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A Splitting Criterion for Vector Bundles on Blowing ups of the Plane

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Abstract. Let $f_s : X_s \to \mathbf{P}^2$ be the blowing-up of s distinct points and E a vector bundle on X_s. Here we give a cohomological criterio which is equivalent to $E \cong f_s^*(A)$ with A a direct sum of line bundles. We also some cohomological characterizations of very particular rank 2 vector bundles on $\mathbf{P}^2$.

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1. Introduction

Fix an integer $s \geq 1$ and s distinct points $P_1, \dots, P_s \in \mathbf{P}^2$. Let $f_s : X_s \to \mathbf{P}^2$ denote the blowing up of the points $P_1, \dots, P_s$. $\mathrm{Pic}(X_s) \cong \mathbb{Z}^{s+1}$ and we will take the line bundles $f_s^*(\mathcal{O}_{\mathbf{P}^2}(1))$ and $D_i := f_s^{-1}(P_i)$, $1 \leq i \leq s$, as free generators of $\mathrm{Pic}(X_s)$. Set $\mathcal{O}_{X_s}(t; a_1, \dots, a_s) := f_s^*(\mathcal{O}_{\mathbf{P}^2}(t))(\sum_{i=1}^s a_i D_i)$. Hence $\mathcal{O}_{X_s}(t; a_1, \dots, a_s) \cdot \mathcal{O}_{X_s}(z; b_1, \dots, b_s) = tz - \sum_{i=1}^s a_i b_i$. For any coherent sheaf A on X_s set $A(t; a_1, \dots, a_s) := A \otimes \mathcal{O}_{X_s}(t; a_1, \dots, a_s)$. Set $X_0 := \mathbf{P}^2$ and $f_0 := \mathrm{Id}_{\mathbf{P}^2}$. In section 2 we will prove the following result.

Theorem 1. *Let E be a rank r torsion free sheaf on X_s which is locally free at each point of $D_1 \cup \cdots \cup D_s$. For every $i \in \{1, \dots, s\}$ let $b_{i,1} \geq \cdots \geq b_{i,r}$ denote the splitting type of $E|D_i$. The following conditions are equivalent:*

(a) $H^1(X_s, E(t; 0, \dots, 0)) = 0$ *for all* $t \in \mathbb{Z}$.

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(b) $b_{i,r} \geq 0$ *for all* $i \in \{1, \dots, s\}$ *and* $f_{s*}(E)$ *is isomorphic to a direct sum of line bundles.*

(c) *There is a direct sum* A *of* r *line bundles on* $\mathbf{P}^2$ *such that* $E \cong f_s^*(A)$.

Remark 1. We will also check that every torsion free sheaf E on X_s such that $H^1(X_s, E(t; 0, \dots, 0)) = 0$ for all $t \ll 0$ is locally free (see Remark 2). Hence we may apply Theorem 1 to this sheaf, without imposing that E is locally free at each point of $D_1 \cup \dots \cup D_s$.

It is essential that we use all multiples (positive and negative) of a " minimal " line bundle $\mathcal{O}_{X_s}(1; 0, \dots, 0)$. In section 3 we will see in the rank 2 case and for the plane X_0 what happens if we take e.g. only twists by even degree line bundles (see Propositions 1 and 2 and Remark 7).

We work over an algebraically closed field $\mathbb{K}$.

2. Proof of Theorem 1

Remark 2. Let X be a smooth and connected projective surface and $R \in \mathrm{Pic}(X)$ such that $|R|$ contains the sum of an effective divisor and of an ample divisor. Let E be a torsion free sheaf on X such that $h^1(X, E \otimes R^{\otimes t}) = 0$ for infinitely many negative integers t. Consider the exact sequence

$$0 \to E \to E^{**} \to E^{**}/E \to 0$$

Since $h^0(X, E^{**} \otimes R^{\otimes t}) = 0$ for $t \ll 0$, E is locally free.

Remark 3. Notice that $h^i(X_s, \mathcal{O}_{X_s}(t; 0, \dots, 0)) = h^i(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(t))$ for all $i = 0, 1, 2$, and all $t \in \mathbb{Z}$. Hence $h^1(X, s, \mathcal{O}_{X_s}(t; 0, \dots, 0)) = 0$ for all $t \in \mathbb{Z}$. When $t \geq 0$ we have $h^0(X_s, \mathcal{O}_{X_s}(t; a_1, \dots, a_s)) = h^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(t))$ if and only if $a_i \geq 0$ for all i.

Remark 4. Let U be a smooth and connected quasi-projective surface. Fix $P \in U$. Let $\pi : V \to U$ denote the blowing up of P. Set $Y := \pi^{-1}(P)$. Let $\mathcal{I}$ denote the ideal sheaf of Y in V. Hence $Y \cong \mathbf{P}^1$ and $\mathcal{I}/\mathcal{I}^2$ is (as an $\mathcal{O}_Y$-sheaf) isomorphic to the degree 1 line bundle of Y. For every integer $n \geq 0$ let $Y^{(n)}$ denote the n-th infinitesimal neighborhood of Y in V, i.e. the closed subscheme of V with $\mathcal{I}^n$ as its indeal sheaf. Hence $Y^{(0)} = 0$. Let $\widehat{Y}$ denote the formal completion of Y in V, i.e. the formal scheme $\mathrm{proj}\lim_n Y^{(n)}$. Fix any rank r vector bundle E on $\widehat{Y}$. Let $b_1 \geq \dots \geq b_r$ be the splitting type of $E|Y$. For every integer $n \geq 0$ we have an exact sequence

$$(1) \qquad 0 \to (\mathcal{I}/\mathcal{I}^2)^{\otimes n} \otimes E|Y \to E|Y^{(n)} \to E|Y^{(n-1)} \to 0$$

Since $\dim(\widehat{Y}) = 1$, we get that for every integer $n \geq 1$ the restriction map $H^1(Y^{(n)}, E|Y^{(n)}) \to H^1(Y^{(n)}, E|Y^{(n-1)})$ is surjective. Hence the restriction map $H^1(\widehat{Y}, E) \to H^1(Y, E|Y)$ is surjective. Since $h^1(Y, E|Y) = 0$ if and only if $b_r \geq -1$, we get $H^1(\widehat{Y}, E) \neq 0$ if $b_r \leq -2$. Now assume $b_r \geq -1$. Since $(\mathcal{I}/\mathcal{I}^2)^{\otimes n}$ is a degree n line bundle on $Y \cong \mathbf{P}^1$, the tower of exact sequences

(1) gives $h^1(Y^{(n)}, E|Y^{(n)}) = 0$ Hence $H^1(\widehat{Y}, E) = 0$ if and only if $b_r \geq -1$. Now assume that E is the restriction to $\widehat{Y}$ of a vector bundle F on V. The formal function theorem ([2], III.11.1) gives that $R^1\pi_*(F) = 0$ if and only if $b_r \geq -1$. Since every fiber of the proper map π has dimension at most 1, $R^j\pi_*(F) = 0$ for all $j \geq 2$ (e.g. by the formulal function theorem ([2], III.11.1, and Nakayama's lemma). For any splitting type $b_1 \geq \cdots \geq b_r$ the sheaf $\pi_*(F)$ is torsion free and its restriction to $U\backslash\{P\}$ is locally free. The natural map $\tau_F : \pi^*\pi_*(F) \to F$ is an isomorphism outside Y. It easy to check that if $b_r < 0$, then $\pi_*(F)$ is not locally free. We have $b_1 = \cdots = b_r = 0$ if and only if the natural map $\tau_F : \pi^*\pi_*(F) \to F$ is an isomorphism.

Lemma 1. *Take the set-up of Remark 4. Let F be a rank r vector bundle on V and let $b_1 \geq \cdots \geq b_r$ be the splitting type of $F|Y$. Assume $b_r \geq 0$. $\pi_*(F)$ is locally free if and only if $b_1 = \cdots b_r = 0$.*

Proof. We claimed the " if " part at the end of Remark 4. Assume $b_1 > 0$. Since $b_r \geq 0$, $h^1(Y, (F|Y) \otimes (\mathcal{I}/\mathcal{I}^2)^{\otimes n}) = 0$ for all $n \geq 0$. Hence (1) implies that the restriction map $H^0(Y^{(n)}, F|Y^{(n)}) \to H^0(Y^{(n-1)}, F|Y^{(n-1)})$ is surjective for every $n > 0$. The formal function theorem ([2], III.11.1) implies that the fiber of $\pi_*(F)$ over P has dimension at least $h^0(Y, F|Y) = r + b_1 + \cdots + b_r > r$. Since $\pi_*(F)$ has rank r, it is not locally free. □

Remark 5. Let F be any vector bundle on U. Since $\dim(U) = \dim(V) = 2$, $H^i(U, \mathcal{F}) = 0$ (resp. $H^i(V, \mathcal{F}) = 0$) for all integers $i \geq 3$ and all coherent sheaves $\mathcal{F}$ on U (resp. V). Since each fiber of the proper map π has dimension at most 1, $R^j\pi_*(F) = 0$ for all $j \geq 2$. Hence the Leray spectral sequence of π gives an exact sequence

$$(2) \qquad 0 \to H^1(U, \pi_*(F)) \to H^1(V, F) \to H^0(U, R^1\pi_*(F)) \to H^2(U, \pi_*(F))$$

([1], p. 31). Hence if $H^1(V, F) = 0$ and $H^2(U, \pi_*(F)) = 0$, then $H^0(U, R^1\pi_*(F)) = 0$. Since $R^1\pi_*(F)$ is supported by P, $H^0(U, R^1\pi_*(F)) = 0$ if and only if $R^1\pi_*(F) = 0$. The same relation is true if we blow up more than one point. Hence we get the following observation. Let A be a rank r vector bundle on X_s. Assume $h^1(X_s, A(t; 0, \ldots, 0))$ for all $t \gg 0$. Notice that for fixed A we have $H^1(\mathbf{P}^2, f_{s*}(A)(t)) = H^2(\mathbf{P}^2, f_{s*}(A)(t)) = 0$ for $t \gg 0$, while the integer $h^0(\mathbf{P}^2, R^1 f_{s_*}(A)(t))$ does not depend from t, because the sheaf $R^1 f_{s*}(A)$ is supported by the finite set $\{P_1, \ldots, P_s\}$. We get $R^1 f_{s*}(A) = 0$. Let $b_{i,1} \geq \cdots \geq b_{i,r}$ be the splitting type of $A|D_i$. Since $R^1 f_{s*}(A) = 0$, $b_{i,r} \geq 0$ for all i (Remark 4).

Proofs of Theorem 1 and of Remark 1. Obviously, (c) implies (a) (Remark 3). The projection formula gives that (c) implies (b). Assume that (a) holds, but only assuming that E is torsion free. The line bundle $\mathcal{O}_{X_s}(2s; -1, \ldots, -1)$ is ample. Hence $\mathcal{O}_{X_s}(2s; 0, \ldots, 0)$ is the tensor product of an ample line bundle and of a line bundle with a non-zero section. Remark 2 gives that E is locally free. Remark 5 gives $R^1 f_{s*}(E) = 0$ and

$b_{i,r} \geq 0$ for all i. Fix a line $D \subset \mathbf{P}^2$ such that $\{P_1, \dots, P_s\} \cap D = \emptyset$. Hence $D \cong f^{-1}(D)$ of X_s. Let $t_1 \geq \dots \geq t_r$ be the splitting type of $E|f^{-1}(D)$. Let $\epsilon : H^0(X_s, E(-t_1; 0, \dots, 0)) \otimes \mathcal{O}_{X_s} \to E(-t_1; 0, \dots, 0)$ denote the evaluation map. Let b the maximal integer such that $1 \leq b \leq r$ and $t_b = t_1$. Hence $h^0(f^{-1}(D), E(-t_1)|f^{-1}(D)) = b$. Since $f^{-1}(D) \in |\mathcal{O}_{X_s}(1; 0, \dots, 0)|$, we have an exact sequence

(3)
$$0 \to E(-t-1; 0, \dots, 0) \to E(-t; 0, \dots, 0) \to E(-t; 0, \dots, 0)|f^{-1}(D) \to 0$$

Look at the cases $t = t_1$ and $t = t_1 + 1$ of (3). Our cohomological assumption on E gives $h^0(X_s, E(-t_1; 0, \dots, 0) = b$ and that ϵ has rank b at each point of $f^{-1}(D)$. Hence $\mathrm{Im}(\epsilon)$ is a rank b torsion free sheaf spanned by a b-dimensional linear space of global sections. Hence $\mathrm{Im}(\epsilon) \cong \mathcal{O}_{X_s}^{\oplus b}$. We also get that the restriction of the inclusion $u : \mathrm{Im}(\epsilon) \to E(-t_1; 0, \dots, 0)$ to the fiber over every $P \in X_s \backslash D_1 \cup \dots \cup D_s$ has rank b. If $s = 0$ we also get that $\mathcal{O}_{\mathbf{P}^2}(t_1)^{\oplus b}$ is a subbundle E' of E such that $E/E'|D$ has splitting type $t_{b+1} \geq \dots \geq t_r$. After finitely many steps we get (unfortunately, with the classical proof) the case $s = 0$, i.e. that an ACM torsion free sheaf on $\mathbf{P}^2$ is a direct sum of line bundles. Hence we may assume $s > 0$. Since $R^1 f_{s*}(E) = 0$, the projection formula and the Leray spectral sequence of f_s give $h^1(\mathbf{P}^2, f_{s*}(E)(t)) = 0$ for all $t \in \mathbb{Z}$. Hence the torsion free sheaf $f_{s*}(E)$ is ACM. We saw in the proof of the case $s = 0$ that $f_{s*}(E)$ is a direct sum of line bundles and in particular it is locally free. Thus $b_{i,j} = 0$ for all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, r\}$ (Lemma 1). Since E is locally free, these equalities are equivalent to the existence of a vector bundle A on $\mathbf{P}^2$ such that $E \cong f_s^*(A)$. Since $A \cong f_{s*}(f_s^*(A))$, we get that (a) implies (c) and (b). Assume (b). Lemma 1 gives $b_{i,j} = 0$ for all $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, r\}$. Hence $E \cong f_s^*(f_{s*}(E))$. Thus (b) implies (c).
Alternatively, one can use descent to get $E \cong f_s^*(M)$ for some torsion free sheaf M on $\mathbf{P}^2$, then show that M is locally free using Remark 2 and then show that M is isomorphic to a direct sum of line bundles. □

3. Rank 2 vector bundles on $\mathbf{P}^2$

Proposition 1. *Let E be a rank 2 torsion free sheaf on $\mathbf{P}^2$ such that $h^1(\mathbf{P}^2, E(t)) = 0$ for every even integer t. Then either $E \cong \Omega_{\mathbf{P}^2}(z)$ for some odd integer z or E is a direct sum of two line bundles.*

Proof. The Euler's sequence gives $h^1(\mathbf{P}^2, \Omega_{\mathbf{P}^2}(t)) = 0$ for $t \neq 0$ and $h^1(\mathbf{P}^2, \Omega_{\mathbf{P}^2}) = 1$. Hence the " if " part is obvious. Now assume $h^1(\mathbf{P}^2, E(t)) = 0$ for every even integer t. Remark 2 gives that E is locally free. Let w be the first integer t such that $h^0(\mathbf{P}^2, E(t)) \neq 0$. Fix any $\sigma \in H^0(\mathbf{P}^2, E(w)) \backslash \{0\}$. The minimality of w shows that σ induces an exact sequence

(4) $$0 \to \mathcal{O}_{\mathbf{P}^2}(t) \to E(w+t) \to \mathcal{I}_Z(c_1 + 2w + t) \to 0$$

where $c_1 := c_1(E)$ and Z is a zero-dimensional subscheme of $\mathbf{P}^2$. The minimaly of w is equivalent to $h^0(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-1)) = 0$. Hence either $c_1+2w-1 < 0$ or $z := \mathrm{length}(Z) \geq (c_1+2w+1)(c_1+2w)/2$. First assume w even. Taking $t = -2$ in (4) and use $h^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-2)) = 0$ we get $h^1(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-2)) = 0$. E splits if and only if $z = 0$. Assume $z > 0$. If $c_1 + 2w - 1 < 0$, then $h^1(\mathbf{P}^2, \mathcal{I}_Z(c_1 + 2w - 2)) = z > 0$. If $c_1 + 2w - 1 \geq 0$ we get $h^1(\mathbf{P}^2, \mathcal{I}_Z(c_1 + 2w - 2)) \geq (c_1+2w+1)(c_1+2w)/2-(c_1+2w)(c_1+2w-1)/2 = c_1+2w > 0$, contradiction. Now assume w odd. Take $t = -3$ in (4). Since $h^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-3)) = 1$, we get that either $z = 1$ or $c_1 + 2w = 1$. If $c_1 + 2w = 1$, then the minimality of w and (4) gives $z = 1$. Since $z \geq (c_1 + 2w + 1)(c_1 + 2w)/2$, in both cases we get $c_1 + 2w = 1$. If $z = 1$ and $c_1 + 2w = 0$, then (4) gives that $E(w)$ is a stable rank two reflexive sheaf with $c_1(E(w)) = 1$ and $c_2(E(w)) = 1$. It is well-known and easy to check that $\Omega_{\mathbf{P}^2}(2)$ is the only such vector bundle. □

Proposition 1 immediately implies the following result, which also follows from Beilinson spectral sequence.

Corollary 1. *Fix an integer m. Let E be a rank 2 torsion free sheaf on $\mathbf{P}^2$ such that $h^1(\mathbf{P}^2, E(t)) = 0$ for all $t \in \mathbb{Z}\backslash\{m\}$. Then either $E \cong \Omega_{\mathbf{P}^2}(-m)$ or E is a direct sum of two line bundles.*

Proposition 2. *Let E be a rank 2 torsion free sheaf on $\mathbf{P}^2$ such that $h^1(\mathbf{P}^2, E(t)) = 0$ for every integer t such that $t \equiv 0 \pmod 3$. Let w be the first integer x such that $h^0(\mathbf{P}^2, E(x)) > 0$. Set $c_1 := c_1(E)$. Then E is isomorphic to one of the following vector bundles:*

(i) *a direct sum of two line bundles.*
(ii) $\Omega_{\mathbf{P}^2}(2 - w)$.
(iii) *$c_1 + 2w = 2$, E is stable and it fits in an exact sequence*

$$0 \to \mathcal{O}_{\mathbf{P}^2} \to E(w) \to \mathcal{I}_Z(2) \to 0 \tag{5}$$

in which Z is a curvilinear zero-dimensional scheme of length 3 not contained in a line..

Conversely, any vector bundle E as in (i), (ii) or (iii) has the property that $h^1(\mathbf{P}^2, E(t)) = 0$ for every integer t such that $t \equiv 0 \pmod 3$. In case (iii) we have $h^1(\mathbf{P}^2, E(z)) = 0$ for all $z \notin \{w-3, w-2\}$, $h^1(\mathbf{P}^2, E(w-2)) = 2$ and $h^1(\mathbf{P}^2, E(w-3)) = 2$.

Proof. Remark 2 gives that E is locally free. Let b denote the only integer such that $1 \leq b \leq 3$ and $w \equiv -b \pmod 3$. Fix any $\sigma \in H^0(\mathbf{P}^2, E(w))\backslash\{0\}$. The minimality of w shows that σ induces an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^2}(t) \to E(w+t) \to \mathcal{I}_Z(c_1 + 2w + t) \to 0 \tag{6}$$

where $c_1 := c_1(E)$ and Z is a locally complete intersection zero-dimensional subscheme of $\mathbf{P}^2$. The minimalityy of w is equivalent to $h^0(\mathbf{P}^2, \mathcal{I}_Z(c_1 + 2w - 1)) = 0$. Hence either $c_1 + 2w - 1 < 0$ or $z := \mathrm{length}(Z) \geq (c_1 + 2w + 1)(c_1 + 2w)/2$. If $z = 0$, then E is a direct sum of two line bundles. Hence we may assume $z > 0$. First assume $b = 1$. Taking $t = -1$ in (6) and using $h^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-1)) = 0$ we get $h^1(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-1)) = 0$. Since $z > 0$, we get $c_1 + 2w - 1 \geq 0$ and $z \leq (c_1+2w+1)(c_1+2w)/2$. Hence $z = (c_1+2w+1)(c_1+2w)/2$. Take $t = -4$ in (6) and using $h^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-4)) = 3$ we get $h^1(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-4)) \leq 3$. Since $c_1+2w-1 \geq 0$ and $z = (c_1 + 2w+1)(c_1+2w)/2$, we get $1 \leq c_1+2w \leq 2$. First assume $c_1+2w = 1$ and hence $z = 1$. Thus $c_2(E(w)) = 1$ and $c_1(E(w)) = 1$. The exact sequence (6) gives the stability of $E(w)$. Hence we are in case (ii). If $c_1+2w = 2$, then we are in case (iii); here we use $h^0(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-1)) = 0$ to see that Z is contained in no line. Now assume $b = 2$. Taking $t = -2$ in (6) and using $h^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-2)) = 0$ we get $h^1(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-2)) = 0$. Hence $c_1+2w-2 \geq 0$ and $z \leq (c_1+2w)(c_1+2w-1)/2$, contradicting the inequality $z \geq (c_1+2w+1)(c_1+2w)/2$. Now assume $b = 3$. Taking $t = -3$ in (6) and using $h^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-3)) = 1$ we get $h^1(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-3)) \leq 1$. Hence $c_1+2w-2 \geq 0$ and $z \leq (c_1+2w-1)(c_1+2w-2)/2+1$, contradicting the inequality $z \geq (c_1+2w+1)(c_1+2w)/2$. The " converse " part is obvious for cases (i) and (ii). Take E as in case (iii) (i.e. as in (6) with $c_1+2w = 2$, $w \equiv -1 \pmod 2$, length$(Z) = 3$ and Z not contained in a line), without assuming the local freeness of E. Since Z is not contained in a line, from (6) we get $h^1(\mathbf{P}^2, E(z)) = 0$ for all $z \geq w-1$. Now assume E locally free. Serre duality gives $h^1(\mathbf{P}^2, E^*(y)) = 0$ for all $y \leq -2-w$. Since rank$(E) = 2$ and E locally free, $E^* \cong E(-c_1)$. Since $c_1+2w = 2$, we get $h^1(\mathbf{P}^2, E(z)) = 0$ for all $z \notin \{w-2, w-3\}$. From (6) we get $h^1(\mathbf{P}^2, E(w-2)) = 2$ and $2 \leq h^1(\mathbf{P}^2, E(w-3)) \leq 3$. By (6) we have $h^1(\mathbf{P}^2, E(w-3)) = 2$ if and only if $h^2(\mathbf{P}^2, E(w-3)) = 0$. Since $c_1(E(w-3)) = -4$, $(E(w-3))^* \cong E(w-3)(4)$. Thus $h^2(\mathbf{P}^2, E(w-3)) = h^0(\mathbf{P}^2, E(w-2)) = 0$. Hence $h^1(\mathbf{P}^2, E(w-3)) = 2$. Now assume that the length 3 scheme Z is curvilinenar, i.e. it is not the first infinitesimal neighborhood of a point of $\mathbf{P}^2$. Since $h^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-1)) = 0$, the Cayley-Bacharach condition is trivially satisfied and hence a general extension (6) with $c_1+2w = 0$ and this curvilinear scheme Z has locally free middle term. □

Remark 6. Let $M(\mathbf{P}^2, c_1, c_2)$ denote the moduli space of rank 2 wector bundle on $\mathbf{P}^2$ with Chern classes c_1, c_2. $M(\mathbf{P}^2, 0, 2)$ is non-empty, irreducible and 5-dimensional. Take E as in case (iii) of Proposition 2. We have $c_1(E) = 0$ and $c_2(E) = c_2(E(1)) - c_1(E(1)) + 1 = 2$ ([3]). We saw that E is stable, i.e. $E \in M(\mathbf{P}^2, 0, 2)$. Take any $F \in M(\mathbf{P}^2, 0, 2)$. Since $c_1(F(1)) = 2$ and $c_2(F(1)) = 3$, Riemann-Roch gives $\chi(F(1)) = (2+3)2/2 + 2 - 3 = 4 > 0$. The stability of F gives $h^0(\mathbf{P}^2, F) = 0$. Hence F fits in the extension (5), i.e. F is described by case (iii) of Proposition 2.

Remark 7. Fix an integer $a \geq 4$ and a rank 2 torsion free sheaf E on $\mathbf{P}^2$ such that $h^1(\mathbf{P}^2, E(t)) = 0$ for all integers t such that $t \equiv 0 \pmod a$. Here we will see that this assumption is very restrictive, but that it seems hopeless to try to classify all such sheaves E. Remark 2 gives that E is locally free. Set

$c_1 := c_1(E)$. Let w be the first integer such that $h^0(\mathbf{P}^2, E(w)) > 0$. Hence we have an exact sequence (6) with Z a zero-dimensional locally complete intersection subscheme. The minimality of w gives $h^0(\mathbf{P}^2, \mathcal{I}_Z(c_1+2w-1)) = 0$. Hence either $c_1 + 2w - 1 < 0$ or $z := \mathrm{length}(Z) \ge (c_1 + 2w + 1)(c_1 + 2w)/2$. If $z = 0$, then E splits. Hence we assume $z > 0$. First assume $c_1 + 2w < 0$. Look at (5). We get $h^1(\mathbf{P}^2, E(t)) > 0$ for all $t < w$ such that either $t + w \le -2$ or $z > (-t - w - 1)(-t - w - 2)/2$. Given any locally complete intersection Z in the case $c_1+2w < 0$ we get an extension (6) with locally free middle term. Now assume $c_1 + 2w \ge 0$. Let b denote the only integer such that $w \equiv -b \pmod a$ and $1 \le b \le a$. First assume $b = 1$ as in the cases $a = 2$ and $a = 3$ we get $z = (c_1 + 2w + 1)(c_1 + 2w)/2$ and $h^i(\mathbf{P}^2, \mathcal{I}_Z(c_1 + 2w - 1)) = 0$, $i = 1, 2$. Taking $t = -1-a$ and using $h^2(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(-1-a)) = a(a-1)/2$ we get $z \le \epsilon+a(a-1)/2$, where $\epsilon = (c_1 + 2w - a + 1)(c_1 + 2w - a)/2$ if $c_1 + 2w \ge a + 1$ and $\epsilon = 0$ if $c_1 + 2w \le a$. We first get $c_1 + 2w \le a$ and then we get $c_1 + 2w \le a - 1$. Now assume $2 \le b \le a$. Taking $t = w - b$ we get $z \le \eta + (b - 1)(b - 2)/2$, where $\eta = 0$ if $c_1 + 2w < b$ and $\eta = (c_1 + 2w - b + 2)(c_1 + 2w - b + 1)/2$ if $c_1 + 2w \ge b$.

We raise the following question.

Question 1. Fix an integer $a \ge 4$ and a rank 2 torsion free sheaf E on $\mathbf{P}^2$ such that $h^1(\mathbf{P}^2, E(t)) = 0$ for all integers t such that $t \equiv 0 \pmod a$. Is it true that $h^1(\mathbf{P}^2, E(t)) \neq 0$ for at most $a - 1$ consecutive integers?

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