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Rank 2 Weakly Arithmetically Cohen-Macaulay Vector Bundles on Hirzebruch Surfaces[1]

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Abstract. Let E be a vector bundle on a Hirzebruch surface F_e, $e \geq 0$. We will say that E is weakly arithmetically Cohen-Macaulay or WACM if $h^1(E \otimes R) = h^1(E \otimes R^*) = 0$ for all ample line bundles L on F_e. Here we classify all WACM rank 2 vector bundle on F_e: there are many families formed by vector bundles which are not extensions of two line bundles.

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1. Introduction

Let X be an integral n-dimensional projective variety, $n \geq 2$. Let η_+ or $\eta_+(X)$ denote the ample cone of $\mathrm{Pic}(X)$ and η_- its opposite. Let η_0 (resp. $\widetilde{\eta}_0$) denote the set of all line bundles on X algebraically equivalent to $\mathcal{O}_X$ (resp. numerically trivial). Set $\eta := \eta_+ \cup \eta_-$, $\gamma := \eta \cup \eta_0$ and $\widetilde{\gamma} := \eta \cup \widetilde{\eta}_0$. Let E be a vector bundle on X. We will say that E is ACM or *arithmetically Cohen-Macaulay* (resp. say that E is WACM or *weakly arithmetically Cohen-Macaulay*, resp. SACM or *strongly arithmetically Cohen-Macaulay*) if $H^i(X, E \otimes L) = 0$ for all $1 \leq i \leq n-1$ and all $L \in \gamma$ (resp. $L \in \eta$, resp. $L \in \widetilde{\gamma}$). In this paper X will always be a Hirzebruch surface F_e, $e \geq 0$. In [1] we described the ACM and the WACM line bundles on F_e (see Remark 3 below), and proved that any ACM rank 2 vector bundle on F_e is an extension of two ACM line bundles. Here we will give a complete classification of all WACM rank 2 vector bundle on F_e. Many of them are not extension of two line bundles and to describe them we need to introduce the following notation. Let $\pi : F_e \to \mathbf{P}^1$ denote a ruling, f a fiber of π and h a section of π with

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minimal self-intersection. We have $\mathrm{Pic}(X) \cong \mathbb{Z}h + \mathbb{Z}f$, $h^2 = -e$, $h \cdot f = 1$, $f^2 = 0$. In this note we prove the following result.

Theorem 1. *Let E be a rank 2 vector bundle on the Hirzebruch surface F_e, $e \ge 0$. E is WACM if and only if either it is an extension of two WACM line bundles on F_e or there are integers y, c, β such that $\beta > 0$, $2y - c \ge \beta$, and $y - c + e + 2 \ge \beta$, a curvilinear zero-dimensional subscheme $Z \subset F_e$ such that $\pi|Z$ is an isomorphism onto the degree β effective divisor $\pi(Z) \subset \mathbb{P}^1$ and E is any locally free middle term of an extension*

$$0 \to \mathcal{O}_{F_e}(-h + cf) \to E \to \mathcal{I}_Z(-h + (y - c)f) \to 0 \tag{1}$$

For any y, c, β as above there is E locally free and WACM with these invariants (see Remark 1 for a description of the family of all such bundles and the stability properties of any such E).

For a description of all WACM and all ACM line bundles on F_e, see Example 3. Hence the known cohomology of the line bundles of F_e gives a complete description of all extensions of two WACM line bundles, the dimension of their families, and check when these vector bundle are stable or semistable with respect to some (or to all) polarizations.

Remark 1. Fix integers y, c, β such that $\beta > 0$ and a length β zero-dimensional subscheme $Z \subset F_e$. Assume the existence of an exact sequence (1). Then Z is a locally complete intersection zero-dimensional scheme. If $\pi|Z$ is an isomorphim onto its image, then Z is curvilinear. Now we fix a zero-dimensional locally complete intersection $Z \subset F_e$ with length $\beta > 0$. Since $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h - (e+2)f)$, $h^0(F_e, Hom(\mathcal{O}_{F_e}(-h + cf), \mathcal{O}_{F_e}(-h + cf)) \otimes \omega_{F_e}) = 0$. Hence the Cayley-Baracharach property is satisfied ([2]). Thus a general extension (1) has locally free middle term. The set of all vector bundles E fitting in (1) is parametrized by a non-empty open subset of a projective space of dimension $\max\{0, 2c - y - 1\} + \beta - 1$. Fix any such E. If $y > 2c$ (resp. $y = 2c$, resp. $y < 2c$), then E is slope stable (resp. slope properly semistable. resp. not slope semistable) with respect to any polarization of F_e. The exact sequence (1) gives $c_1(E) = \mathcal{O}(-2h + yf)$ and $c_2(E) = -y - e + \beta$. Take a zero-dimensional $Z \subset F_e$ such that $\mathrm{length}(Z) = \beta > 0$. If $\beta = 1$, then Z is a point and hence $\pi|Z$ is an isomorphism. If $\beta \ge 2$, then there is schemes Z such that $\pi|Z$ is an isomorphism, but $\pi|Z$ is an isomorphism for a non-empty open subset of the Hilbert scheme of all length β closed subschemes of F_e.

We work over an algebraically closed field $\mathbb{K}$.

2. Proof of Theorem 1

Set $\mathcal{O} := \mathcal{O}_{F_e}$. For any sheaf G on F_e and any integer $i \ge 0$ set $H^i(G) := H^i(F_e, G)$ and $h^i(G) := \dim(H^i(G))$.

Remark 2. Let Y be any integral projective variety. Any extension of two ACM (resp. WACM, resp. SACM) vector bundles on Y is ACM (resp. WACM, resp. SACM).

The following elementary remark is contained in [1].

Remark 3. Fix an integer $e \ge 0$. We have $\mathrm{Pic}(F_e) \cong \mathbb{Z}h + \mathbb{Z}f$, $h^2 = -e$, $h \cdot f = 1$, $f^2 = 0$ and $\omega_{F_e} \cong \mathcal{O}(-2h - (e+2)f)$. $\mathcal{O}(ah + bf) \in \eta_+$ if and only if $a > 0$ and $b > ae$. Set $R := \mathcal{O}(ah + bf)$. Use the Leray spectral sequence of π to compute the cohomology of R. If $a \ge 0$, then $h^1(R) = 0$ if and only if $b \ge ae - 1$. If $a = -1$, then $h^1(R) = 0$ for any b. Assume $a \le -2$. Serre duality gives $h^1(R) = h^1(\mathcal{O}((-a-2)h + (-b-e-2)f)$. Notice that $-b-e-2 \ge e(-a-2)-1$ if and only if $b \le ea+e-1$ Thus $h^1(R)) = 0$ if and only if either $a \ge 0$ and $b \ge -1$ or $a = -1$ or $a \le -2$ and $b \le ea+e-1$. Now we check for which values of the integers a, b the line bundle R is WACM or ACM. First assume $a \ge 0$. If $b \ge ea-1$ (resp. $b \ge ea-2$), then $h^1(R \otimes L) = 0$ for all $L \in \eta_+ \cup \{\mathcal{O}\}$ (resp. $L \in \eta_+$). If $a > 0$, then $h^1(R(-h - xf)) > 0$ if $x \gg 0$. Hence if $a > 0$, then R is not WACM. Now assume $a = 0$. $h^1(R) = 0$ if and only if $b \ge -1$. $h^1(R \otimes L) = 0$ for all $L \in \eta_+$ if and only if $b \ge -2$. Fix integers $t \ge 2$, $c \ge 0$. Serre duality gives $h^1(R(-th-(te+c+1)f)) = h^1(\mathcal{O}((t-2)h+(-b+te+c+1-e-2)f))$. Hence $h^1(\mathcal{O}(R(-th-(te+c+1)f)) = 0$ if and only if $-b+te+c+1-e-2 \ge e(t-2)-1$. Hence $h^1(R(-th-(te+c+1)f)) = 0$ for all $t \ge 2$ and all $c \ge 0$ if and only if $b \le e$. Since $h^1(O_X) - h + zf)) = 0$ for any z, $\mathcal{O}(bf)$ is ACM (resp. WACM) if and only $-1 \le b \le e$ (resp. $-2 \le b \le e$. Now assume $a = -1$. Hence $h^1(R) = 0$. $h^1(R \otimes L) = 0$ for all $L \in \eta_+$ if and only if $b \ge -e-2$. $h^1(R \otimes L^*) = 0$ for all $L \in \eta_+$ if and only if $h^1(\mathcal{O}(-(t+1)h + (b-te-1-c)f)) = 0$ for all $t \ge 1$ and all $c \ge 0$, i.e. (Serre duality) if and only if $h^1(\mathcal{O}((t-1)h + (te+c+1-b-e-2)f)) = 0$, i.e. if and only $-b-1 \ge -1$, i.e. if and only if $b \le 0$. Hence if $a = -1$ R is ACM if and only if it is WACM if and only if $-e-2 \le b \le 0$. Now assume $a = -2$. $h^1(R) = 0$ if and only if $b \le -e-1$. It is easy to check that $h^1(R \otimes L) = 0$ for all $L \in \eta_+$ if and only if $h^1(R(2h + (2e+1)f)) = 0$, i.e. if and only if $b \ge -2e-2$. It is easy to check that $h^1(R \otimes L) = 0$ for all $L \in \eta_-$ if and only if $h^1(R(-h-(e+1)f)) = 0$, i.e. if and only if $b - e - 1 \le -3a + e - 1$, i.e. if and only if $b \le -e$. Hence $\mathcal{O}(-2h + bf)$ is ACM (resp. WACM) if and only if $-2e-2 \le b \le -e-1$ (resp. $-2e-2 \le b \le -e$). Now assume $a \le -3$. Since $h^1(R(h + xf)) > 0$ for all $x \gg 0$, R is not WACM.

Proof of Theorem 1. For the existence part of a locally free E fitting in an extension (1), see Remark 1. Let E be a rank 2 vector bundle on X. Set $\mathcal{O}(xh+yf) := \det(E)$. We first assume that E is WACM. There is a maximal integer z such that E has a rank 1 subsheaf $O_X(zh+cf)$ for some c. Taking maximal the integer c (for fixed z) we obtain an exact sequence on X:

$$(2) \qquad 0 \to \mathcal{O}(zh+cf) \to E \to \mathcal{I}_Z((x-z)h + (y-c)f) \to 0$$

in which Z is a locally complete intersection zero-dimensional subscheme of Z. We saw in [1] that $-2 \le x \le 0$, $-4 \le z \le -2$, that if $Z \ne \emptyset$, then E is an extension of two WACM line bundles. We also saw in [1] that if $Z \ne \emptyset$, then $(x, z) = (-2, -1)$.

Hence for now on we assume $Z \ne \emptyset$ and $(x, z) = (-2, -1)$. Hence the extension (2) is just the extension (1). Set $\beta := \mathrm{length}(Z) > 0$, $A := \mathcal{O}(-h + cf)$ and $B := \mathcal{O}(-h + (y - cf))$. Fix integers $a > 0$ and $b > ae$. Since $h^i(A \otimes \mathcal{O}(ah + bf)) = 0$ for $i = 1, 2$, $h^1(E(ah + bf)) = 0$ if and only if $\mathcal{I}_Z((a-1)h + (b + y - c)F) = 0$. Since $\mathcal{O}((a-1)h + (b - e)f)$ is spanned, the last inequality is true for all $a > 0$, $b > ae$ of and only if it is true for the pair $(a, b) := (1, e + 1)$. Hence $h^1(E \otimes R) = 0$ for all $R \in \eta_+$ if and only if $\mathcal{I}_Z((y - c + e + 1)) = 0$, i.e. if and only if $\pi|Z$ is an isomorphism and

$$y - c + e + 2 \ge \beta. \tag{3}$$

Now we study the condition $h^1(E \otimes R^*) = 0$, i.e. the condition $h^1(E(2h + yf) \otimes \omega_X \otimes R) = 0$ (Serre duality and $E \cong E^* \otimes \det(E)^*$), i.e. the condition $h^1(E \otimes M) = 0$, where $M \cong \mathcal{O}(ah + (b + y - e - 2)f)$. Since $h^i(A \otimes M) = 0$ for $i = 1, 2$, $h^1(E \otimes M) = 0$ if and only if $h^1(\mathcal{I}_Z((a-1)h + (b + 2y - c - e - 2)f)) = 0$. Again, this is true for all $a > 0$ and all $b > ae$ if and only if this is true for the pair $(a, b) := (1, e + 1)$. Again, we find that this is true if and only if $\pi|Z \cong Z$ and

$$2y - c \ge \beta \tag{4}$$

Thus (modulo [1]) the proof of Theorem 1 is over. □

References

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