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On Additive Mappings in Rings with Identity Element

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Abstract

Let R be an associative ring with identity element, $F : R \to R$, $D : R \to R$ and $T : R \to R$ all additive mappings and $n \geq 1$ a fixed integer. We prove that (i) if R is $(n+1)!$-torsion free any ring such that $F(x^{n+1}) = F(x)x^n + xD(x)x^{n-1} + x^2D(x)x^{n-2} + \cdots + x^nD(x)$ holds for all $x \in R$, then D is a Jordan derivation and F is a Jordan generalized derivation; (ii) if R is a $(n+2)!$-torsion free semiprime ring such that $(n+1)T(x^{n+1}) = x^nT(x) + x^{n-1}T(x)x + \cdots + T(x)x^n$ holds for all $x \in R$, then there exists $a \in Z(R)$ such that $T(x) = ax$ for all $x \in R$.

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In this article R denotes an associative ring. A ring R is called n-torsion free if $nx = 0$ for $x \in R$ implies that $x = 0$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$, and is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. Clearly, every derivation is a Jordan derivation. The converse is not true in general. A well known result of Herstein [5] states that every Jordan derivation on a prime ring of characteristic different from two is a derivation. Brešar and Vukman proved this result briefly in [3]. Further, Cusack [4] has

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generalized this result for semiprime ring stating that every Jordan derivation of a 2-torsion free semiprime ring is a derivation.

An additive mapping $T : R \to R$ is said to be a left (right) centralizer if $T(xy) = T(x)y$ (respectively $T(xy) = xT(y)$) holds for all $x, y \in R$ and Jordan left (right) centralizer if $T(x^2) = T(x)x$ (correspondingly $T(x^2) = xT(x)$) holds for all $x \in R$. An additive mapping which is both left and right centralizer is called simply a centralizer mapping. Obviously, every left (right) centralizer mapping is a Jordan left (right) centralizer mapping, but the converse is not true in general. Zalar [10] has proved that every Jordan left (right) centralizer mapping in a 2-torsion free semiprime ring is a left (right) centralizer mapping.

In [2], Brešar has introduced the generalized derivation mapping which covers both the concepts of derivation and left centralizer mapping. An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $D : R \to R$ such that $F(xy) = F(x)y + xD(y)$ holds for all $x, y \in R$. An additive mapping $F : R \to R$ is said to be a Jordan generalized derivation if there exists a Jordan derivation $D : R \to R$ such that $F(x^2) = F(x)x + xD(x)$ holds for all $x \in R$. In [1], Ashraf and Rehman showed that in a 2-torsion free ring R which has a commutator nonzero divisor, every Jordan generalized derivation on R is a generalized derivation. Recently, Vukman [9] has proved that every generalized Jordan derivation on a 2-torsion free semiprime ring is a generalized derivation. In the present paper, our aim is to improve above results by considering the identity $F(x^{n+1}) = F(x)x^n + xD(x)x^{n-1} + x^2D(x)x^{n-2} + \cdots + x^nD(x)$ for all $x \in R$, where R is any ring with identity element and F, D are two additive mappings. More precisely, we prove the following:

Theorem 1. *Let $n \geq 1$ be a fixed integer and let R be a $(n+1)!$-torsion free any ring with identity element. If $F : R \to R$ and $D : R \to R$ are two additive mappings such that $F(x^{n+1}) = F(x)x^n + xD(x)x^{n-1} + x^2D(x)x^{n-2} + \cdots + x^nD(x)$ holds for all $x \in R$, then D is a Jordan derivation and F is a Jordan generalized derivation.*

In [8], Vukman and Kosi-Ulbl proved that if R is a 2-torsion free semiprime rings with extended centroid C and $T : R \to R$ is an additive mapping such that $3T(xyx) = T(x)yx + xT(y)x + xyT(x)$ holds for all $x, y \in R$, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$. Recently, Kosi-Ulbl [6] showed that if R is a $(m+n+2)!$-torsion free semiprime ring with the identity element, where $m \geq 1, n \geq 1$ are fixed integers, and $T : R \to R$ is an additive mapping such that $T(x^{m+n+1}) = x^mT(x)x^n$ holds for all $x \in R$, then T is a centralizer mapping.

These results of Vukman and Kosi-Ulbl motivate us to consider the situation when $(n+1)T(x^{n+1}) = x^nT(x) + x^{n-1}T(x)x + \cdots + T(x)x^n$ for all $x \in R$, where R is a semiprime ring with identity element and T is an additive mapping. More precisely, we prove the following:

Theorem 2. *Let $n > 0$ be a fixed integer and let R be a $(n+2)!$-torsion free semiprime ring with identity element. If $T : R \to R$ is an additive mapping such that $(n+1)T(x^{n+1}) = x^n T(x) + x^{n-1}T(x)x + \cdots + T(x)x^n$ for all $x \in R$, then there exists $a \in Z(R)$ such that $T(x) = ax$ for all $x \in R$.*

Proof of Theorem 1. We have the relation

$$\begin{aligned} F(x^{n+1}) &= F(x)x^n + xD(x)x^{n-1} + x^2D(x)x^{n-2} + \cdots + x^nD(x) \\ &= F(x)x^n + \sum_{i=1}^{n} x^iD(x)x^{n-i} \end{aligned} \tag{1}$$

for all $x \in R$. Let e be the identity element of R, then replacing x by e in (1), we get $F(e) = F(e) + nD(e)$ which implies $nD(e) = 0$. Since R is n-torsion free, we may conclude that $D(e) = 0$.

Now replacing x by $x + ke$ in (1), where k is any positive integer, we get

$$F((x + ke)^{n+1}) = F(x + ke)(x + ke)^n + \sum_{i=1}^{n}(x + ke)^iD(x + ke)(x + ke)^{n-i}$$

for all $x \in R$. Expanding the power values of $(x + ke)$ and using the fact $D(e) = 0$, we have

$$\begin{aligned} &F\Big[x^{n+1} + \tbinom{n+1}{1}kx^n + \cdots + \tbinom{n+1}{n-1}k^{n-1}x^2 + \tbinom{n+1}{n}k^nx + k^{n+1}e\Big] \\ &= (F(x) + kF(e))\Big\{x^n + \cdots + \tbinom{n}{n-2}k^{n-2}x^2 + \tbinom{n}{n-1}k^{n-1}x + k^ne\Big\} \\ &\quad + \sum_{i=1}^{n}\Big\{x^i + \cdots + \tbinom{i}{i-2}k^{i-2}x^2 + \tbinom{i}{i-1}k^{i-1}x + k^ie\Big\}D(x) \\ &\quad \Big\{x^{n-i} + \cdots + \tbinom{n-i}{n-i-2}k^{n-i-2}x^2 + \tbinom{n-i}{n-i-1}k^{n-i-1}x + k^{n-i}e\Big\} \end{aligned} \tag{2}$$

for all $x \in R$. Using relation (1), this can be written as

$$kf_1(x, e) + k^2f_2(x, e) + \cdots + k^nf_n(x, e) = 0 \tag{3}$$

for all $x \in R$. Now, replacing k by $1, 2, 3, \ldots, n$ in turn, and considering the resulting system of n homogeneous equations, we see that the coefficient matrix of the system is a Van der Monde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2^2 & 2^3 & \cdots & 2^n \\ \vdots & \vdots & \vdots & & \vdots \\ n & n^2 & n^3 & \cdots & n^n \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than n, and since R is $(n+1)!$-torsion free, it follows immediately that $f_i(x,e)=0$ for all $x\in R$, $i=1,\dots,n$. Now, $f_n(x,e)=0$ implies that

$$(n+1)F(x)=F(x)+nF(e)x+nD(x)$$

that is $nF(x)=nF(e)x+nD(x)$. Since R is n-torsion free, $F(x)=F(e)x+D(x)$ for all $x\in R$. Now, $f_{n-1}(x,e)=0$ gives

$$\binom{n+1}{2}F(x^2)=nF(x)x+\binom{n}{2}F(e)x^2+\sum_{i=1}^{n}\Big\{ixD(x)+(n-i)D(x)x\Big\}.$$

Multiplying both sides by 2, it reduces to

$$\begin{aligned} n(n+1)F(x^2) &= 2nF(x)x+n(n-1)F(e)x^2 \\ &\quad +n(n+1)xD(x)+n(n-1)D(x)x. \end{aligned} \tag{4}$$

Since R is n-torsion free, this gives

$$\begin{aligned} (n+1)F(x^2) &= 2F(x)x+(n-1)F(e)x^2 \\ &\quad +(n+1)xD(x)+(n-1)D(x)x. \end{aligned} \tag{5}$$

Using $F(x)=F(e)x+D(x)$, this implies

$$\begin{aligned} (n+1)\Big\{F(e)x^2+D(x^2)\Big\} &= 2\Big\{F(e)x^2+D(x)x\Big\}+(n-1)F(e)x^2 \\ &\quad +(n+1)xD(x)+(n-1)D(x)x \end{aligned}$$

which is $(n+1)D(x^2)=(n+1)(D(x)x+xD(x))$ for all $x\in R$. Again, since R is $(n+1)$-torsion free, we have $D(x^2)=D(x)x+xD(x)$ for all $x\in R$ which means D is a Jordan derivation in R. Therefore, from $F(x)=F(e)x+D(x)$, we obtain that $F(x^2)=F(e)x^2+D(x)x+xD(x)=F(x)x+xD(x)$ for all $x\in R$, implying F a Jordan generalized derivation in R. Thus the proof is complete.

Proof of Theorem 2. We have the relation

$$\begin{aligned} (n+1)T(x^{n+1}) &= x^nT(x)+x^{n-1}T(x)x+\cdots+T(x)x^n \\ &= \sum_{i=0}^{n}x^iT(x)x^{n-i} \end{aligned} \tag{6}$$

for all $x\in R$. Let e be the identity element in R. Now replacing x by $x+ke$ in (6), where k is any positive integer, we get

$$\begin{aligned} &(n+1)T\Big[x^{n+1}+\binom{n+1}{1}kx^n+\cdots+\binom{n+1}{n-1}k^{n-1}x^2+\binom{n+1}{n}k^nx+k^{n+1}e\Big] \\ &=\sum_{i=0}^{n}\Big\{x^i+\cdots+\binom{i}{i-2}k^{i-2}x^2+\binom{i}{i-1}k^{i-1}x+k^ie\Big\}T(x+ke) \\ &\quad\Big\{x^{n-i}+\cdots+\binom{n-i}{n-i-2}k^{n-i-2}x^2+\binom{n-i}{n-i-1}k^{n-i-1}x+k^{n-i}e\Big\} \end{aligned} \tag{7}$$

for all $x \in R$, where we denote $\binom{n}{k} = 0$ for $k < 0$ and for $k > n$. Using relation (6), this can be written as

$$kf_1(x,e) + k^2 f_2(x,e) + \cdots + k^n f_n(x,e) = 0 \tag{8}$$

for all $x \in R$. Then by the similar argument as in the proof of Theorem 1, we have that $f_i(x,e) = 0$ for all $x \in R$, $i = 1, \ldots, n$. Now, $f_n(x,e) = 0$ implies,

$$(n+1)\binom{n+1}{n}T(x) = \sum_{i=0}^{n}\left\{T(x) + \binom{i}{i-1}xT(e) + \binom{n-i}{n-i-1}T(e)x\right\}$$

which gives multiplying both sides by 2 that

$$2(n+1)^2T(x) = 2(n+1)T(x) + n(n+1)(xT(e) + T(e)x).$$

Since R is $(n+1)$-torsion free,

$$2(n+1)T(x) = 2T(x) + n(xT(e) + T(e)x)$$

which is $2nT(x) = n(xT(e) + T(e)x)$. Again, since R is n-torsion free, we obtain $2T(x) = xT(e) + T(e)x$ for all $x \in R$. Now, $f_{n-1}(x,e) = 0$ gives

$$\begin{aligned} (n+1)\binom{n+1}{2}T(x^2) &= \sum_{i=0}^{n}\left\{\binom{i}{i-1}xT(x) + \binom{n-i}{n-i-1}T(x)x\right. \\ &\quad \left. + \binom{i}{i-2}x^2T(e) + \binom{n-i}{n-i-2}T(e)x^2 + \binom{i}{i-1}\binom{n-i}{n-i-1}xT(e)x\right\} \\ &= \sum_{i=0}^{n}\left\{i(xT(x) + T(x)x) + \binom{i}{2}(x^2T(e) + T(e)x^2) + i(n-i)xT(e)x\right\}. \end{aligned} \tag{9}$$

Since, $\sum_{i=0}^{n}\binom{i}{2} = \sum_{i=0}^{n} i(n-i) = \frac{(n+1)n(n-1)}{6}$, multiplying both sides by 6 in (9), we have

$$\begin{aligned} 3(n+1)^2nT(x^2) &= 3n(n+1)(xT(x) + T(x)x) \\ &\quad +(n+1)n(n-1)\left\{x^2T(e) + xT(e)x + T(e)x^2\right\}. \end{aligned}$$

Since R is $(n+2)!$-torsion free,

$$\begin{aligned} 3(n+1)T(x^2) &= 3(xT(x) + T(x)x) \\ &\quad +(n-1)\left\{x^2T(e) + xT(e)x + T(e)x^2\right\}. \end{aligned} \tag{10}$$

Now using $2T(x) = xT(e) + T(e)x$, (10) implies,

$$\begin{aligned} 3(n+1)(x^2T(e) + T(e)x^2) &= 3x(xT(e) + T(e)x) + 3(xT(e) + T(e)x)x \\ &\quad +2(n-1)\left\{x^2T(e) + xT(e)x + T(e)x^2\right\}. \end{aligned}$$

By simple manipulation, this reduces to $(n+2)[[T(e),x],x] = 0$ for all $x \in R$. Since R is $(n+2)$-torsion free, $[[T(e),x],x] = 0$ for all $x \in R$. Since R is semiprime ring, by [7, Theorem 2], $T(e) \in Z(R)$. Thus (**??**) implies that, $T(x) = T(e)x$, as desired.

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