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Lattices of Fuzzy Submodules

S. K. Bhambri

Department of Mathematics
Kirori Mal College, (University of Delhi)
Delhi-110 007, India
skbhambri@rediffmail.com

Pratibha Kumar

Department of Mathematics
Kirori Mal College, (University of Delhi)
Delhi-110 007, India

Abstract. A technique of generating a fuzzy submodule by a given arbitrary fuzzy set is provided. It is shown that (i) the sum of two fuzzy submodules of a module M is the fuzzy submodule generated by their union and (ii) the set of all fuzzy submodules of a given module forms a complete lattice. Consequently it is established that the collection of all fuzzy submodules, having the same value at zero, of M constitute a complete sublattice of the lattice of fuzzy submodules of M. Interrelationship of these finite range sublattices is established. Finally, it is shown that the lattice of all submodules of M can be embedded into a lattice of fuzzy submodules of M.
Throughout this note, M denotes an R-module, where R is a commutative ring with unity.

Keywords: Fuzzy submodules, level submodules, Fuzzy submodule generated by a fuzzy set, sum of two fuzzy submoudles, lattice of fuzzy submodules

1. Introduction and Results

Rosenfeld [8] initiated the study of fuzzy subgroups. Liu [6] introduced the notion of fuzzy set products and fuzzy subrings while Kuroki [5] persued the study of fuzzy semigroups. Pan [7] and Katsaras and Liu [3] introduced the concept of fuzzy submodules and fuzzy vector spaces respectively. More recently in [1], the notion of set products is discussed in details and in [2], the lattice theoretical aspect of fuzzy subgroups and fuzzy normal subgroups are explored.

The formation of a lattice of submodules of a module is well known feature in classical algebra. However, the same has not been explored in fuzzy setting. In order to initiate such studies, the concept of fuzzy submodule generated by an arbitrary fuzzy set is formulated in this note. Using this concept we construct various types of lattices of fuzzy submodules and establish an embedding of the lattice of all submodules of a module M into the lattice of fuzzy submodules of M.

1.1. Definition. Let μ be a fuzzy subset of a set S and $t \in [0,1]$. Then the set

$$\mu_t = \{x \in S \mid \mu(x) \geq t\}$$

is called a level subset of μ.

1.2. Definition. A fuzzy subset μ of M is called a fuzzy submodule of M if the following conditions are satisfied :

(i) $\mu(a+b) \geq \min\{\mu(a), \mu(b)\}$, for all $a, b \in M$, and
(ii) $\mu(xa) \geq \mu(a)$, for all $a \in M$, $r \in R$.

For a fuzzy submodule of μ of M the level subset $\mu_t = \{a \in M \mid \mu(a) \geq t\}$, $t \in \operatorname{Im}\mu$, are submodules of M called the *level submodules* of M.

The following result can be easily proved.

1.3. Lemma. *A fuzzy subset μ of M is a fuzzy submodule of M if and only if each level subset $\mu_t, t \in Im\mu$, is a submodule of M.*

1.4. Definition. Let μ be a fuzzy subset of M. Define a fuzzy subset $\langle\mu\rangle$ of M as follows :

$$\langle\mu\rangle(x) = \sup\{k \mid x \in \langle\mu_k\rangle\}\,, \qquad x \in M\,.$$

$\langle\mu\rangle$ is called the *fuzzy subset of M generated by μ*. (Here $\langle\mu_k\rangle$ is the submodule of M generated by the level subset μ_k).

1.5. Theorem. *Let μ be a fuzzy submodule of M. Then the fuzzy subset $\langle\mu\rangle$ is a fuzzy submodule of M generated by μ. Moreover $\langle\mu\rangle$ is the smallest fuzzy submodule containing μ.*

Proof. Let $x, y \in M$ and let $\langle\mu\rangle(x) = t_1$, $\langle\mu\rangle(y) = t_2$ and $\langle\mu\rangle(x+y) = t$.
Let, if possible,

$$\begin{aligned} t &= \langle\mu\rangle(x+y) \\ &< \min\{\langle\mu\rangle(x), \langle\mu\rangle(y)\} \\ &= \min\{t_1, t_2\} \\ &= t_1 \qquad\qquad \text{(say)}. \end{aligned}$$

Then $t_1 = \langle\mu\rangle(x) = \sup\{k \mid x \in \langle\mu_k\rangle\} > t$, therefore there exists $k_1 > t$ such that $x \in \langle\mu_{k_1}\rangle$.
Also $t_2 = \langle\mu\rangle(y) = \sup\{k \mid y \in \langle\mu_k\rangle\} \geq t_1 > t$, therefore there exists $k_2 > t$ such that $y \in \langle\mu_{k_2}\rangle$.

Without loss of generality we may assume that $k_1 \geq k_2$, so that $\langle \mu_{k_1} \rangle \subseteq \langle \mu_{k_2} \rangle$. Then $x, y \in \langle \mu_{k_2} \rangle$, that is $x + y \in \langle \mu_{k_2} \rangle$, which is a contradiction since $k_2 > t$. Therefore $t \geq t_1$.
Consequently

$$\langle \mu \rangle (x + y) \geq \min\{\langle \mu \rangle (x), \langle \mu \rangle (y)\} . \tag{1}$$

Now let, if possible, $t_3 = \langle \mu \rangle (rx) < \langle \mu \rangle (x) = t_1$.
Then $t_1 = \langle \mu \rangle (x) = \sup\{k \,|\, x \in \langle \mu_k \rangle\} > t_3$, therefore there exists k such that $x \in \langle \mu_k \rangle$ and $t_1 > k > t_3$.
So that $rx \in \langle \mu_k \rangle \subseteq \langle \mu_{t_1} \rangle$, which is a contradiction.
Hence

$$t_3 = \langle \mu \rangle (rx) \geq \langle \mu \rangle (x) = t_1 . \tag{2}$$

Consequently conditions (1) and (2) yield that $\langle \mu \rangle$ is a fuzzy submodule of M. Finally, to show that $\langle \mu \rangle$ is the smallest fuzzy submodule containing μ, let us assume θ to be a fuzzy subbmodule of M such that $\mu \subseteq \theta$ and show that $\langle \mu \rangle \subseteq \theta$.
Let, if possible, $t = \langle \mu \rangle (x) > \theta(x)$ for some $x \in M$.
Let $\varepsilon > 0$ be given. Then

$$t - e < t = \sup\{k \,|\, x \in \langle \mu_k \rangle\} .$$

Therefore there exists k such that $x \in \langle \mu_k \rangle$ and $t - \varepsilon < k < t$. So that $x \in \langle \mu_k \rangle \subseteq \langle \mu_{t-\varepsilon} \rangle$, for all $\varepsilon > 0$. Now $x = \alpha_1 x_1 + \ldots + \alpha_n x_n$, $\alpha_i \in R$, $x_i \in \mu_{t-\varepsilon}$. $x_i \in \mu_{t-\varepsilon}$ implies $\mu(x_i) \geq t - \varepsilon$, that is $\theta(x_i) \geq t - \varepsilon$, for all $\varepsilon > 0$. Therefore

$$\begin{aligned} \theta(x) &\geq \min\{\theta(x_1), \ldots , \theta(x_n)\} , \\ &\geq t - \varepsilon , \qquad \text{for all } \ \varepsilon > 0 . \end{aligned}$$

Hence $\theta(x) = t$, which is a contradiction to our supposition. □

Thus the result follows.

1.6. Definition. Let μ and θ be fuzzy submodules of an R-module M. Then the sum of μ and θ denoted by $\mu + \theta$ is defined as :

$$(\mu + \theta)(x) = \sup_{x=a+b} \{\min(\mu(a), \theta(b))\}, \qquad x \in M .$$

Clearly $\mu + \theta$ is a fuzzy subset of M.

1.7. Theorem ([4]). *Let μ and θ be fuzzy submodules of M. Then the sum $\mu + \theta$ is a fuzzy submodule of M.*

1.8. Proposition. *Let μ, θ be fuzzy submodules of M such that $\mu(0) = \theta(0)$. Then $\mu \subseteq \mu + \theta$, $\theta \subseteq \mu + \theta$.*

Proof. Let $x \in M$. Then

$$\begin{aligned}(\mu+\theta)(x) &= \sup_{x=a+b}\{\min(\mu(a),\theta(b))\} \\ &\geq \min(\mu(x),\theta(0)) \\ &= \min(\mu(x),\mu(0)) \\ &= \mu(x)\,.\end{aligned}$$

Similarly $(\mu+\theta)(x) \geq \theta(x)$. □

1.9. *Remark.* Proposition 1.8 is not true if $\mu(0) \neq \theta(0)$, for let $\mu(0) > \theta(0)$, then $(\mu+\theta)(0) = \theta(0) < \mu(0)$ and so $\mu \nsubseteq \mu+\theta$.

The following results shows that the sum of two fuzzy submodules is the fuzzy submodule generated by their union when they have same value at zero.

1.10. Theorem. *Let μ and θ be fuzzy submodules of M such that $\mu(0) = \theta(0)$. Then $(\mu+\theta) = \langle \mu \cup \theta \rangle$.*

Proof. Let $x \in M$ and let

$$t_1 = (\mu+\theta)(x) = \sup_{x=a+b}\{\min(\mu(a),\theta(b))\}\,.$$

Let $\varepsilon > 0$ be given. Then

$$t_1 - \varepsilon < \min(\mu(a),\theta(b)), \qquad \text{for some } a, b \in M$$

such that $x = a+b$, so that $t_1 - \varepsilon < \mu(a)$, $t_1 - \varepsilon < \theta(b)$.
But $\mu, \theta \subseteq (\mu \cup \theta) \subseteq \langle \mu \cup \theta \rangle$, therefore

$$\begin{aligned}t_1 - \varepsilon &< \min\{\langle \mu \cup \theta \rangle(a), \langle \mu \cup \theta > (b)\} \\ &\leq \langle \mu \cup \theta \rangle(a+b) \\ &= \langle \mu \cup \theta \rangle(x)\,, \qquad \text{for all } \ \varepsilon > 0\,.\end{aligned}$$

Hence $t_1 \leq \langle \mu \cup \theta \rangle(x) = t$ (say).
Let if possible

$$\begin{aligned}(\mu+\theta)(x) &= t_1 < t = \langle \mu \cup \theta \rangle(x) \\ &= \sup\{k \,|\, x \in \langle (\mu \cup \theta)_k \rangle\}\,.\end{aligned}$$

Therefore there exists k such that

$$x \in \langle (\mu \cup \theta)_k \rangle \quad \text{and} \quad t_1 < k < t\,.$$

Then $(\mu \cup \theta)_t \subseteq (\mu \cup \theta)_k \subseteq (\mu \cup \theta)_{t_1}$. so that $\langle (\mu \cup \theta)_t \rangle \subseteq \langle (\mu \cup \theta)_k \rangle \subseteq \langle (\mu \cup \theta)_{t_1} \rangle$ implying $x \in (\mu \cup \theta)_{t_1} \rangle$, which is a contradiction.
Hence $t_1 = (\mu+\theta)(x) = \langle \mu \cup \theta \rangle(x) = t$. □

1.11. Corollary. *For any fuzzy submodule μ of M, $\mu + \mu = \mu$.*

1.12. *Remark.* Theorem 1.10 is not true if $\mu(0) \neq \theta(0)$, for let $\mu(0) > \theta(0)$, then $(\mu \cup \theta)(0) = \mu(0) > (\mu \cup \theta)(0)$.
So, $\langle \mu \cup \theta \rangle(0) > (\mu \cup \theta)(0)$. Hence $(\mu+\theta) \neq \langle \mu \cup \theta \rangle$.

Let $\mathcal{M}$ be the set of all fuzzy submodules of M. Then clearly the intersection of an arbitrary family of fuzzy submodules of M is a fuzzy submodule and Theorem 1.10 shows the existence of the least fuzzy submodule containing the union of an arbitrary family of fuzzy submodules of M. These facts give rise to the following :

1.13. Proposition. *$\mathcal{M}$ is a lattice under the usual ordering of fuzzy set inclusion. Moreover $\mathcal{M}$ is a complete lattice.*

Next, let t be an arbitrary but fixed real number in $]0,1]$ and let us denote by $\mathcal{M}_t$, the subset of all fuzzy submodules θ of M such that $\theta(0) = t$.
Then we have the following :

1.14. Proposition. *$\mathcal{M}_t$ is a complete sublattice of $\mathcal{M}$.*

Let $\mathcal{M}_t$ denote the set of all fuzzy submodules θ of M such that $\mathrm{Im}\theta = \{\theta(x) \,|\, x \in M\}$ is finite (That is θ has finite range).
Then again we have,

1.15. Proposition. *$\mathcal{M}_t$ is a sublattice of $\mathcal{M}$. Let $\mathcal{M}_t$ denote the set of all fuzzy submodules θ of M such that $\theta(0) = t$ and Imθ is finite.*

1.16. Proposition. *$\mathcal{M}_{f_t}$ is a sublattice of $\mathcal{M}$.*

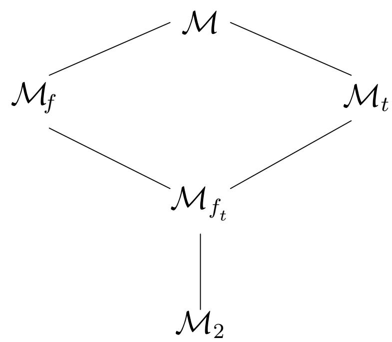
Proof. Since $\mathcal{M}_{f_t} = \mathcal{M}_f \cap \mathcal{M}_t$ and intersection of sublattices is a sublattice, M_{f_t} is a sublattice of $\mathcal{M}$. □

Finally, we intend to demonstrate an embedding of well known lattice of all submodules of M into the lattice $\mathcal{M}$ of all fuzzy submodules of $\mathcal{M}$. This we accomplish by considering the set $\mathcal{M}_2$ of all fuzzy submodules defined as follows :

$$\mathcal{M}_2 = (\theta \,|\, \theta \in \mathcal{M} \ \text{ s.t. } \ \mathrm{Im}\theta = (r,t), r < t).$$

The following is obvious.

1.17. Proposition. *$\mathcal{M}_2$ is a sublattice of $\mathcal{M}_{f_t}$ and hence of $\mathcal{M}$. We have the following diagram :*



1.18. Proposition. *$\mathcal{L}(M)$, the lattice of all submodules of M can be embedded in $\mathcal{M}_2$.*

Proof. Let $A \in \mathcal{L}(M)$. Define $\theta : M \to [0,1]$, such that

$$\begin{aligned} \theta(x) &= t \quad \text{if } x \in A \\ &= 0 \quad \text{if } x \notin A\,. \end{aligned}$$

Then θ is a fuzzy submodule of M (by Lemma 1.3) such that $\text{Im}\theta = (0,t), 0 < t$.
Therefore $\theta \in \mathcal{M}_2$.
Define $\sigma : \mathcal{L}(M) \to \mathcal{M}_2$,, such that $\sigma(A) = \theta$.Then σ is clearly well defined.
Let $\sigma(A) = \sigma(B)$. Then $\theta = \mu$, implies $A = B$. So σ is one-one.
Let $A, B \in \mathcal{L}(M)$. Let $\sigma(A) = \theta$, $\sigma(B) = \mu$. Then

$$\begin{aligned} \sigma(A) + \sigma(B) &= \theta + \mu \\ &= \langle \theta \cup \mu \rangle\,, \qquad \text{(by 1.10)} \\ &= \sigma(A + B) \end{aligned}$$

and $\sigma(A \cap B) = \theta \cap \mu = \sigma(A) \cap \sigma(B)$.
Thus σ is a lattice homomorphism. Hence σ is a lattice embedding. □

1.19. *Remark.* The above result justifies our modification of Pan' s definition [7] of fuzzy submodules wherein he assumes that the fuzzy submodules necessarily assumes value 1 at zero of given module.

Since fuzzy submodules are fuzzy normal subgroups, the following result follows from the corresponding Theorem in [2].

1.20. Theorem. *$\mathcal{M}_t$ is a modular sublattice of $\mathcal{M}$ for each $t \in [0,1]$.*

1.21. Corollary. *The set $\mathcal{L}(M)$ of all submodules of M forms a modular lattice.*

Proof. Let $\mathcal{M}_\chi$ denote the set of characteristic functions of all the submodules of M. It is easy to verify that $\mathcal{M}_\chi$ is a sublattice of $\mathcal{M}_t$ for $t = 1$. Moreover,

$\mathcal{L}(M)$ is isomorphic to $\mathcal{M}_\chi$ under the map $A \to \chi_A$ as $\chi_{A+B} = \chi_A + \chi_B$ and $\chi_{A\cap B} = \chi_A \cap \chi_B$. Since sublattice of a modular lattice is a modular lattice, $\mathcal{M}_t$ is modular and hence, $\mathcal{L}(M)$ is modular. □

References

[1] N. Ajmal, Set product and Fuzzy subgroups, *Proc. of IFSA World Congress* (1991) 3-7.

[2] N.Ajmal, K.V. Thomas, Lattices of Fuzzy subgroups and Fuzzy normal subgroups, *Information Sciences*, 76 (1994) 1-11.

[3] A.K. Katsaras and D.B. Liu, Fuzzy vector spaces and Fuzzy topological vector spaces, *J.Math. Anal. Appl.* 8(1977) 135-146.

[4] R.Kumar, S.K. Bhambri, Pratibha Kumar, Fuzzy submodules: Some Analogues and Deviations, *Fuzzy Sets and Systems*, 70 (1995) 125-130.

[5] N.K. Kuroki, On Fuzzy semigroups, *Information Sciences*, 53 (1991) 203-236.

[6] W.J. Liu, Operations on Fuzzy Ideals, *Fuzzy Sets and Systems*, 11 (1983) 31-41.

[7] Fu-Zheng Pan, Fuzzy finitely generated modules, *Fuzzy Sets and Systems*, 21 (1987) 105-113.

[8] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* 35(1971) 512-517.

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