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On the Secant Varieties of the Osculating Varieties of Veronese Embeddings of $\mathbf{P}^n$: An Asymptotic Result

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Abstract. Fix Integers $n > 0, k > 0$. Here we prove the existence on an integer $d(n,k)$ with the following property. Fix any integer $d \geq d(n,k)$. Let $O_{k,n,d} \subset \mathbf{P}^N$, $N := \binom{n+d}{n} - 1$, be the order k osculating variety of the order d Veronese embedding of $\mathbf{P}^n$. Then all the secant varieties of $O_{k,n,d}$ have the expected dimension.

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1. Introduction

We work over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$. Fix integers $n > 0$ and $d > k > 0$. Let $V_{n,d} \subseteq \mathbf{P}^N$, $N := \binom{n+d}{n} - 1$, be the order d Veronese embedding of $\mathbf{P}^n$ and $O_{k,n,d} \subseteq \mathbf{P}^N$ the order k osculating variety of $V_{n,d}$ in $\mathbf{P}^N$, i.e. the closure in $\mathbf{P}^N$ of the union of the general k-osculating linear spaces of $V_{n,d}$. $\dim(O_{k,n,d}) = \min\{N, \binom{n+k}{n} + n - 1\}$, i.e. $O_{k,n,d}$ has the expected dimension ([2], Lemma 3.3). For any integral m-dimensional projective subvariety $X \subseteq \mathbf{P}^N$ and any integer $s \geq 1$ let $S^{s-1}(X)$ denote the closure in $\mathbf{P}^N$ of all $(s-1)$-dimensional linear subspaces of $\mathbf{P}^N$ spanned by s points of X. $S^{s-1}(X)$ is an integral variety and $\dim(S^{s-1}(X)) \leq \min\{N, s(m+1)-1\}$. If $\dim(S^{s-1}(X)) = \min\{N, s(m+1)-1\}$, then we will say that $S^{s-1}(X)$ has the expected dimension. X is said to be *ordinary* if all its secant varieties have the expected dimension. A. Bernardi, M. V. Catalisano, A.V. Geramita and A. Gimigliano studied in [2] the dimension of the secant varieties of $O_{k,n,d}$.

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To prove their results they introduced the following zero-dimensional schemes. Fix $P \in \mathbf{P}^n$ and a line $D \subseteq \mathbf{P}^n$ such that $P \in D$. For any integer $m > 0$ let mP denote the infinitesimal neighborhood of order $m-1$ of P in $\mathbf{P}^n$, i.e. the closed subscheme of $\mathbf{P}^n$ with $(\mathcal{I}_P)^m$ has its ideal sheaf. Hence $(mP)_{red} = \{P\}$ and length$(mP) = \binom{n+m-1}{n}$. mP will be called an m-point of $\mathbf{P}^n$. Fix a homogeneous coordinate system $x_0, \dots, x_n$ such that $P = (1; 0; \dots; 0)$ and $D = \{x_2 = \cdots = x_n = 0\}$. Write $z_i := x_i/x_0$, $1 \le i \le n$, and use $z_1, \dots, z_n$ as affine coordinates around P. Let $Z(n,k;P,D)$ be the closed subscheme defined by the equations given by all monomials in the variables $z_1, \dots, z_n$ of degree at least $k+2$ and by all monomials of degree $k+1$, except the n monomials $z_1^k z_i$, $1 \le i \le n$. Hence $(k+1)P \subset Z(n,k;P,D) \subseteq (k+2)P$, length$(Z(n,k;P,D)) = \binom{n+k}{n} + n$ and $Z(n,k;P,D) = (k+2)P$ if and only if $n = 1$. We will say that $Z(n,k;P,D)$ is a (k+1,k+2)-point of $\mathbf{P}^n$. Notice that $Z(n,k;P,D)$ only depends from n, k, P and the tangent vector at P determined by D, but not from the coordinate system, because if instead of z_1 we write $z_1 + M$ with M a power series in $z_1, \dots, z_n$ with no term of degree ≤ 1, then all terms $(z_1+M)^k z_i - z_1^k z_i$, $1 \le i \le n$, are zero modulo the ideal sheaf of $(k+2)P$. We will say that $Z(n,k;P,D)$ is a $(k+1,k+2)$-point of $\mathbf{P}^n$. Let $Z(n,k;P,D)'$ be the closed subscheme defined by the equations given by all monomial in the variables $z_1, \dots, z_n$ of degree at least $k+2$ and by all monomials of degree $k+1$, except the monomial z_1^{k+1}. The scheme $Z(n,k;P,D)'$ depends only from n, k, P and the tangent dircetion ot P determined by D, length$(Z(n,k;P,D)') = \binom{n+k}{n} + 1$, and $(k+1)P \subset Z(n,k;P,D)' \subseteq Z(n,k;P,D)$. We will say that $Z(n,k;P,D)'$ is a (k+1;1)-point of $\mathbf{P}^n$. Now we list our main results.

Theorem 1. *For all positive integers n, k there is a positive integer $a(n,k)$ with the following properties. Fix non-negative integers s, d, b, a_m, $2 \le m \le k+1$, such that $d \ge a(n,k)$. Let $Z \subset \mathbf{P}^n$ be a general union of s (k+1,k+2)-points, b (k;1)-points, and a_m m-points of $\mathbf{P}^n$ for all $2 \le m \le k+1$. Then either $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ (case length$(Z) \le \binom{n+d}{n}$) or $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ (case length$(Z) \ge \binom{n+d}{n}$).*

Corollary 1. *For all positive integers n, k there is a positive integer $d(n,k)$ with the following properties. Fix integers $s > 0$ and $d \ge d(n,k)$. Let $Z \subset \mathbf{P}^n$ be a general union of s (k+1,k+2)-points of $\mathbf{P}^2$. Then either $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$ (case $s\binom{n+k}{n} + sn \le \binom{n+d}{n}$) or $h^0(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$ (case $s\binom{n+k}{n} + sn \ge \binom{n+d}{n}$), i.e. $O_{k,n,d}$ is ordinary. We may take $d(2,k) = (k+2)^4$.*

The “ i.e. ” part (i.e. that $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ if and only if $\dim(S^{s-1}(O_{k,n,d})) = \binom{n+d}{n} - 1$ and that $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ if and only if $\dim(S^{s-1}(O_{k,n,d})) = s(\binom{n+k}{n} + n) - 1)$ is proved in [2], §3. The case $n = 2$ of Corollary 1 is not covered by Theorem 1. We will directly prove it. However, to prove Corollary 1 for an integer $n \ge 3$ we will use Theorem 1 for the integer $n' := n - 1$.

The case $n = 2$ of the following result seems to be rather strong (see section 3 for its proof and for a less interesting result with an upper bound for $h^1(\mathbf{P}^n, \mathcal{I}_Z(d))$).

Proposition 1. *Fix positive integers n, k, d, s. Let $Z \subset \mathbf{P}^n$ (resp. $W \subset \mathbf{P}^n$) be a general union of s $(k+1, k+2)$-points (resp. $(k+1)$-points). If $h^0(\mathbf{P}^n, \mathcal{I}_W(d)) \le s$, then $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$, i.e.* $\dim(S^{s-1}(O_{k,n,d})) = \binom{n+d}{n} - 1$.

Question 1. Fix an integer $k \ge 2$. Is there an integer $d(k)$ not depending from n such that either $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ (case $s(k+2)(k+1)/2 + 2s \le (d+2)(d+1)/2)$ or $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$ for all integers $n \ge 2$, $k > 0$ and $d \ge d(k)$, where Z is a general union of s (k+1,k+2)-points of $\mathbf{P}^n$? Is it possible to take $d(k) = 3(k+1)$?

It should be easy to disprove or get numerical evidence for the very optimistic bound $d(k) = 3(k+1)$. The huge difference between $3(k+1)$ and $(k+2)^4$ explains why we did not tried to get a little better bound for $d(2, k)$.

2. Preliminary results

For any smooth and connected variety A, any $P \in A$ and any integer $m > 0$ let $\{mP, A\}$ denote the closed subscheme of A with $(\mathcal{I}_{P,A})^m$ as its ideal sheaf. Fix an effective Cartier divisor D of A. For any closed subscheme Z of A let $\mathrm{Res}_D(Z)$ denote the residual of Z with respect to D, i.e. the closed subscheme of A with $\mathcal{I}_Z : \mathcal{I}_A$ as its ideal sheaf. For any $L \in \mathrm{Pic}(A)$ we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z),A} \otimes L(-D) \to \mathcal{I}_{Z,A} \otimes L \to \mathcal{I}_{Z \cap D, D} \otimes (L|D) \to 0 \qquad (1)$$

Notation 1. Fix a hyperplane $H \subset \mathbf{P}^n$, $P \in H$ and a zero-dimensional scheme $A \subset \mathbf{P}^n$ such that $A_{red} = \{P\}$. Set $A_0 := A$. Define inductively the schemes A_i and B_i, $i \ge 1$, by the formulas $A_i := \mathrm{Res}_H(A_{i-1})$ and $B_i := A_i \cap H$. For evey integer $i \ge 1$, set $c_i := \mathrm{length}(B_{i-1})$. We will say that A has type $(c_1, c_2, \dots$ with respect to H. Since A is zero-dimensional and $A_{red} \subseteq H$, we have $c_i = 0$ for $i \gg 0$, $c_{i+1} \le c_i$ for all i, and $\mathrm{length}(A) = \sum_{i \ge 1} c_i$. Instead of a sequence $(c_1, c_2, \dots$ we will usually write a finite string $(c_1, \dots, c_s)$ if $c_{s+1} = 0$, i.e. if $c_i = 0$ for all $i > s$.

Remark 1. Fix a hyperplane $H \subset \mathbf{P}^n$, $n \ge 2$, an integer $k > 0$, $P \in H$ and a line $D \subseteq \mathbf{P}^n$ such that $P \in D$. Set $Z_0 := Z(n, k; P, D)$ and $W_0 := Z_0 \cap H$. Define inductively the schemes Z_i and W_i, $i \ge 1$, by the formulas $Z_i := \mathrm{Res}_H(Z_{i-1})$ and $W_i := Z_i \cap H$. First assume $D \subseteq H$. We have $W_0 = Z(n-1, k; P, D)$, $Z_1 = Z(n, k-1; P, D)'$, $W_1 = Z(n-1, k-1; P, D)'$, $Z_i = (k-i)P$ and $W_i = \{(k+2-i)P, H\}$ for $2 \le i \le k+1$, and $Z_i = W_i = \emptyset$ for all $i \ge k+2$. Now assume $D \nsubseteq H$. We have $W_i = \{(k+1-i)P, H\}$ for $0 \le i \le k$, $W_{k+1} = \{2P, H\}$, $W_{k+2} = \{P\}$, $W_i = \emptyset$ for all $i \ge k+3$, $Z_i = Z(n, k-i; P, D)$ for $1 \le k$, $Z_{k+1} = 2P$, $Z_{k+2} = \{P\}$ and $Z_i = \emptyset$ for all $i \ge k+3$. If $n = 2$ and $D \ne H$, then $Z(2, k; P, D)$ has type $(c_1, \dots$ with respect to H, where

$c_i = k+2-i$ for $1 \le i \le k$, $c_{k+1} = 2$, $c_{k+2} = 1$ and $c_j = 0$ for all $j \ge k+3$. $Z(2,k;P,H)$ has type $(c_1,\dots$ with respect to H, where $c_i = k+3-i$, for $i =, 2$, $c_i = k+2-i$ for $3 \le i \le k+1$ and $c_j = 0$ for all $j \ge k+2$. If $D \subseteq H$, then we will say that $Z(n,k;P,D)$ is *strongly supported* by H. We will say that a (k+1;1)-point $Z(n,k;Q,R)'$ is *strongly supported* by H if $Q \in H$ and $D \subseteq H$. Now we fix an integer $m > 0$. Notice that $mP|H = \{mP, H\}$ and $\mathrm{Res}_H(mP) = (m-1)P$, with the convention $0P = \emptyset$.

Remark 2. Here we will explain three easy cases of the Differential Horace Lemma ([1]). Take the set-up of Remark 1. Assume $P \in D \subseteq H$. Let $A \subset \mathbf{P}^2$ be any zero-dimensional subscheme such that $P \notin A_{red}$. To prove $h^1(\mathbf{P}^2, \mathcal{I}_{A\cup Z(n,k;P,D)}(d)) = 0$ (resp. $h^0(\mathbf{P}^n, \mathcal{I}_{A\cup Z(n,k;P,D)}(d)) = 0$), it is sufficient to prove $h^1(H, \mathcal{I}_{A\cap H\cup\{P\}}(d)) = 0$ and $h^1(\mathbf{P}^n, \mathcal{I}_{A\cup B}(d-1)) = 0$ (resp. $h^0(H, \mathcal{I}_{A\cap H\cup\{P\}}(d)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_{A\cup B}(d-1)) = 0$), where B is a virtual scheme of type $(c_1,\dots$ with $c_1 = 1$, $c_2 = \binom{n+k}{n-1} + n - 1$, $c_3 = \binom{n+k-1}{n-1} + 1$, $c_i = \binom{n+k+2-i}{n-1}$ for $4 \le i \le k+2$ and $c_j = 0$ for all $j \ge k+3$. Notice $c_{k+2} = n$ and $c_{k+3} = 0$. Roughly speaking, we insert in H first P, then $Z(n-1,k;P,D)$, then $Z(n-1,k-1;P,D)'$ and then $\{(k+1-i)P, H\}$ for $1 \le i \le k-1$. In this case we will say that we applied the Differential Horace Lemma with respect to the sequence $(1, \binom{n+k}{n-1} + n - 1, \dots, n)$. Now we consider $Z(n,k-1;P,D)'$. We assume $D \subseteq H$. In this case we insert in H first P, then $Z(n-1,k-1;P,D)'$ and then $\{(k-i)P, H\}$ for $1 \le i \le k-2$. In this case we will say that we applied the Differential Horace Lemma with respect to the sequence $(1, \binom{n+k-1}{n-1} + 1, \dots, n)$. Now we fix an integer $m \ge 2$. Instead of mP we first insert in H the point P and then $\{(m+1-i)P, H\}$ for $1 \le i \le m+2$. In this case we will say that we applied the Differential Horace Lemma with respect to the sequence $(1, \binom{n+m-1}{n-1}, \dots, n)$.We may apply the Differential Horace Lemma simultaneously with respect to several distinct points of H.

Lemma 1. *Fix integers $d > 0$, $a \ge 0$, $b \ge 0$, $c > 0$, a zero-dimensional scheme $E \subset \mathbf{P}^n$ and a hyperplane $H \subset \mathbf{P}^n$. Let G be the union of E and c general points of H. Then:*

- (i) $h^0(\mathbf{P}^n, \mathcal{I}_G(d)) \le a$ *if and only if* $h^0(\mathbf{P}^n, \mathcal{I}_F(d)) \le a+c$ *and* $h^0(\mathbf{P}^n, \mathcal{I}_{Res_{H(E)}}(d-1)) \le a$.
- (ii) $h^1(\mathbf{P}^n, \mathcal{I}_G(d)) \le b$ *if and only if* $h^1(\mathbf{P}^n, \mathcal{I}_F(d)) \le b$ *and* $h^0(\mathbf{P}^n, \mathcal{I}_{Res_{H(E)}}(d-1)) \le \binom{n+d}{n} - length(E) - c + b$.

Proof. We will first check part (i). The " only if " part follows from the residual exact sequence (1), because $\mathrm{Res}_H(E) = \mathrm{Res}_H(G)$. Assume $h^0(\mathbf{P}^n, \mathcal{I}_E(d)) \le a+c$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(E)}}(d-1)) \le a$ Let $S \subset \mathbf{P}^n$ be a general union of a points of $\mathbf{P}^n$. In particular we have $S \cap H = \emptyset$, $S \cap E_{red} = \emptyset$, $h^0(\mathbf{P}^n, \mathcal{I}_{E\cup S}(d)) \le c$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(E)}\cup S}(d-1)) = 0$. To check part (i) it is sufficient to prove $h^0(\mathbf{P}^n, \mathcal{I}_{G\cup S}(d) = 0)$. Let $\rho : H^0(\mathbf{P}^n, \mathcal{I}_{E\cup S}(d)) \to H^0(H, \mathcal{I}_{E\cap H}(d))$ be the restriction map. Since $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(E)\cup S}}(d-1)) = 0$, ρ is injective. Hence $\dim(\mathrm{Im}(\rho)) \le c$. The generality of $G\backslash$ for fixed $E \cup S$ gives that no non-zero element of $\mathrm{Im}(\rho)$ vanishes at all points of F, concluding the proof of part (i). Since $h^0(\mathbf{P}^n, \mathcal{I}_G(d)) - h^1(\mathbf{P}^n, \mathcal{I}_G(d)) = \mathrm{length}(E) + c + \binom{n+d}{n}$, parts (i) and (ii) are equivalent (for suitable integers a, b). □

3. The proofs

Proof of Proposition 1. Order the connected components of Z and of W. Let Z_i, $0 \le i \le s$, be the union of the first i connected components of Z and the last $s-i$ connected components of W. It is sufficient to prove $h^0(\mathbf{P}^n, \mathcal{I}_{Z_i}(d)) \le s-i$ for all i. By assumption this inequality is satisfied for $i = 0$. Hence it is sufficient to prove $h^0(\mathbf{P}^n, \mathcal{I}_{Z_i}(d)) \le \max\{0, h^0(\mathbf{P}^n, \mathcal{I}_{Z_{i-1}}(d)) - 1\}$ for all $1 \le i \le s$. Fix an integer i such that $1 \le i \le s$. To prove the last inequality we may assume $h^0(\mathbf{P}^n, \mathcal{I}_{Z_{i-1}}(d)) > 0$. Let P denote the support of the $(s-i-1)$-th connected component of W. Let E denote the union of the first $i-1$ connected components of Z, the last $s-i$ connected components of W and a general scheme contained between $(k+1)P$ and $(k+2)P$ and with length $\binom{n+k}{n}+1$, i.e. the minimal length among the schemes strictly containing $(k+1)P$. $h^0(\mathbf{P}^n, \mathcal{I}_{Z_{i-1}}(d)) > h^0(\mathbf{P}^n, \mathcal{I}_E(d))$ ([4], Proposition 2.2). For general P we may also assume that E and Z_i have the same support. We cannot claim $E \subset Z_i$ if $k \ge 2$ and $n \ge 2$, because, even moving the line through P giving the corresponding connected component of Z_i, Z_i does not contain a general enlargment of $(k+1)P$ by a length one scheme. Take again the set up of the definition of $Z(n, k; P, D)$ with the affine coordinates $z_1, \dots, z_n$. Since we are in characteristic zero, the $(k+1)$-powers L^{k+1} of all linear forms in $z_1, \dots, z_n$ spans the set of all homogeneous degree $k+1$ polynomials in the variables $z_1, \dots, z_n$. Since $h^0(\mathbf{P}^n, \mathcal{I}_{Z_{i-1}}(d)) > h^0(\mathbf{P}^n, \mathcal{I}_E(d))$, there is at least one index j such that $1 \le j \le n$ and $h^0(\mathbf{P}^n, \mathcal{I}_E(d)) = h^0(\mathbf{P}^n, \mathcal{I}_F(d))$, where F is the scheme obtained from Z_{i-1} taking instead of $(k+1)P$ the zero-set of all monomials of degree $\ge k+2$ and the ones of degree $k+1$ except z_j^{k+1}. Take $D = \{z_a = 0 \text{ for all } a \ne j\}$. With this choice of the line D we have $F \subset Z_i$ and hence $h^0(\mathbf{P}^n, \mathcal{I}_{Z_i}(d) < h^0(\mathbf{P}^n, \mathcal{I}_{Z_{i-1}}(d))$. □

The same proof gives the following result.

Proposition 2. *Fix positive integers n, k, d, s. Let $Z \subset \mathbf{P}^n$ (resp. $W \subset \mathbf{P}^n$) be a general union of s $(k+1, k+2)$-point (resp. $(k+1)$-points). If $h^0(\mathbf{P}^n, \mathcal{I}_W(d)) \ge s$, then $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \le h^1(\mathbf{P}^n, \mathcal{I}_W(d)) + (n-1)s$.*

Lemma 2. *Fix integers $d > k > 0$, $s \ge 0$, $b \ge 0$, $a_i \ge 0$, $1 \le i \le k$ and e such that $0 \le e \le k+1$, $d \ge (k+2)^3$ and*

$$(k+2)s + (k+1)b + \sum_{i=1}^{k} i a_i + e \le d+1 \tag{2}$$

Then

$$(k+1)s + (k-1)b + \sum_{i=2}^{k}(i-1)a_i + e(k+2) \le d \tag{3}$$

Proof. Increasing if necessary a_1 we may assume

$$(k+2)s + (k+1)b + \sum_{i=1}^{k} ia_i + e = d+1 \tag{4}$$

By (4) and (3) it is sufficient to prove that

$$s + 2b + \sum_{i=1}^{k} a_i \ge 1 + (k+2)(k+1) \tag{5}$$

The last inequality is true because $d \ge (k+2)^2(k+1)+1$ and we assumed the equality (4). □

We won't try to make explicit and small the integers $a(n,k)$ and $d(n,k)$ in the statements of Theorem 1 and Corollary 1, because for fixed k we didn't get as $d(n,k)$ a polynomial function of n. Hence we will not give an explicit bound for the integer $b(n,k)$ in the next lemma.

Lemma 3. *For all integers $n \ge 2$ and $k > 0$ there is an integer $b(n,k) > 0$ with the following property. Fix non-negative integers x, d, a, a_m, e, $2 \le m \le k+1$, such that $e \le \binom{n+k}{n-1} + n - 2$, $d \ge b(n,k)$ and*

$$x(\binom{n+k}{n-1} + n - 1) + a(\binom{n+k-1}{n-1} + 1) + \\ + \sum_{m=2}^{k+1} a_m \binom{n+m-1}{n-1} + e \le \binom{n+d-1}{n-1} \tag{6}$$

Then

$$x(\binom{n+k-1}{n-1} + 1) + a\binom{n+k-2}{n-1} + \\ + \sum_{m=2}^{k+1} a_m \binom{n+m-2}{n-1} + e(\binom{n+k}{n-1} + n - 1) \le \binom{n+d-2}{n-1} \tag{7}$$

Proof. Increasing if necessary a_2 we may assume

$$x(\binom{n+k}{n-1} + n - 1) + a(\binom{n+k-1}{n-1} + 1) + \\ + \sum_{m=2}^{k+1} a_m \binom{n+m-1}{n-1} + e \ge \binom{n+d-1}{n-1} - n + 2 \tag{8}$$

Under this additional assumption it is sufficient prove the following inequality

$$x(\binom{n+k-1}{n-2}+n-2)+a(\binom{n+k-2}{n-2}+1)+\sum_{m=2}^{k+1}a_m\binom{n+m-2}{n-2} \tag{9}$$
$$\geq (\binom{n+k}{n-1}+n-2)^2+n+\binom{n+d-2}{n-2}+n-2$$

Set $e_0 := (\binom{n+k}{n-1}+n-1)/(\binom{n+k-1}{n-1}+1)$, $e_1 := (\binom{n+k-1}{n-1}+1)/(\binom{n+k-2}{n-1})$ and $e_m := \binom{n+m-1}{n-1}/\binom{n+m-2}{n-1} = (n+m-1)/(m-1)$ for all $2 \leq m \leq k+1$. Hence $e_i \geq (n+k)/k$ for all i. Since $\binom{n+d-1}{n-1}/\binom{n+d-2}{n-1} = (n+d-1)/d$, we conclude. □

Proof of Corollary 1 for $n = 2$. Fix a line $H \subset \mathbf{P}^2$. Set $\epsilon := (d+2)(d+1)/2 - s(k+2)(k+1)/2 + 2s$. Increasing or decreasing the integer s we see that (for a fixed degree d) it is sufficient to check all cases with $-(k^2+3k)/2 \leq \epsilon \leq (k^2+3k)/2$. From now on we will assume that these inequalities are satisfied. Until part (e) we will also assume $\epsilon \geq 0$, i.e. we will check that $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$.

(a) Write $u_1 := \lfloor (d+1)/(k+2) \rfloor$ and $v_1 := d+1-u_1(k+2)$. If $s < u_1+v_1$, then we will say that we stopped at the degree d. Assume $s \geq u_1+v_1$. We specialize Z to a general union Z' of $s-u_1-v_1$ $(k+1,k+2)$-points, u_1 $(k+1,k+2)$-points strongly supported by H and v_1 virtual schemes obtained applying Differential Horace Lemma with respect to the sequence $(1,k+2,k+1,k-1,\cdots,2)$, i.e. applying Remark 2 to the scheme $Z(2,k;P,H)$. Since $\text{length}(H \cap Z') = (k+2)u_1+v_1 = d+1$ and $H \cong \mathbf{P}^1$, $h^i(\mathbf{P}^2, \mathcal{I}_{Z'}(d)) = h^i(\mathbf{P}^2, \mathcal{I}_{\text{Res}_H(Z')}(d))$ by the Differential Horace Lemma. The virtual residue $Z_1 := \text{Res}_H(Z')$ intersects H in u_1 connected schemes with length $k+1$ and v_1 connected schemes with length $k+2$. Lemma 2 and our assumption on d gives $e_1 := \text{length}(Z_1 \cap H) \leq d$. Set $u_2 := \lfloor (d-e_1)/(k+2) \rfloor$ and $v_2 := d-e_1-(k+2)u_2$. If $u_1+v_1 \leq s < u_1+v_1+u_2+v_2$, then we will say that we stopped at the degree $d-1$. Assume $s \geq u_1+v_1+u_2+v_2$. We degenerate Z_1 to a general union Z_1' of $s-u_1-v_1-u_2-v_2$ (k+1,k+2)-points, the connected components of Z_1 intersecting Z, u_2 (k+1,k+2)-points strongly supported by H and v_2 virtual schemes obtained applying Differential Horace Lemma with respect to the sequence $(1,k+2,k+1,k-1,\cdots,2)$. Set $Z_2 := \text{Res}_H(Z_1')$ and $e_2 := \text{length}(Z_2 \cap H)$. Since $d-1 \geq (k+2)^2(k+1)$, Lemma 2 gives $e_2 \leq d-1$. Set $u_3 := \lfloor (d-1-e_2)/(k+2) \rfloor$ and $v_3 := d-1-e_2-(k+2)u_3$. And so on, up to the degree $d-k-1$ defining each time the integers u_i, v_i, e_{i-1}, unless we stopped, say at the degree x, because $\sum_{i=1}^{d-x-1}(u_i+v_i) \leq s < \sum_{i=1}^{d-x}(u_i+v_i)$. Fix an integer i such that Z_i is defined. Notice that the supports of the connected components of $Z_1 \cap H$ are general in H in the following very strong sense: we may fix the support of all except except one prescribed in advance and then take as the support of the choosen component a general point of H. Since we are in characteristic zero, a general lemma say that $Z_i \cap H$ (for general supports) imposes the maximal possible number of components of a prescribed linear system Λ on H ([5]).

(b) Assume that we never stopped until the degree $d-k-1$. Z_{k+1} contains u_1 reduced connected components, all of them contained in H. Let E_{k+1} be the union of the unreduced components of Z_{k+1}. By Remark 1 to prove $h^1(\mathbf{P}^2, \mathcal{I}_{Z_{k+1}}(d-k-1)) = 0$ and hence to prove $h^1(\mathbf{P}^2, \mathcal{I}_Z(d)) = 0$, it is sufficient to prove $h^1(\mathbf{P}^2, \mathcal{I}_{E_{k+1}}(d-k-1)) = 0$ and $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(E_{k+1})}}(d-k-2)) \leq \epsilon$. Here we will check the h^1-vanishing leaving the h^0-inequality (and all the other h^0-inequalities which will soon appear) to part (d). Set $e_{k+1} := \mathrm{length}(E_{k+1} \cap H)$, $u_{k+2} := \lfloor (d-k-e_{k+1})/(k+2) \rfloor$ and $v_{k+2} := d-k-e_{k+1}-(k+2)u_{k+2}$. If $s < \sum_{i=1}^{k+2}(u_i+v_i)$, then we will say that we stopped at the degree $d-k-1$. Now assume $s \geq \sum_{i=1}^{k+2}(u_i+v_i)$. We degenerate E_{k+1} to a general union E'_{k+1} of $s - \sum_{i=1}^{k+2}(u_i+v_i)$ (k+1,k+2)-points, all the connected components of E_{k+1} intersecting H, u_{k+2} (k+1,k+2)-points strongly supported by H and v_2 virtual schemes obtained applying Differential Horace Lemma with respect to the sequence $(1, k+2, k+1, k-1, \cdots, 2)$. Hence $\mathrm{length}(E'_{k+1} \cap H) = d-k$. $\mathrm{Res}_H(E'_{k+2})$ contains u_2 reduced connected components, all of them contained in H. Let E_{k+2} denote the union of the unreduced connected components of $\mathrm{Res}_H(E'_{k+2})$. By Remark 1 to prove $h^1(\mathbf{P}^2, \mathcal{I}_{E_{k+1}}(d-k-1)) = 0$ it is sufficient to prove $h^1(\mathbf{P}^2, \mathcal{I}_{E_{k+2}}(d-k-2)) = 0$ and $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(E_{k+2})}}(d-k-3)) \leq \epsilon + u_1$. And so on, defining each time the integers u_i, v_i, e_{i-1} and the schemes E_i, E'_i, unless we stopped, say at the degree x, because $\sum_{i=1}^{d-x-1}(u_i+v_i) \leq s < \sum_{i=1}^{d-x}(u_i+v_i)$. We need to check $h^1(\mathbf{P}^2, \mathcal{I}_{E_i}(d-i)) = 0$ for at least one index i, but we need to prove $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(E_i)}}(d-i-1)) \leq \epsilon + \sum_{j=1}^{i-k} u_j$ for all integers i for which E_i is defined. Since at each step we applied Lemma 2, E_i is defined only for $i \leq d-1-(k+2)^3$. Here we want to check that we stopped before arriving at the degree $d-1-(k+2)^3$, i.e. we want to check that $s < \sum_{i=1}^{d-1-(k+2)(k+1)}(u_i+v_i)$. For each $i \geq k = 1$ (resp. $1 \leq i \leq k+1$) we have $(d+2-i)(d+1-i)/2 - \epsilon = \mathrm{length}(E_i) + \sum_{j=1}^{i-k-1} u_j$ (resp. $(d+2-i)(d+1-i)/2 - \epsilon = \mathrm{length}(Z_i)$). Notice that $s \geq \sum_{j=1}^{i}(u_j+v_j)$ if E_i or Z_i are defined, $v_j \leq k+1$ for all j, and $s((k+2)(k+1)/2+2) = (d+2)(d+1)/2 - \epsilon$. Hence if E_i or Z_i are defined, then $\sum_{j=1}^{i} u_j \geq (k+1)i + ((d+2)(d+1)/2-\epsilon)/((k+2)(k+1)/2+2)$. Thus if $E_{d-1-(k+2)^3}$ is defined, then $\mathrm{length}(E_{d-1-(k+2)^3-k-1} \leq (d-k-1)(k+1) - ((d+2)(d+1)/2-\epsilon)/((k+2)(k+1)/2+2) < 0$ (just use that $|\epsilon| \leq (k^2+3k)/2$ and the assumption $d \geq (k+2)^4$), contradiction.

(c) Now assume that we stopped at the degree x. Hence we defined either Z_{d-x+1} (case $x \geq d-k$) or E_{d-x-1} (case $x \leq d-k-1$). Call F_{d-x-1} the scheme we defined. We defined the integers e_{d-x-1}, u_{d-x} and v_{d-x} and $\sum_{i=1}^{d-x-1}(u_i+v_i) \leq s < \sum_{i=1}^{d-x}(u_i+v_i)$. Write $a := \min\{u_{d-x}, s - \sum_{i=1}^{d-x-1}(u_i+v_i)\}$ and $b := s - a - \sum_{i=1}^{d-x-1}(u_i+v_i)$. We degenerate F_{d-x-1} to a general union G of the connected components of F_{d-x-1} intersecting H, a (k+1,k+2)-points strongly supported by H and virtual schemes obtained applying Differential

Horace Lemma with respect to the sequence $(1, k+2, k+1, k-1, \cdots, 2)$. We apply $k+1$ times the Horace Differential Lemma and reduce to the statement that the empty set have good coomology. We may apply Lemma 2 at the first step because $x \ge (k+2)^3$. We do not need to apply it in the next k steps, because neither in the first step nor in the following ones we insert any new virtual scheme of type $(1, k+2, k+1, k-1, \cdots, 2)$ (it is taking the residual of one of them which compels us to use Lemma 2 with $e \ne 0$). Call β the number of steps we used in part (c) to arrive to $E_\beta \emptyset$. If $i \ge \beta$, then $F_i = \mathrm{Res}_H(F_i) = \emptyset$ and the h^0-inequalities are satisfied (see part (c)). If $i < \beta$ after at most $\beta - i$ steps every h^0-inequality coming from $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(F_i)}}(d-i-1)) \le \epsilon + \sum_{j=1}^{i-k} u_j$ (or a similar one for F_i) is proved as in part (c).

(d) At each step in part (b) we got a new h^0-inequality. We need to show that we may control simultaneously all these inequalities. All these inequalities are of the form $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(E_i)}}(d-k-i)) \le \epsilon + \sum_{j=1}^{i} u_j$. Let F_i be the union of all unreduced components of $\mathrm{Res}_H(E_i)$. $\mathrm{Res}_H(E_i) \backslash F_i$ is a general union of $x_i \ge 0$ points of H. Hence $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(E_i)}}(d-k-i)) \le \epsilon + \sum_{j=1}^{i} u_j$ if $h^0(\mathbf{P}^2, \mathcal{I}_{F_i}(d-k-i)) \le \epsilon + \sum_{j=1}^{i} u_j + x_i$ and $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(F_i)}}(d-k-i-1)) \le \epsilon + \sum_{j=1}^{i} u_j$. We iterate the same game for each of these inequalities. We specialize F_i to F_i' (using also the Differential Horace Lemma) in such a way that $\mathrm{length}(F_i' \cap H) = d-k-i+1$ and applyx_i is the number of 2-points of E_i with support on H. At the next step using F_i' the new x_{i+1} is the number of 3-points of E_i with support on H.

(e) Here we assume $-(k^2+3k)/2 \le \epsilon \le 0$. We take $u_i, v_i.e_{i-1}, Z_i, Z_i'$ as in part (a) and E_{k+1} as in part (b). Now we need to check $h^0(\mathbf{P}^2, \mathcal{I}_{E_{k+1}}(d-k-1)) = 0$ and $h^0(\mathbf{P}^2, \mathcal{I}_{\mathrm{Res}_{H(E_{k+1})}}(d-k-1)) = 0$. Then we continue as in part (d), except that there is no h^1-vanishing and in all h^0-inequalities listed to be checked the term " ϵ " does not appear. The other h^0-inequalities appearing for $\epsilon \le 0$ are of the form $h^0(\mathbf{P}^2, \mathcal{I}_{E_i}(d-k-i)) \le \sum_{j=1}^{i-1} u_j$ and are handled in the same way. □

Proof of Theorem 1. Set $\epsilon := \epsilon(n, k, d, s, b, a_1, \ldots, a_{k+1}) := s(\binom{n+k}{n} + n) + b(\binom{n+k-1}{n} + 1) + \sum_{i=2}^{k+1} \binom{n+i}{n} - \binom{n+d}{n}$. Hence $\epsilon = \mathrm{length}(Z) - \binom{n+d}{n}$. Fix a hyperplane $H \subset \mathbf{P}^n$. Since Theorem 1 is trivially true when $n = 1$ taking $a(1,k) = 1$, we may assume that the result is true for $n' := n-1$ and in particular there is a finite integer $a(n-1,k)$. Fix an integer $d \ge \max\{a(k-1,n) + 2k + 2, b(n,k) + 2k + 2\}$, where $b(n,k)$ is any positive integer satisfying the thesis of Lemma 3. Later, we will impose $d \gg 0$ we no explict allowable integer d. Take Z as in the statement of Theorem 1. In parts (a), (b), (c), (d) and (e) we will assume $\epsilon \ge 0$, i.e. we these parts we will check that $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = 0$.

(a) Let s_1 be the maximal non-negative integer such that $s_1(\binom{n+k}{n-1} + n - 1) \le \binom{n+d-1}{n-1}$. Let b_1 be the maximal non-negative integer $\le b$ such that

$b_1(\binom{n+k}{n-1}+1) \le \binom{n+d-1}{n-1} - s_1(\binom{n+k}{n-1}+n-1)$. We define the integers $a_{m,1}$, $1 \le m \le k+1$, by decreasing induction on m. Let $a_{k+1,1}$ be the maximal non-negative integer $\le a_{k+1}$ such that $a_{k+1,1}\binom{n+k}{n-1} \le \binom{n+d-1}{n-1} - s_1(\binom{n+k}{n-1}+n-1) - b_1(\binom{n+k}{n-1}+1)$. Assume defined the integers $a_{j,1}$ for $k+2 \ge j \ge m$ for some integer $m \ge 3$. Let $a_{m-1,1}$ be the maximal non-negative integer $\le a_{m-1}$ such that $a_{m-1,1}\binom{n+m-2}{n-1} \le \binom{n+d-1}{n-1} - s_1(\binom{n+k}{n-1}+n-1) - b_1(\binom{n+k}{n-1}+1) - \sum_{j=m}^{k+1} a_{j,1}\binom{n+j-1}{n-1}$. Set $v_1 := \binom{n+d-1}{n-1} - s_1(\binom{n+k}{n-1}+n-1) - b_1(\binom{n+k}{n-1}+1) - \sum_{j=2}^{k+1} a_{j,1}\binom{n+j-1}{n-1}$. If $v_1 < s - s_1 + b - b_1 + \sum_{j=2}^{k+2}(a_j - a_{j,1})$, then we will say that we stopped at the degree $d-1$. Assume $v_1 \ge s - s_1 + b - b_1 + \sum_{j=2}^{k+2}(a_j - a_{j,1})$. Set $y_1 := \min\{v_1, s - s_1\}$, $f_1 := \min\{b - b_1, v_1 - y_1\}$ and $g_{k+1,1} := \min\{a_{k+1} - a_{k+1,1}, v_1 - e_1 - f_1\}$. Define by decreasing induction the integers $g_{j,1}$ for all $k \ge j \ge 2$ in a similar way. Hence $v_1 \ge y_1 + f_1 + \sum_{j=2}^{k+1} g_{j,1}$ and if some $g_{j,1} > 0$, then $s = s_1 + y_1$, $b = b_1 + f_1$ and $a_j = a_{j,1} + g_{j,1}$ for all $m+1 \le j \le k+1$. If $v_1 > y_1 + f_1 + \sum_{j=2}^{k+1} g_{j,1}$, then we will say that we stopped at the degree $(d,1)$. Notice that in this case we have $s = s_1 + y_1$, $b = b_1 + f_1$ and $a_j = a_{j,1} + g_{j,1}$ for all j. Assume $v_1 = e_1 + f_1 + \sum_{j=2}^{k+1} g_{j,1}$. We degenerate Z' to a general union of $s - s_1 - y_1$ (k+1,k+2)-points, $b - b_1 - f_1$ (k,1)-points, $a_m - a_{m,1} - g_{m,1}$ m-points for all $2 \le m \le k+1$, and v_1 virtual schemes, y_1 of them obtained applying the Differential Horace Lemma (i.e. Remark 2) with respect to the sequence $(1, \binom{n+k}{n-1}+n-1, \dots, n)$, f_1 of them with respect to the sequence $(1, \binom{n+k}{n-1}+1, \dots, n)$ and $g_{m,1}$, $2 \le m \le k+2$, of them with respect to the sequence $(1, \binom{n+m-1}{n-1}, \dots, n)$. Hence $\mathrm{length}(Z' \cap H) = \binom{n+d-1}{n-1}$. Since $d \ge a(n-1,k)$ we may apply the inductive assumption and get $h^1(H, \mathcal{I}_{Z'\cap H}(d)) = 0$, $i = 0,1$. Hence $h^i(\mathbf{P}^n, \mathcal{I}_Z(d)) \le h^i(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z')}}(d-1))$, $i = 0,1$. Let E_1 be the union of the unreduced components of $\mathrm{Res}_H(Z')$. $\mathrm{Res}_H(Z')\backslash H$ is a general union of $a_{2,1}$ points of H. In this case we will write $x_1 := a_{2,1}$. By Lemma 1 to prove $h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(Z')}}(d-1)) = 0$ it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{E_1}(d-1)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_{H(E_1)}}(d-1)) \le \epsilon$. Here we will only check the h^1-vanishing, leaving this h^0-inequality and all the h^0-inequalities which will soon arrive to steps (d) and (e). Set $e_1 := \mathrm{length}(E_1 \cap H)$. Lemma 3 and the assumption $d \ge b(n,k){+}{+}$ gives $e_1 + x_1 \le \binom{n+d-2}{n-1}$. Let s_2 be the maximal non-negative integer $\le s - s_1$ such that $s_2(\binom{n+k}{n-1}+n-1) \le \binom{n+d-2}{n-1} - e_1$. Let b_2 be the maximal non-negative integer $\le b - b_1$ such that $b_2(\binom{n+k}{n-1}+1) \le \binom{n+d-2}{n-1} - s_2(\binom{n+k}{n-1}+n-1) - e_1$. We define the integers $a_{m,2}$, $1 \le m \le k+1$, by decreasing induction on m. Let $a_{k+1,2}$ be the maximal non-negative integer $\le a_{k+1} - a_{k+1,1}$ such that $a_{k+1,2}\binom{n+k}{n-1} \le \binom{n+d-2}{n-1} - s_2(\binom{n+k}{n-1}+n-1) - b_2(\binom{n+k}{n-1}+1) - e_1$. Assume defined the integers $a_{j,2}$ for $k+2 \ge j \ge m$ for some integer $m \ge 3$ let $a_{m-1,2}$ be the maximal non-negative integer $\le a_{m-1} - a_{m-1,1}$ such that $a_{m-1,2}\binom{n+m-2}{n-1} \le \binom{n+d-1}{n-1} - s_2(\binom{n+k}{n-1}+n-1) - b_2(\binom{n+k}{n-1}+1) - \sum_{j=m}^{k+1} a_{j,2}\binom{n+j-1}{n-1} - e_1$. Set $v_2 := \binom{n+d-1}{n-1} - e_1 - s_2(\binom{n+k}{n-1}+n-1) - b_2(\binom{n+k}{n-1}+1) - \sum_{j=2}^{k+1} a_{j,2}\binom{n+j-1}{n-1}$. If

$v_2 < (s-s_1-s_2)+(b-b_1-b_2)+\sum_{j=2}^{k+2}(a_j-a_{j,1}-a_{j,2})$, then we will say that we stopped at the degree $d-2$. Assume $v_2 \ge (s-s_1-s_2)+(b-b_1-b_2)+\sum_{j=2}^{k+2}(a_j-a_{j,1}-a_{j,2})$. Set $y_2 := \min\{v_2, s-s_1-s_2\}$, $f_2 := \min\{b-b_1-b_2, v_2-y_2\}$ and $g_{k+1,2} := \min\{a_{k+1}-a_{k+1,1}, v_2-y_2-f_2\}$. Define by decreasing induction the integers $g_{j,2}$ for all $k \ge j \ge 2$ in a similar way. Hence $v_2 \ge y_2+f_2+\sum_{j=2}^{k+1} g_{j,2}$ and if some $g_{m,2} > 0$, then $s = s_1+y_1+y_2$, $b = b_1+f_1+f_2$ and $a_j = a_{j,1}+g_{j,1}+g_{j,2}$ for all $m+1 \le j \le k+1$. If $v_1 > y_2+f_2+\sum_{j=2}^{k+1} g_{j,2}$, then we will say that we stopped at the degree $(d-1,1)$. Notice that in this case we have $s = s_1+y_1+y_2$, $b = b_1+f_1+f_2$ and $a_j = a_{j,1}+g_{j,1}+g_{j,2}$ for all j. Assume $v_2 = y_2+f_2+\sum_{j=2}^{k+1} g_{j,2}$. We degenerate E_1 to a general union E_1' of all connected components of E_1 intersecting H, $s-s_1-y_1-s_2-y_2$ (k+1,k+2)-points, $b-b_1-f_1-b_2-f_2$ (k,1)-points, $a_m-a_{m,1}-g_{m,1}-a_{m,2}-g_{m,2}$ m-points for all $2 \le m \le k+1$, and v_2 virtual schemes, y_2 of them obtained applying the Differential Horace Lemma with respect to the sequence $(1, \binom{n+k}{n-1}+n-1, \dots, n)$, f_2 of them with respect to the sequence $(1, \binom{n+k}{n-1}+1, \dots, n)$ and $g_{m,2}$, $2 \le m \le k+2$ of them with respect to the sequence $(1, \binom{n+m-1}{n-1}, \dots, n)$. Hence $\mathrm{length}(Z' \cap H) = \binom{n+d-1}{n-1}$. Since $d-1 \ge a(n-1,k)$ we may apply the inductive assumption and get $h^1(H, \mathcal{I}_{E_1' \cap H}(d-1)) = 0$, $i = 0,1$. Hence $h^i(\mathbf{P}^n, \mathcal{I}_Z(d)) \le h^i(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(E_1')}(d-2))$, $i = 0,1$. Let E_2 be the union of the unreduced components of $\mathrm{Res}_H(E_1')$. $\mathrm{Res}_H(E_1')\backslash E_2$ is a general union of x_2 general points of H, where x_2 is described in the following way. If $k \ge 3$, then $x_2 = a_{3,1}+a_{2,2}$. If $k = 2$, then $x_2 = a_{3,1}+a_{2,2}+b_1$. If $k = 1$, then $x_2 = a_{2,2}+y_1$. By Lemma 1 to check that $h^1(\mathbf{P}^n, \mathcal{I}_{E_1}(d-1)) = 0$ it is sufficient to prove that $h^1(\mathbf{P}^n, \mathcal{I}_{E_2}(d-2)) = 0$ and $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(E_2)}(d-3)) \le \epsilon + x_1$. And so on defining each time integers $s_i, b_i, a_{m,i}, y_i, f_i, g_{m,i}$, $i \ge 1$, and then the integers e_i, x_i until either we stopped at the degree $d-i+1$ or at the degree $(d-i,1)$ or we cannot apply Lemma 3 because $d-i-1 = b(n,k)$.

(b) Here we assume that we stopped at the degree $(d-i+1,1)$. We assume $d-i \ge \max\{b(n,k), a(n-1,k)+k+2\}$. Even in this case we have defined the scheme E_i', but now the inductive assumption and the inequality $d-i \ge b(n,k)$ only give $h^1(H, \mathcal{I}_{H\cap E'}(d-i)) = 0$. This is enough to prove $h^1(\mathbf{P}^n, \mathcal{I}_{E_i'}(d-i)) \le h^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}(E_i')}(d-i-1))$. Set $G_1 := \mathrm{Res}(E_i')$ and define inductively G_j, $j \ge 2$, by the formula $G_j := \mathrm{Res}_H(G_{j-1})$. Since each connected component of E_i' intersects H, we have $G_{k+2} = 0$. The inequality $d-i \ge a(n-1,k)+k+2$ gives $h^1(H, \mathcal{I}_{G_j \cap H}(d-i-j)) = 0$, for all $j > 0$. Hence $h^1(\mathbf{P}^n, \mathcal{I}_{E_i'}(d-i))$.

(c) Here we assume that we stopped at the degree $d-i+1$. We assume $d-i \ge a(n-1,k)+k+2$. Here we are in the situation of part (b), but E_i' has no virtual scheme and hence we do not need to use Lemma 2.

(d) Here we assume $d \gg 0$ and we prove that we must stop either at the degree $d-i+1$ or at the degree $(d-i+1,1)$ for some i such that $d-i \ge$

$\max\{b(n,k), a(n-1,k)+k+2\}$. We also assume $d-i \geq k+3$. If E_i is defined, then $\binom{d-i+1}{n} - \mathrm{length}(E_i) = \epsilon + \sum_{j=1}^{i} x_j$. Hence $\epsilon + \sum_{j01}^{i} \leq \binom{d-i+1}{n}$. Between the degree d and the degree $d-i+k+3$ we inserted in H at least $\lceil (\binom{n+d}{n} - \binom{n+d-i+k+3}{n})/(\binom{n+k}{n}+n)\rceil$ connected components (virtual or not) and each non-virtual component contributes to the integer $\sum_{j=1}^{i} x_j$. Since $v_j \leq \binom{n+k}{n}+n-2$ for all j, we get $\sum_{j=1}^{i} x_j \geq \lceil (\binom{n+d}{n} - \binom{n+d-i+k+3}{n})/(\binom{n+k}{n}+n)\rceil - (d-i+k+2)(\binom{n+k}{n}+n-2)$. For $i \gg 0$ we get $\sum_{j=1}^{i} x_j \gg \binom{d-i+1}{n}$; more precisely, there is an integer $c_{n,k}$ such that if $i \geq c_{n,k}$, then $\sum_{j=1}^{i} x_j > \binom{n+d-i+1}{n}$, contradiction.

(e) Here we will check all the h^0-inequalities. In the case $\epsilon \geq 0$ we got inequalities of the form $h^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}(E'_y)}(d-y-1)) \leq \epsilon + \sum_{j=1}^{y-1} x_j$. In part (f) (case $\epsilon \leq 0$) we also need inequalities of type $h^0(\mathbf{P}^n, \mathcal{I}_{E_y}(d-y)) \leq \sum_{j=1}^{y-1} x_j$. To check all of them we degenerate several times $\mathrm{Res}(E'_y)$ or E_y, until we arrive at a stopping degree, either of type $(x,1)$ or just a stopping degree x. At each time we apply Lemma 1 and hence we get two new h^0-inequalities intead of the old one. We conclude as in part (d), because we only need $\sum_{j=1}^{x} x_j > \binom{n+x}{n}$ and this is true for $d \gg 0$ and depending only from d, not from y: we may choose the same x for all y coming from a fixed d.

(f) Here we assume $\epsilon \leq 0$. We just get the same inequalities, except we omit the term ϵ in these inequalities. □

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