An Engel Condition with Generalized Derivations on k-th Commutators

Francesco Rania

Department of Mathematics
University of Messina, Salita Sperone 31
98166 Sant’Agata Messina, Italy
rania@dipmat.unime.it

Abstract

Let $R$ be a non-commutative ring of characteristic different from 2, with center $Z(R)$, Utumi quotient ring $U$ and extended centroid $C$. Let $G$ be a non-zero generalized derivation of $R$, $k \geq 1$ a fixed integer, such that $[G([r_1, r_2]_k), [r_1, r_2]_k] = 0$ for all $r_1, r_2 \in R$. Then one of the following holds:

1. there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;
2. $R$ satisfies the standard identity $s_4(x_1, \ldots, x_4)$ and there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax + xa + \alpha x$, for all $x \in R$.

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1 Introduction

Let $R$ be an associative prime ring with center $Z(R)$, extended centroid $C$ and Utumi quotient ring $U$. Recall that an additive mapping $d$ of $R$ into itself is a derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. The well known theorem of Posner established that a prime ring $R$ must be commutative if it admits a derivation $d$ such that $[d(x), x] \in Z(R)$, for all $x \in R$ [12]. Later Lanski generalized this result to left ideals. More precisely in [7] he proved that if $R$ is a semiprime ring, $I$ a nonzero left ideal, $d$ a nonzero derivation on $R$ and $n, t_0, t_1, \ldots, t_n$ positive integers such that the extended commutator $[d(x^{t_0}), x^{t_1}, x^{t_2}, \ldots, x^{t_n}]$ is zero for all $x \in I$, then either $d(I) = \{0\}$ or the ideal of $R$ generated by $d(I)$ and $d(R)I$ lies in the center of $R$. Hence, if $R$ is prime, then $R$ is commutative.
Recently, in [10], Lee studied an Engel condition with derivation for polynomials on right (left) ideals of $R$. If you fix the attention on multilinear polynomials, the Lee’s result has the following flavour: let $I$ a non-zero right ideal of $R$ and $f(x_1, ..., x_n)$ a non-zero multilinear polynomial over $C$ such that, for any $r_1, ..., r_n \in I$, $[d(f(r_1, ..., r_n)), f(r_1, ..., r_n)]_k = 0$, then there exists an idempotent element $e \in soc(RC)$ such that either $f(x_1, ..., x_n)$ is central valued on the central simple algebra $eRCe$ or $\text{char}(R) = 2$ and $eRCe$ satisfies the standard identity $s_4$.

In [5] Hvala introduced the notion of generalized derivation in rings. An additive mapping $G$ from $R$ to $R$ is called a generalized derivation if there exists a derivation $d$ from $R$ to $R$ such that $G(xy) = g(x)y + xd(y)$ holds for all $x, y \in R$. Thus the generalized derivation covers both the concepts of derivation and left multiplier mapping. The left multiplier mapping means an additive mapping $G$ from $R$ to $R$ satisfying $G(xy) = G(x)y$ for all $x, y \in R$.

In view of this, one might wonder what can be said in the case an Engel condition with generalized derivations is satisfied by the elements of a suitable subset of $R$. In [3] De Filippis proves the following result: if $G$ is a non-zero generalized derivation of $R$, $f(x_1, ..., x_n)$ a multilinear polynomial over $C$, $I$ a non-zero right ideal of $R$ such that $[G(f(r_1, ..., r_n), f(r_1, ..., r_n))] = 0$, for all $r_1, ..., r_n \in I$, then either $G(x) = ax$, with $(a - \gamma)I = 0$ and a suitable $\gamma \in C$ or there exists an idempotent element $e \in soc(RC)$ such that $IC = eRC$ and one of the following holds:

- $f(x_1, ..., x_n)$ is central valued in $eRCe$;
- $G(x) = cx + xb$, where $(c + b + \alpha)e = 0$, for $\alpha \in C$, and $f(x_1, ..., x_n)^2$ is central valued in $eRCe$;
- $\text{char}(R) = 2$ and $s_4(x_1, x_2, x_3, x_4)$ is an identity for $eRCe$.

Our purpose here is to continue on this line of investigation and consider the case when the multilinear polynomial polynomial $f(x_1, ..., x_n)$ is replaced by the k-th commutator $[x_1, x_2]_k$, for a fixed integer $k \geq 1$, which is not multilinear.

We show that:

**Theorem 1.1** Let $R$ be a non-commutative ring of characteristic different from 2, with center $Z(R)$, Utumi quotient ring $U$ and extended centroid $C$. Let $G$ be a non-zero generalized derivation of $R$, $k \geq 1$ a fixed integer. If $[G(r_1, r_2)_k], [r_1, r_2]_k = 0$ for all $r_1, r_2 \in R$, then one of the following holds:

1. there exists $\alpha \in C$ such that $G(x) = \alpha x$, for all $x \in R$;
2. $R$ satisfies the standard identity $s_4(x_1, ..., x_4)$ and there exist $a \in U$, $\alpha \in C$ such that $G(x) = ax + xa + \alpha x$, for all $x \in R$. 
2 The Result

In all that follows we assume that $R$ is an associative prime ring with characteristic different from 2. We begin with:

Lemma 2.1 Let $c,q \in U$. If $R$ satisfies $[c x_1, x_2]_k + [x_1, x_2]_k q, [x_1, x_2]_k]$, then either both $c$ and $q$ are central elements of $U$ or $R$ satisfies the standard identity $s_4(x_1, \ldots, x_4)$ and $c - q \in C$.

Proof. If both $c$ and $q$ are central elements of $U$, we are done. On the other hand, in case either $c \notin C$ or $q \notin C$, then $[c x_1, x_2]_k + [x_1, x_2]_k q, [x_1, x_2]_k]$ is a non-trivial generalized polynomial identity satisfied by $R$. By [1] and [2] $R$ and $U$ satisfy the same generalized polynomial identities. Therefore $U$ satisfies $[c x_1, x_2]_k + [x_1, x_2]_k q, [x_1, x_2]_k]$. Therefore we assume that $U$ satisfies some non-trivial generalized polynomial identity. Moreover $U$ and $U \otimes_C \mathcal{C}$ are both centrally closed algebras ([4]) and, in case $C$ is infinite, they satisfy the same generalized polynomial identities.

By Martindale’s theorem in [11], $U$ is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$.

Suppose first that $dim_C(V) = \infty$. As in lemma 2 in [13], the set $[U, U]_k$ is dense on $U$ and so from $[c r_1, r_2]_k + [r_1, r_2]_k q, [r_1, r_2]_k]$ for all $r_1,r_2 \in U$, we have $[c r + r q, r]$ for all $r \in U$. In this case, as a reduction of the above cited result in [3], we get the contradiction that $R$ must be commutative.

Hence consider the case when $dim_C(V) = m$, a finite integer, hence $U = M_m(C)$. Here $e_{rs}$ denotes the usual matrix unit with 1 in $(r, s)$-entry and zero elsewhere. Say $c = \sum c_{rs} e_{rs}, q = \sum q_{rs} e_{rs}$, where $c_{rs}, q_{rs} \in C$. In the identity $[c x_1, x_2]_k + [x_1, x_2]_k q, [x_1, x_2]_k]$, choose

$$[x_1, x_2]_k = [e_{ij} + e_{ii}, e_{ji} + e_{ij}]_k = 2^{2k-2}(e_{ii} - e_{jj} + e_{ij} - e_{ji}).$$

Thus, by calculations and since $char(R) \neq 2$, we have

$$(e_{ii} - e_{jj} + e_{ij} - e_{ji})(q - c)(e_{ii} - e_{jj} + e_{ij} - e_{ji}) = 0$$

in particular the $(i, i)$-entry of the previous matrix is zero, that is

$$q_{ii} - c_{ii} - q_{ij} + c_{ij} + q_{ji} - c_{ji} - q_{jj} + c_{jj} \quad (1).$$

Analogously for

$$[x_1, x_2]_k = [e_{ij} - e_{ii}, e_{ji} + e_{ij}]_k = 2^{2k-2}(e_{ii} - e_{jj} - e_{ij} + e_{ji}).$$

Thus, by calculations and since $char(R) \neq 2$, we have

$$(e_{ii} - e_{jj} - e_{ij} + e_{ji})(q - c)(e_{ii} - e_{jj} - e_{ij} + e_{ji}) = 0$$
in particular the \((i, i)\)-entry of the previous matrix is zero, that is
\[
q_{ii} = c_{ii} + q_{ij} - c_{ij} - q_{ji} + c_{ji} - q_{jj} + c_{jj} \quad (2)
\]
By summing (1) and (2) and again since \(\text{char}(R) \neq 2\), we get \(q_{ii} - c_{ii} = q_{jj} - c_{jj}\). Finally let \([x_1, x_2]_k = [e_{ij}, e_{jj} - e_{ii}]_k = 2^ke_{ij}\). By calculations it follows \(2^ke_{ij}(q - c)e_{ij} = 0\), that is \(q_{ji} - c_{ji} = 0\).

All the previous facts says that \(q - c\) is a central matrix in \(M_m(C)\), that is \(c = q + \alpha\), for \(\alpha \in Z(M_m(C))\); moreover in this situation we have that \(M_m(C)\) satisfies \([q[x_1, x_2]_k + [x_1, x_2]_k q, [x_1, x_2]_k]\), that is \([q, ([x_1, x_2]_k)^2]\). By Lemma 5 and Theorem 6 in [10] it follows that either \(q \in Z(M_m(C))\), and we are done, or \(([[x_1, x_2]_k]^2\) is central valued in \(M_m(C)\). Notice that for \(m = 2\) we obtain one of the required conclusions, hence we may consider \(m \geq 3\). In this case, for \(x_1 = e_{32} + e_{31}, x_2 = e_{13} + e_{31}\) we have:

- if \(k\) is odd, \([x_1, x_2]_k = 2^{k-1}(e_{11} - e_{33}) - e_{12} + (2^{k-2}(e_{11} + e_{33}) - 2^{k-1}e_{12} \notin Z(M_m(C))\);
- if \(k\) is even, \([x_1, x_2]_k = 2^{k-1}(e_{13} - e_{31}) - e_{32} + (2^{k-2}(-e_{11} - e_{33}) - 2^{k-1}e_{12} \notin Z(M_m(C))\).

In both cases we get a contradiction. \(\Box\)

We are ready to prove the main result:

**Theorem 2.2** Let \(R\) be a non-commutative ring of characteristic different from 2, with center \(Z(R)\), Utumi quotient ring \(U\) and extended centroid \(C\). Let \(G\) be a non-zero generalized derivation of \(R\), \(k \geq 1\) a fixed integer. If \([G([r_1, r_2]_k), [r_1, r_2]_k] = 0\) for all \(r_1, r_2 \in R\), then one of the following holds:

1. there exists \(\alpha \in C\) such that \(G(x) = \alpha x\), for all \(x \in R\);
2. \(R\) satisfies the standard identity \(s_4(x_1, \ldots, x_4)\) and there exist \(a \in U\), \(\alpha \in C\) such that \(G(x) = ax + xa + \alpha x\), for all \(x \in R\).

**Proof.** By Theorem 3 in [8] every generalized derivation \(F\) on a dense right ideal of \(R\) can be uniquely extended to the Utumi quotient ring \(U\) of \(R\), and thus we can think of any generalized derivation of \(R\) to be defined on the whole \(U\) and to be of the form \(f(x) = ax + d(x)\) for some \(a \in U\) and \(d\) a derivation on \(U\). Thus we may assume that there exist \(a \in U\) and \(d\) a derivation on \(U\) such that \(G(x) = ax + d(x)\) for all \(x \in R\).

This means that \(R\) satisfies the differential identity \([a[x_1, x_2]_k + d([x_1, x_2]_k), [x_1, x_2]_k]\), that is \(R\) satisfies
\[
[a[x_1, x_2]_k + \sum_{h=0}^k (-1)^h \binom{k}{h} \sum_{i+j=h-1} x_i^j d(x_2)x_i^j x_2^{k-h}, [x_1, x_2]_k] +
\]
without loss of generality, in order to prove our results we may assume that $G$ satisfies the same differential identities \[9\], then without loss of generality, in order to prove our results we may assume that $U$ satisfies (3). Assume $d(x) = [q, x]$ is an inner derivations in $U$, that is $G(x) = ax + [q, x] = (a + q)x + x(q)$, for a suitable element $q \in U$. In this case we are done. Assume now that $d$ is not an inner derivation of $U$. In this case, by Kharchenko’s Theorem (see [6]) and by (3) we have that $U$ satisfies

$$[a[x_1, x_2]_k + \sum_{h=0}^{k} (-1)^h \binom{k}{h} x_2^h d(x_2) x_2^{k-h}, [x_1, x_2]_k] + [a[x_1, x_2]_k + \sum_{h=0}^{k} (-1)^h \binom{k}{h} x_2^h x_1^2 x_1 x_2^{k-h}, [x_1, x_2]_k]$$

and in particular $U$ satisfies the blended component $[a[x_1, x_2]_k, [x_1, x_2]_k]$. Again by Lemma 1, this implies that $a \in C$, therefore $U$ satisfies $[d([x_1, x_2]_k), [x_1, x_2]_k]$. In this case, by [10] and since $\text{char}(R) \neq 2$, we have that either $d = 0$ or $[x_1, x_2]_k$ is central valued on $U$. In the first case we get $G(x) = ax$, for $a \in C$ and we are done; in the second one, it is well known by Posner’s Theorem that there exists a field $F$ and an integer $t \geq 1$ such that $U$ and $M_t(F)$, the ring of $t \times t$ matrices over $F$, satisfy the same polynomial identities. In particular, for $t \geq 2$, $x_1 = e_{12}$ and $x_2 = e_{22} - e_{11}$, it follows the contradiction $[e_{12}, e_{22} - e_{11}]_k = 2^k e_{12} \notin Z(M_t(F))$.

\[\square\]

References


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