International Mathematical Forum, 4, 2009, no. 16, 795 - 804

The Wronskian and the Ermakov-Lewis Invariant

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Abstract

The Ermakov-Lewis invariant is shown to be a quadratic form that emerges from the Wronskian linear form. This invariant is shown to be, in general, positive semidefinite. The Wronskian is expressed in terms of different sets of variables. The nonlinear superposition principle for the amplitude and phase variables is discussed. The normalized Wronskian is shown to produce a positive definite Ermakov invariant. Appropriate representations for diverse physical problems are presented.

Mathematics Subject Classification: 34A30, 34A34, 70H33

Keywords: Amplitude and phase representations, Invariants, Ordinary differential equations, Time dependent harmonic oscillator, Wronskian

1 Introduction

The Ermakov-Lewis invariant has received considerable interest in recent years [1]. This invariant arises from the time dependent harmonic oscillator (TDHO) equation for the coordinate variable q

$$\ddot{q} + \Omega^2 (t) q = 0, \tag{1}$$

where the over dot represents differentiation with respect to time and $\Omega^2 (t)$ is a time dependent parameter. It may be derived in a variety of ways [2] leading to the result

$$I = \frac{1}{2} \left[\frac{q^2}{\rho^2} + (q\dot{\rho} - \rho\dot{q})^2 \right], \tag{2}$$

where the function ρ fulfills the auxiliary equation

$$\ddot{\rho} + \rho\Omega^2 = \rho^{-3}. \tag{3}$$

Pairs of equations, where (1) and (3) are one such example, lead to invariants that were first studied by Ermakov. This formal description is encountered in many physical problems ranging from mechanics and electromagnetism to cosmological models [3].

On the other hand, the determinant of a matrix involving a set of functions and their corresponding derivatives in order to produce a square matrix is referred as the Wronskian after Josef Hoene-Wronski (1778-1853). This quantity is used as a measure of linear dependence when it vanishes over the entire range of the variables. In particular, the Wronskian generated from two solutions q_1, q_2 of the TDHO equation is constant and is given by

$$W \equiv q_1 \dot{q}_2 - q_2 \dot{q}_1, \tag{4}$$

where the real functions q_1, q_2 are linearly independent solutions if $W \neq 0$ and orthogonal if they fulfill the Sturm Liouville orthogonal functions integral $\int q_1 q_2 dt = 0$. From the above equation, given a solution q_1, the linearly independent solution is given by

$$q_2 = q_1 \int \frac{W}{q_1^2} dt. \tag{5}$$

The purpose of this communication is threefold. Firstly to show the relationship between the Ermakov-Lewis invariant and the Wronskian. Secondly, to demonstrate that the Ermakov-Lewis invariant is, in general, not positive definite or equivalently that the Wronskian is sign indefinite. Furthermore, it will be shown under which circumstances this invariant is positive definite. Thirdly, to establish some related invariants and represent them in terms of different sets of variables.

2 Ermakov-Lewis invariant derived from the Wronskian

Lemma 2.1 *The Ermakov-Lewis invariant is equal to one half the square of the Wronskian.*

Proof. Consider the time dependent function ρ defined in terms of two linearly independent solutions of the TDHO equation as

$$q_1^2 + q_2^2 = \rho^2. \tag{6}$$

The Wronskian (4) in terms of $q = q_1$ and ρ may be obtained by evaluating the function q_2 from the above equation

$$q_2 = \sqrt{\rho^2 - q^2}, \qquad \dot{q}_2 = \frac{\rho\dot{\rho} - q\dot{q}}{\sqrt{\rho^2 - q^2}}.$$

Then

$$W = \frac{\rho}{\sqrt{\rho^2 - q^2}} \left(q\dot{\rho} - \rho\dot{q} \right), \tag{7}$$

which may be rearranged as

$$I = \frac{1}{2} W^2 = \frac{1}{2} \left(W^2 \frac{q^2}{\rho^2} + \left(q\dot{\rho} - \rho\dot{q} \right)^2 \right). \tag{8}$$

This expression has the form of the Ermakov invariant for the TDHO equation. In order to complete the proof, consider the derivative of the above equation recalling that W is constant

$$\left(\dot{\rho} q - \rho\dot{q} \right)^2 - \frac{\rho}{q} \left(\rho^2 - q^2 \right) \left(q\ddot{\rho} - \rho\ddot{q} \right) = 0.$$

An equation for the function ρ alone is obtained by invoking the invariant relationship (7) and the oscillator equation (1), thus

$$W^2 - \rho^3 \left(\ddot{\rho} + \rho\Omega^2 \right) = 0. \tag{9}$$

But this is the auxiliary Ermakov equation (3) when $W = 1$. □

The dimensionless equation with this constant normalized to one is often quoted in the literature although the generalization $W \neq 1$ has been pointed out before [4]. The present procedure is, in fact, a simple derivation of the Ermakov-Lewis invariant without invoking *a priori* the auxiliary equation (9). It should be noted that the constant multiplying the q^2/ρ^2 term in equation (8) has been described as an arbitrary constant k [1]. It is indeed arbitrary since the solutions q_1, q_2 may be multiplied by arbitrary constants k_1, k_2 respectively such that $k = \sqrt{k_1 k_2}$. However, this constant is in fact the square of the Wronskian once the scaled functions are introduced.

The Ermakov Lewis invariant is a quadratic form that arises from the Wronskian linear form when these quantities are expressed in terms of coordinate variables since

$$I = \frac{1}{2} \left(q_1\dot{q}_2 - q_2\dot{q}_1 \right)^2, \tag{10}$$

as may be seen from (4) and (8). Some variants of this result may arise depending on the particular linearly independent solutions that are used to construct the Wronskian. The invariants derived from complex valued solutions and the relationships involving only real valued solutions have been shown to be entirely equivalent. The ansatz introduces the complex solution $\tilde{q} = q_1 + iq_2$ and the real Wronskian is written as $W = (1/2i) \left(\tilde{q}^* \dot{\tilde{q}} - \tilde{q} \dot{\tilde{q}}^* \right)$ where the star represents the complex conjugate [5]. Classical invariants of Ermakov systems from complex Wronskian invariants involving particular solutions have been derived by Thylwe [6]. Their work has also been extended to a system of coupled linear oscillators [7]. In either case, the generalized invariants have the form of one half the Wronskian times its complex conjugate. This issue will be further discussed in the last section.

3 Semipositive definite quantities

Recall that a positive definite quantity J, dependent on variables abridged by g, is defined by two conditions i) $J(g) \geq 0$ and ii) $J(g) = 0 \Leftrightarrow g = 0$. If the latter condition is not fulfilled it is then semipositive [8].

Lemma 3.1 *The Ermakov-Lewis invariant is positive semidefinite.*

Proof. It is clear that for real variables q, ρ the Ermakov-Lewis invariant (8) cannot be negative. However, this quantity may still be semipositive if it is zero for nonzero q, ρ. This condition is fulfilled if $q\dot{\rho} = \rho\dot{q}$. $\square$

To elucidate this result, let us cast the Wronskian in terms of the amplitude and phase variables. The terms involving derivatives in the Wronskian (7) may be rewritten as $q\dot{\rho} - \rho\dot{q} = -\rho^2 d/dt\,(q/\rho)$ and the equation

$$W = -\rho^2 \left(1 - \frac{q^2}{\rho^2}\right)^{-\frac{1}{2}} \frac{d}{dt}\left(\frac{q}{\rho}\right) \tag{11}$$

is readily integrated to obtain the coordinate variable in terms of the function ρ

$$q = \rho \cos\left(\int \frac{W}{\rho^2} dt\right). \tag{12}$$

The function ρ in terms of the coordinate variable is obtained by invoking $q\dot{\rho} - \rho\dot{q} = q^2 d/dt\,(\rho/q)$ to obtain

$$\rho = q\sqrt{\left(\int \frac{W}{q^2} dt\right)^2 + 1}. \tag{13}$$

This result is equally obtained from (5) and (6). Define the function γ as the argument of the cosine function in Eq. (12)

$$\gamma = \int \frac{W}{\rho^2} dt; \tag{14}$$

Upon differentiation, the Wronskian is then given by

$$W = \rho^2 \dot{\gamma}. \tag{15}$$

The variables ρ and γ are physically interpreted as amplitude and phase quantities. The product $q = \rho\cos\gamma$ recreates the coordinate variable. The amplitude and phase representation is thus concomitant to Ermakov systems with exact invariants. The Wronskian written in terms of coordinate and phase variables is

$$W = q^2\omega + \left(\frac{1}{2}q\dot{\omega} + \omega\dot{q}\right)^2 \frac{1}{\omega^3}, \tag{16}$$

where the auxiliary equation for the frequency is now required. The frequency defined as the derivative of the phase function $\dot{\gamma} \equiv \omega$ obeys the equation

$$\omega\ddot{\omega} - \frac{3}{2}\dot{\omega}^2 + 2\left[\omega^2 - \Omega^2\right]\omega^2 = 0. \tag{17}$$

This phase equation is the starting point of the phase integral approximation [9] or phase-amplitude method [10] used in quantum mechanics.

From expression (15), it is clear that the Wronskian is sign indeterminate since the derivative of the phase may acquire, in general, positive or negative values. This issue has been a long standing problem in the Klein-Gordon-Schrödinger scalar wave equation and has been recently discussed invoking two complementary fields [5]. The Ermakov Lewis invariant in terms of these variables is

$$I = \frac{1}{2}W^2 = \frac{1}{2}\rho^4\dot{\gamma}^2. \tag{18}$$

If the trajectory function q is proportional to the amplitude function ρ, the phase γ is constant. The Wronskian and hence the Ermakov-Lewis invariant are then zero.

3.1 Nonlinear superposition principle

Given particular amplitude and phase solutions A and s, the general solution in terms of ρ, γ may be constructed from the nonlinear superposition principle [11]. This principle is a translation of the linear superposition principle in the linear coordinate equation to the nonlinear equations for the amplitude and the phase. It is instructive to show that the inverse procedure is also possible. Namely, that the governing equations for A and s may be obtained from the general solution. To this end, allow for a general complex valued solution of the form

$$\tilde{q}_g = \rho e^{i\gamma} = Ae^{is} + \sigma Ae^{-is}. \tag{19}$$

where σ is a constant. These variables correspond to opposite phase solutions $\pm s$ whose generalization to time dependent σ proves useful in obtaining analytical solutions for some classes of time dependent parameter Ω^2 [12]. The general solution in terms of the particular amplitude A and phase s solutions is a statement of the nonlinear superposition principle for the amplitude

$$\rho = A\sqrt{1 + \sigma^2 + 2\sigma\cos(2s)} \tag{20}$$

and phase functions

$$\gamma = \arctan\left[\frac{1-\sigma}{1+\sigma}\tan(s)\right]. \tag{21}$$

The consequences of the indeterminacy introduced in these functions by the arbitrary constant σ has been recently addressed [13]. The TDHO differential equation for the general solution (19) then reads

$$\ddot{A} + i\left(2\dot{A}\dot{s} + A\ddot{s}\right)\frac{\left(e^{is} - \sigma e^{-is}\right)}{\left(e^{is} + \sigma e^{-is}\right)} - A\dot{s}^2 + \Omega^2 A = 0.$$

The imaginary part of this equation

$$\left(2\dot{A}\dot{s} + A\ddot{s}\right)\frac{\left(1+\sigma^2\right)i}{1+\sigma^2+2\sigma\cos 2s} = 0,$$

leads to the invariant

$$W_p = A^2\dot{s}. \tag{22}$$

This invariant is equal to the Wronskian of the functions $q_1 = A\cos\dot{s}$, $q_2 = A\sin\dot{s}$. The real part of the differential equation

$$\ddot{A} + \left(2\dot{A}\dot{s} + A\ddot{s}\right)\frac{-2\sigma\sin 2s}{1+\sigma^2+2\sigma\cos 2s} - A\dot{s}^2 + \Omega^2 A = 0,$$

together with the condition imposed by the imaginary result leads to an Ermakov auxiliary equation

$$\ddot{A} + \Omega^2 A - \frac{W_p^2}{A^3} = 0 \tag{23}$$

but the invariant is now W_p^2 instead of W^2 (see Eq. 9). The relationship between the constants in the amplitude equations for ρ and A may be obtained from equations (20) and (21) inserted in equation (15)

$$W = A^2\dot{s}\left(1-\sigma^2\right) = W_p\left(1-\sigma^2\right). \tag{24}$$

Substitution of the amplitude in terms of the phase yields the phase equation

$$\dot{s}s^{(3)} - \frac{3}{2}\ddot{s}^2 + 2\left[\dot{s}^2 - \Omega^2\right]\dot{s}^2 = 0. \tag{25}$$

The differential equations for the particular phase function s and the general phase function γ are identical. Notice that from Eq. (24), the Ermakov-Lewis invariant $I = \frac{1}{2}W^2$ vanishes for $\sigma = 1$.

3.2 Normalization

The Ermakov Lewis invariant (8) is often stated with the auxiliary equation coefficient W in (9) set to one as mentioned before. To this end, the procedure is to scale the amplitude ρ by the factor $\rho_0 = \sqrt{W} = \sqrt{W_p\,(1-\sigma^2)}$ given by (24). The scaled amplitude for the general solution is then

$$\rho_n = \frac{\rho}{\rho_0} = \frac{A}{\sqrt{W_p\,(1-\sigma^2)}}\sqrt{1+\sigma^2+2\sigma\cos{(2s)}}. \tag{26}$$

The Ermakov-Lewis invariant (2) is thus obtained and this quantity is positive definite since I is zero if and only if q is zero. Nonetheless, the price that is payed is that in the limit when $\sigma \to 1$ the amplitude is divergent. On the other hand, if $\dot{s} < 0$, the amplitude becomes imaginary. This re-scaling forces the Wronskian to unity since $W = \rho_n^2\dot{\gamma} = 1$. The positively defined quadratic form of the real solutions to the amplitude equation in this normalized case has been discussed by Matzkin [14].

3.3 Related invariants

Several invariants have been proposed regarding the TDHO differential equation [15]. The two functions required to generate a nonvanishing Wronskian may be a combination of a particular and the general solution provided that they are linearly independent. Furthermore, in Thylwe's work [6], similar invariants are also constructed for the amplitude equation, often referred as the Milne equation [16]. It is possible, for example, to combine the particular amplitude solution A with the complex valued general solution $\tilde{q}_g = \rho e^{i\gamma}$ rather than with its counterpart $\tilde{q} = Ae^{is}$. An invariant with the Ermakov Lewis form is then also obtained [17]

$$I_\sigma = \frac{2\sigma W^2}{(1-\sigma^2)^2} = 2\sigma W_p^2 = \frac{1}{2}\left[W_p^2\frac{\tilde{q}_g^2}{A^2} + \left(\tilde{q}_g\dot{A} - A\dot{\tilde{q}}_g\right)^2\right], \tag{27}$$

where the constant W_p has been left with an arbitrary value rather than normalized to one as in the previously cited work. This invariant I_σ is proportional to σ and, for this reason, has been interpreted as twice the ratio of the opposite phase solutions [18]. In this case, the Wronskian is constructed from the general solution $\tilde{q}_1 = \tilde{q}_g$ but instead of taking the linearly independent solution $\tilde{q}_2$ from (5) a particular scaled solution is introduced

$$\tilde{q}_p = \frac{A}{i\sqrt{\sigma}}e^{is}. \tag{28}$$

The Wronskian is then

$$W_{gp} = \tilde{q}_g\dot{\tilde{q}}_p - \tilde{q}_p\dot{\tilde{q}}_g = 2\sqrt{\sigma}A^2\dot{s}, \tag{29}$$

so that its square is equal to twice the Ermakov-Lewis invariant $I_\sigma = \frac{1}{2}W_{gp}^2$.

4 Conclusions

The Ermakov Lewis invariant (10) has been shown to be a quadratic form that is equal to one half the square of the Wronskian. An economical derivation of this invariant has been presented without invoking *a priori* the auxiliary amplitude equation. The Wronskian is sign indefinite and therefore the Ermakov Lewis invariant has been shown to be, in general, positive semidefinite. In the restricted case where the amplitude is re-scaled so that the Wronskian is set to unity, the Ermakov-Lewis invariant becomes positive definite.

The coordinate and phase equations are identical for either the general or a particular coordinate and phase solutions. However, the amplitude equations for the general (9) or a particular solution (23) involve different constants. The relationship between the Wronskians is $W = W_p\left(1-\sigma^2\right)$ and the amplitudes are related by (20). This relationship is an assertion of the nonlinear (amplitude) superposition principle. The relationship between the phase variables is $\gamma = \arctan\left[\frac{1-\sigma}{1+\sigma}\tan\left(s\right)\right]$; Following the same vein, this equation may be coined as a statement of the nonlinear phase superposition principle.

The Wronskian has been expressed in terms of different sets of variables. i) The well known linearly independent coordinate solutions. It is worth mentioning that W becomes a quadratic form if the linearly independent solutions are related by $q_1 = \dot{q}_2$, the Wronskian is then $W = \dot{q}_2^2 + \Omega^2 q_2^2$ and is positive definite for real Ω. ii) In amplitude and phase variables $W = \rho^2\dot{\gamma}$. iii) In coordinate and amplitude variables commonly used in the literature for the Ermakov-Lewis invariant. iv) Finally, the coordinate and phase representation $W = q^2\omega + \left(\frac{1}{2}q\dot{\omega} + \omega\dot{q}\right)^2\omega^{-3}$, where the coordinate fulfills the TDHO equation and the frequency, defined as the derivative of the phase, obeys equation (17). One of these representations may be more appropriate depending on the specific problem. In quantum mechanics, the coordinate and amplitude representation is nicely tailored to describe minimum uncertainty states [19]. In physical optics, the amplitude and phase representation is often better suited to describe wave propagation [20].

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Received: August, 2008