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Noetherian Rings of Regular Functions

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Abstract. Let X be a pure n-dimensional quasi-projective locally Cohen-Macaulay scheme such that $\dim(H^i(X, \mathcal{O}_X)) < +\infty$ for all $1 \le i \le n-1$. Set $A := H^0(X, \mathcal{O}_X)$. For any $P \in X$ set $Z(P) := \{Q \in X : f(Q) = f(P)$ for all $f \in A\}$. Assume that each $Z(P)$, $P \in X$, is finite. Then $h^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ and A is Noetherian. For every maximal ideal $J \subsetneq A$ there is $P \in X$ such that $J = \{f \in A : f(P) = 0\}$.

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Here we prove the following result.

Theorem 1. *Let X be a pure n-dimensional quasi-projective locally Cohen-Macaulay scheme such that $\dim(H^i(X, \mathcal{O}_X)) < +\infty$ for all $1 \le i \le n-1$. Set $A := H^0(X, \mathcal{O}_X)$. For any $P \in X$ set $Z(P) := \{Q \in X : f(Q) = f(P)$ for all $f \in A\}$. Assume that each $Z(P)$, $P \in X$, is finite. Then $h^i(X, \mathcal{O}_X) = 0$ for all $i > 0$ and A is Noetherian. For every maximal ideal $J \subsetneq A$ there is $P \in X$ such that $J = \{f \in A : f(P) = 0\}$.*

To prove the last assertion of Theorem 1 we will use the following well-known lemma.

Lemma 1. *Let X be any scheme such that $H^i(X, \mathcal{O}_X) = 0$ for all $i \ge 1$. Set $A := H^0(X, \mathcal{O}_X)$. If $F_1, \dots, F_s$ are finitely many elements of A with no common zero, then there are $G_1, \dots, G_s \in A$ such that $\sum_{i=1}^{s} G_i F_i \equiv 1$.*

Proof. Since $F_1, \dots, F_s$ have no common zero, their Koszul complex is a finite exact complex (starting and ending with the zero sheaf and with $s+1$ non-zero terms) in which eacch non-zero term is a free and finitely generated sheaf on X. Since $H^i(X, \mathcal{O}_X) = 0$ for all $i \ge 1$, splitting this exact sequence into short exact

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sequences we get the surjectivity the map $\psi : A^{\oplus s} \to A$ defined by the formula $\psi((f_1, \dots, f_s)) := \sum_{i=1}^{s} f_i F_i$. Take $(G_1, \dots, G_s)$ such that $\psi((G_1, \dots, G_s))$ is the constant function 1. □

Remark 1. Lemma 1 is true with the same proof in the complex-analytic category (see [2]).

Lemma 2. *Let X be any finite-dimensional scheme such that $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq 1$. Set $A := H^0(X, \mathcal{O}_X)$. Let $I \subseteq A$ be any ideal such that $V(I) := \{P \in X : f(P) = 0$ for all $f \in I\} = \emptyset$. Then $I = A$.*

Proof. If I is finitely generated, then we may apply I. In the general case the finite-dimensionality of X and a dimensional count show that for every ideal J of A there is a finitely-generated ideal $J' \subseteq A$ such that $V(J') = V(I)$. Use this observation for I and apply Lemma 1. □

Proof of Theorem 1. To prove that A is Noetheriae will use induction on n, the cases $n \leq 1$ being obvious. Indeed, in these cases A is even a finitely generated $\mathbb{K}$-algebra. Assume $n \geq 2$. Since X is not complete and $\dim(X) = n$, $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq n$. The proof of the analytic case given in [1] works in the algebraic case and gives that $H^i(X, \mathcal{O}_X) = 0$ for all $1 \leq i \leq n-1$. Fix an indeal $I \subseteq A$, $I \neq 0$. If $V(I) = \emptyset$, then $I = A$ (Lemma 1). Assume $I \neq A$. Since $I \neq 0$, there is $f \neq 0$ vanishing at some $P \in X$. Set $Z(f) := \{Q \in X : f(Q) = P)$. Let Z be the scheme $(Z(f), \mathcal{O}_Z)$ with $\mathcal{O}_Z := \mathcal{O}_X / f\mathcal{O}_X | Z(f)$. Hence $Z(f)$ is a locally Cohen-Macaulay with pure dimension $n-1$ (not necessarly reduced, even if X is reduced). Since X is locally Cohen-Macaulay, the multiplication $\times f : \mathcal{O}_X \to \mathcal{O}_X$ by f is injective. Hence $f\mathcal{O}_X \cong \mathcal{O}_X$. Hence $H^1(X, f\mathcal{O}_X) = 0$. From the exact sequence

$$0 \to f\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{1}$$

we get that the restriction map $\rho_Z : H^0(X, \mathcal{O}_X) \to H^0(Z, \mathcal{O}_Z)$ is surjective. By the inductive assumption the ring $H^0(Z, \mathcal{O}_Z)$ is Noetherian. Hence $\rho_Z(I)$ is finitely generated, say by $g_i \in H^0(Z, \mathcal{O}_Z)$, $1 \leq i \leq s$. Since ρ_Z is surjective, there are $f_i \in I$, $1 \leq i \leq s$, such that $g_i = \rho_Z(f_i)$. Notice that $I = (f, f_1, \dots, f_s)$. For the last assertion, just use Lemma 2. □

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