a fuzzy subalgebra of a d-algebra, and then we investigated several basic properties related to fuzzy dot subalgebras.

2 Preliminaries

A d-algebra is an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(1) $x * x = 0$,

(ii) $0 * x = 0$,

(iii) $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.

Let S be a nonempty subset of a d-algebra X. Then S is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$. A map f from a d-algebra X to a d-algebra Y is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

We now review some fuzzy logic concepts. A fuzzy subset of a set X is a function $\mu : X \to [0,1]$. For any fuzzy subset μ and ν of a set X, we define

$$\mu \subseteq \nu \Leftrightarrow \mu(x) \leq \nu(x) \quad \forall x \in X,$$

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\} \quad \forall x \in X.$$

Let $f : X \to Y$ be a function from a set X to a set Y and let μ be a fuzzy subset of X. The fuzzy subset ν of Y defined by

$$\nu(y) := \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset \quad \forall y \in Y, \\ 0 & \text{otherwise,} \end{cases}$$

is called the *image* of μ under f, denoted by $f[\mu]$. If ν is a fuzzy subset of Y, the fuzzy subset μ of X given by $\mu(x) = \nu(f(x))$ for all $x \in X$ is called the *preimage* of ν under f and is denoted by $f^{-1}[\nu]$.

A fuzzy relation μ on a set X is a fuzzy subset of $X \times X$, that is, a map $\mu : X \times X \to [0,1]$. A fuzzy subset μ of a d-algebra X is called a fuzzy *subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

3 fuzzy dot subalgebras of d-algebras

In what follows let X denote a d-algebra unless otherwise specified.

Definition 3.1. A fuzzy subset μ of X is called a *fuzzy dot subalgebra* of a d-algebra X if $\mu(x * y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in X$.

Example 3.2. Consider a d-algebra $X = \{0, 1, 2\}$ having the following Cayley table:

$*$	0	1	2
0	0	0	0
1	2	0	2
2	1	1	0

Define a fuzzy set μ in X by $\mu(0) = 0.6, \mu(1) = \mu(2) = 0.7$. It is easy to verify that μ is a fuzzy dot subalgebra of a d-algebra X.

Example 3.3. Consider a d-algebra $X = \{0, a, b, c\}$ having the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Define a fuzzy set μ in X by $\mu(0) = \mu(a) = 0.8, \mu(b) = \mu(c) = 0.3$. It is easy to verify that μ is a fuzzy dot subalgebra of a d-algebra X.

Note that every fuzzy subalgebra is a fuzzy dot subalgebra of a d-algebra, but the converse is true. In fact, the fuzzy dot subalgebra μ in Example 3.2 is not a fuzzy subalgebra since

$$\mu(1 * 1) = \mu(0) = 0.6 < 0.7 = \mu(1) = \min\{\mu(1), \mu(1)\}.$$

Proposition 3.4. *If μ is a fuzzy dot subalgebra of a d-algebra X, then we have $\mu(0) \geq (\mu(x))^2$ for all $x \in X$.*

Proof. For every $x \in X$, we have

$$\mu(0) = \mu(x * x) \geq \mu(x) \cdot \mu(x) = (\mu(x))^2,$$

completing the proof. □

Proposition 3.5. *If μ and ν are fuzzy dot subalgebra of a d-algebra X, then so is $\mu \cap \nu$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}(\mu \cap \nu)(x * y) &= \min\{\mu(x * y), \nu(x * y)\} \\ &\geq \min\{\mu(x) \cdot \mu(y), \nu(x) \cdot \nu(y)\} \\ &\geq (\min\{\mu(x), \nu(x)\}) \cdot (\min\{\mu(y), \nu(y)\}) \\ &= ((\mu \cap \nu)(x)) \cdot ((\mu \cap \nu)(y)).\end{aligned}$$

Hence $\mu \cap \nu$ is a fuzzy dot subalgebra of a d-algebra X. □

Let χ_A denote the characteristic function of a non-empty subset A of a d-algebra X.

Theorem 3.6. *Let $A \subseteq X$. Then A is a subalgebra of a d-algebra X if and only if χ_A is a fuzzy dot subalgebra of a d-algebra X.*

Proof. Let $x, y \in A$. Then $x * y \in A$. Hence we have

$$\chi_A(x * y) = 1 \geq \chi_A(x) \cdot \chi_A(y).$$

If $x \in A$ and $y \notin A$ (or $x \notin A$ and $y \in A$), then we have $\chi_A(x) = 1$ or $\chi_A(y) = 0$. This means that

$$\chi_A(x * y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 0 = 0.$$

Conversely, Assume that χ_A is a fuzzy dot subalgebra of a d-algebra X. Now let $x, y \in A$. Then

$$\chi_A(x * y) \geq \chi_A(x) \cdot \chi_A(y) = 1 \cdot 1 = 1,$$

and so $x * y \in A$. This completes the proof. □

Note that a fuzzy subset μ of X is a fuzzy subalgebra of d-algebra X if and only if a nonempty level subset

$$U(\mu;t) := \{x \in X \mid \mu(x) \geq t\}$$

is a subalgebra of X for every $t \in [0,1]$. But we know that if μ is a fuzzy dot subalgebra of X, then there exists $t \in [0,1]$ such that

$$U(\mu;t) := \{x \in X \mid \mu(x) \geq t\}$$

is not a subalgebra of X. In fact, if μ is the fuzzy dot subalgebra of X in Example 3.2, then

$$U(\mu;0.7) = \{x \in X \mid \mu(x) \geq 0.7\} = \{1,2\}$$

is not a subalgebra of X since $2 * 2 = 0 \notin U(\mu;0.7)$.

Proposition 3.7. *If μ is a fuzzy dot subalgebra of a d-algebra X, then*

$$U(\mu;1) := \{x \in X \mid \mu(x) = 1\}$$

is either empty or is a subalgebra of a d-algebra X.

Proof. If x and y belong to $U(\mu;1)$, then $\mu(x * y) \geq \mu(x) \cdot \mu(y) = 1$. Hence $\mu(x*y) = 1$ which implies $x*y \in U(\mu;1)$. Consequently, $U(\mu;1)$ is a subalgebra of a d-algebra X. □

Theorem 3.8. *Let $g : X \to Y$ be an onto homomorphism of d-algebras. If ν is a fuzzy dot subalgebra of X, then the image $f[\mu]$ of μ under f is a fuzzy dot subalgebra of Y.*

Proof. For any $x_1, x_2 \in X$, we have

$$\begin{aligned} g^{-1}[\nu](x_1 * x_2) &= \nu(g(x_1 * x_2)) = \nu(g(x_1) * g(x_2)) \\ &\geq \nu(g(x_1) \cdot \nu(g(x_2)) = g^{-1}[\nu](x_1) \cdot g^{-1}[\nu](x_2). \end{aligned}$$

Thus $g^{-1}[\nu]$ is a fuzzy dot subalgebra of X. □

Theorem 3.9. *Let $f : X \to Y$ be an onto homomorphism of d-algebras. If μ is a fuzzy dot algebra of X, then the image $f[\mu]$ of μ under f is a fuzzy dot subalgebra of Y.*

Proof. For any $y_1, y_2 \in Y$, let $A_1 = f^{-1}(y_1), A_2 = f^{-1}(y_2)$, and $A_12 = f^{-1}(y_1 * y_2)$. Consider the set

$$A_1 * A_2 := \{x \in X \mid x = a_1 * a_2 \text{ for some } x_1 \in A_1, a_2 \in A_2\}.$$

If $x \in A_1 * A_2$, then $x = x_1 * x_2$ for some $x_1 \in A_1$ and $x_2 \in A_2$ so that

$$f(x) = f(x_1 * x_2) = f(x_1) * f(x_2) = y_1 * y_2,$$

that is, $x \in f^{-1}(y_1 * y_2) = A_{12}$. Hence $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} f[\mu](y_1 * y_2) &= \sup_{x \in f^{-1}(y_1 * y_2)} \mu(x) = \sup_{x \in A_{12}} \mu(x) \\ &\geq \sup_{x \in A_1 * A_2} \mu(x) \geq \sup_{x_1 \in A_1, x_2 \in A_2} \mu(x_1 * x_2) \\ &\geq \sup_{x_1 \in A_1, x_2 \in A_2} \mu(x_1) \cdot \mu(x_2). \end{aligned}$$

Since $\cdot : [0,1] \times [0,1]$ is continuous, for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\tilde{x_1} \geq \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\tilde{x_2} \geq \sup_{x_2 \in A_2} \mu(x_1) - \delta$, then $\tilde{x}_1 \cdot \tilde{x}_2 \geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) - \epsilon$. Choose $a_1 \in A_1$ and $a_2 \in A_2$ such that $\mu(a_1) \geq \sup_{x_1 \in A_1} \mu(x_1) - \delta$ and $\mu(a_2) \geq \sup_{x_2 \in A_2} \mu(x_2) - \delta$. Then

$$\mu(a_1) \cdot \mu(a_2) \geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) - \epsilon.$$

Consequently,

$$\begin{aligned} f[\mu](y_1 * y_2) &\geq \sup_{x_1 \in A_1, x_2 \in A_2} \mu(x_1) \cdot \mu(x_2) \\ &\geq \sup_{x_1 \in A_1} \mu(x_1) \cdot \sup_{x_2 \in A_2} \mu(x_2) \\ &= f[\mu](y_1) \cdot f[\mu](y_2), \end{aligned}$$

and hence $f[\mu]$ is a fuzzy dot subalgebra of Y. □

Definition 3.10. Let λ and μ be the fuzzy sets in a set X. The cartesian product $\lambda \times \mu : X \times X \to [0,1]$ is defined by

$$(\lambda \times \mu)(x, y) = \lambda(x) \cdot \mu(y),$$

for all $x, y \in X$.

Theorem 3.11. *If λ and μ are fuzzy dot subalgebras of a d-algebra X, then $\lambda \times \mu$ is a fuzzy dot subalgebra of $X \times X$.*

Proof. For any $x_1, x_2, y_1, y_2 \in X$,

$$\begin{aligned} (\lambda \times \mu)((x_1, y_1) * (x_2, y_2)) &= (\lambda \times \mu)(x_1 * x_2, y_1 * y_2) \\ &= \lambda(x_1 * x_2) \cdot \mu(y_1 * y_2) \\ &\geq ((\lambda(x_1) \cdot \lambda(x_2)) \cdot ((\mu(y_1) \cdot \mu(y_2)) \\ &= ((\lambda(x_1) \cdot \mu(y_1)) \cdot (\lambda(x_2) \cdot \mu(y_2)) \\ &= (\lambda \times \mu)(x_1, y_1) \cdot (\lambda \times \mu)(x_2, y_2), \end{aligned}$$

completing the proof. □

Definition 3.12. Let σ be a fuzzy subset of X. The strongest fuzzy σ-relation on d-algebra X is the fuzzy subset μ_σ of $X \times X$ given by $\mu_\sigma(x, y) = \sigma(x) \cdot \sigma(y)$ for all $x, y \in X$.

Theorem 3.13. *Let μ_σ be the strongest fuzzy σ-relation on d-algebra X, where σ is a fuzzy subset of a d-algebra X. If σ is a fuzzy dot subalgebra of a d-algebra X, then μ_σ is a fuzzy dot subalgebra of $X \times X$.*

Proof. Suppose that σ is a fuzzy dot subalgebra of X. For any $x_1, x_2, y_1, y_2 \in X$, we have

$$\begin{aligned}\mu_\sigma((x_1, y_1) * (x_2, y_2)) &= \mu_\sigma(x_1 * x_2, y_1 * y_2)\\ &= \sigma(x_1 * x_2) \cdot \sigma(y_1 * y_2)\\ &\geq (\sigma(x_1) \cdot \sigma(x_2)) \cdot (\sigma(y_1) \cdot \sigma(y_2))\\ &= (\sigma(x_1) \cdot \sigma(y_1)) \cdot (\sigma(x_2) \cdot \sigma(y_2))\\ &= \mu_\sigma(x_1, y_1) \cdot \mu_\sigma(x_2, y_2),\end{aligned}$$

and so μ_σ is a fuzzy dot subalgebra of $X \times X$. □

Definition 3.14. Let σ be a fuzzy subset of a d-algebra X. A fuzzy relation μ on d-algebra X is called a *fuzzy σ-product relation* if $\mu(x, y) \geq \sigma(x) \cdot \sigma(y)$ for all $x, y \in X$.

Definition 3.15. Let σ be a fuzzy subset of a d-algebra X. A fuzzy relation μ on d-algebra X is called a *left fuzzy relation* on σ if $\mu(x, y) = \sigma(x)$ for all $x, y \in X$.

Similarly, we can define a right fuzzy relation on σ. Note that a left (resp. right) fuzzy relation on σ is a fuzzy σ-product relation.

Theorem 3.16. *Let μ be a left fuzzy relation on a fuzzy subset σ of a d-algebra X. If μ is a fuzzy dot subalgebra of $X \times X$, then σ is a fuzzy dot subalgebra of a d-algebra X.*

Proof. Suppose that a left fuzzy relation μ on σ is a fuzzy dot subalgebra of $X \times X$. Then

$$\begin{aligned}\sigma(x_1 * x_2) =&\mu(x_1 * x_2, y_1 * y_2) = \mu((x_1, y_1) * (x_2 * y_2))\\ &\geq \mu(x_1, y_1) \cdot \mu(x_2, y_2) = \sigma(x_1) \cdot \sigma(x_2)\end{aligned}$$

for all $x_1, x_2, y_1, y_2 \in X$. Hence σ is a fuzzy dot subalgebra of a d-algebra X. □

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Received: July, 2008