

# Lie Symmetries and Solutions of KdV Equation

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## Abstract

Symmetries of a differential equations is one of the most important concepts in theory of differential equations and physics. One of the most prominent equations is KdV (Kortwege-de Vries) equation with application in shallow water theory. In this paper we are going to explain a particular method for finding symmetries of KdV equation, which is called Harrison method. Our tools in this method are Lie derivatives and differential forms, which will be discussed in the first section more precisely. In second chapter we will have some analysis on the solutions of KdV equation and we give a method, which is called first integral method for finding the solutions of KdV equation.

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## 1 Introduction

There are diverse methods for finding symmetries of differential equations. One of the most important one is Lie method. In this method we need a lot of basis, such as Lie group theory, prolongation and ..., which are foundations of Lie's theory of symmetry groups of differential equations. This is a firm method for finding symmetries and it has a lot of applications in differential equations theory. Harrison's method is an another one, which does not need the above necessities. Here we act this method on the KdV equation, a very famous differential equation, and testing the results, specially with the Lie method see [4],[14], [15], [16].

## 2 Harrison's Method

The method proceeds as follows. We consider a set of partial differential equations, defined on a differentiable manifold  $E$  with  $p$  independent variables and  $q$  dependent variables, here we assume  $p = 2$  and  $q = 1$ . We define the partial derivatives of the dependent variables as new variables (prolongation) in sufficient number to write the equation as second order equation, then we can construct a set of differential forms. We speak of the set of forms, representing the equations, as an ideal  $I$ . It is to be closed. (See [6], [7], [11] for more details.)

In this method Lie derivative of a form with respect to a vector field makes an important roll. Lie derivatives of geometrical objects, like differential forms, are associated with symmetries of those objects. The Lie derivative of a geometrical object carries it along a path, determined by a vector field  $\mathbf{v}$ , in its manifold. If the Lie derivative vanishes, then the vector  $\mathbf{v}$  represents the direction of an infinitesimal symmetry in the manifold. We may construct the Lie derivative ( $\mathcal{L}$ ) of differential form in the ideal  $I$ .

Setting the Lie derivative of these forms equal to zero should therefore represent symmetries. We require that the Lie derivatives of the forms in  $I$  to be linear combinations of those forms themselves and when they vanish they Lie derivatives also vanish. We can illustrate this by writing  $\mathcal{L}_{\mathbf{v}}I \equiv 0 \pmod{I}$ , or  $\mathcal{L}_{\mathbf{v}}I \subset I$ , for Lie derivative properties see [12].

### 2.1 Construction of Differential Forms

First of all substitute  $u_x$  by  $w$ , thus the KdV equation  $u_{xxx} + uu_x + u_t = 0$ , reduced to the following second order partial differential equation

$$w_{xx} + uw + u_t = 0. \quad (1)$$

The jet space corresponds to the equation (1) is a 8-dimensional manifold with the coordinate  $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$ . Corresponding to the equation we have the following contact 1-forms

$$\begin{aligned} \theta^1 &= du - u_t dt - u_x dx, \\ \theta^2 &= du_t - u_{tt} dt - u_{tx} dx, \\ \theta^3 &= du_x - u_{tx} dt - u_{xx} dx. \end{aligned}$$

Consequently our required forms are

$$\begin{aligned} \alpha^1 &= \theta^1 \wedge \theta^2 = (u_x u_{tt} - u_t u_{tx}) dx \wedge dt + u_{tx} dx \wedge du - u_x dx \wedge du_t \\ &\quad + u_{tt} dt \wedge du - u_t dt \wedge du_t + du \wedge du_t, \\ \alpha^2 &= \theta^1 \wedge \theta^3 = (u_x u_{tx} - u_t u_{xx}) dx \wedge dt + u_{xx} dx \wedge du - u_x dx \wedge du_x \end{aligned}$$

$$\begin{aligned}
& +u_{tx}dt \wedge du - u_t dt \wedge du_x + du \wedge du_x, \\
\alpha^3 &= \theta^2 \wedge \theta^3 = (u_{tx}^2 - u_{tt}u_{xx})dx \wedge dt + u_{xx}dx \wedge du_t - u_{tx}dx \wedge du_x \\
& +u_{tx}dt \wedge du_t - u_{tt}dt \wedge du_x + du_t \wedge du_x, \\
\alpha^4 &= d\theta^1 = dx \wedge du_x + dt \wedge du_t, \\
\alpha^5 &= d\theta^2 = dx \wedge du_{tx} + dt \wedge du_{tt}, \\
\alpha^6 &= d\theta^3 = dx \wedge du_{xx} + dt \wedge du_{tx}, \\
\alpha^7 &= udt \wedge du - dt \wedge du_{xx} - dx \wedge du.
\end{aligned}$$

Consider a vector field on the assumed jet space in the form of

$$\mathbf{v} = A_1 \frac{\partial}{\partial t} + A_2 \frac{\partial}{\partial x} + A_3 \frac{\partial}{\partial u} + A_4 \frac{\partial}{\partial u_t} + A_5 \frac{\partial}{\partial u_x} + A_6 \frac{\partial}{\partial u_{tt}} + A_7 \frac{\partial}{\partial u_{tx}} + A_8 \frac{\partial}{\partial u_{xx}},$$

where  $A_i$ ,  $i = 1, \dots, 8$ , are smooth functions of  $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$ .

The next step is solving the following huge!! partial differential equations system,

$$\mathcal{L}_{\mathbf{v}}\alpha^i = \lambda_i \alpha^i, \quad \text{for } i = 1, 2, 3, \quad (2)$$

for smooth functions  $\lambda_i$ .

The system (2) has the form

$$\begin{aligned}
& -\left(\frac{\partial}{\partial u_{tt}}A_3\right) + \dots + \left(\frac{\partial}{\partial u_{xx}}A_1\right)u_{tx}^2 = 0, \\
& -u\left(\frac{\partial}{\partial u_{xx}}A_2\right)u_x u_{tx} + \dots - \left(\frac{\partial}{\partial u_x}A_5\right)u_{tx} = 0, \\
& \quad \quad \quad \vdots \\
& -\left(\frac{\partial}{\partial t}A_2\right) + \dots - \left(\frac{\partial}{\partial u_{xx}}A_7\right) = 0, \\
& -\left(\frac{\partial}{\partial u_{xx}}A_4\right) + \dots + \left(\frac{\partial}{\partial u_{tx}}A_2\right)u_{tx} = 0.
\end{aligned}$$

After solving this system with respect to  $A_1, \dots, A_8$ , we have

$$\begin{aligned}
A_1 &= c_1 t + c_2, & A_2 &= c_1 x/3 + c_3 t + c_4, & A_3 &= -2c_1 u/3 + c_3, \\
A_4 &= -c_3 u_x - 5c_1 u_t/3, & A_5 &= -c_1 u_x, & A_6 &= -2c_3 u_{tx} - 8c_1 u_{tt}/3, \\
A_7 &= -c_3 u_{xx} - 2c_1 u_{tx}, & A_8 &= -4c_1 u_{xx}/3,
\end{aligned}$$

where  $c_1, c_2, c_3, c_4$  are arbitrary constants, thus

$$\begin{aligned}
\mathbf{v} &= (c_1 t + c_2) \frac{\partial}{\partial t} + \left(\frac{c_1}{3}x + c_3 t + c_4\right) \frac{\partial}{\partial x} + \left(c_3 - \frac{2c_1}{3}u\right) \frac{\partial}{\partial u} - \left(c_3 u_x + \frac{5c_1}{3}u_t\right) \frac{\partial}{\partial u_t} \\
& -c_1 u_x \frac{\partial}{\partial u_x} - \left(2c_3 u_{tx} + \frac{8c_1}{3}u_{tt}\right) \frac{\partial}{\partial u_{tt}} - \left(c_3 u_{xx} + 2c_1 u_{tx}\right) \frac{\partial}{\partial u_{tx}} - \frac{4c_1}{3}u_{xx} \frac{\partial}{\partial u_{xx}}.
\end{aligned}$$

We claim that the four vector fields which will be constructed from  $\mathbf{v}$ , make four dimensional symmetry group for KdV equation such as we can obtain in Lie method.

$$\begin{aligned}\mathbf{v}_1 &= \frac{\partial}{\partial x}, \\ \mathbf{v}_2 &= \frac{\partial}{\partial t}, \\ \mathbf{v}_3 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial u} - u_x \frac{\partial}{\partial u_t} - 2u_{tx} \frac{\partial}{\partial u_{tt}} - u_{xx} \frac{\partial}{\partial u_{tx}}, \\ \mathbf{v}_4 &= x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u} - 5u_t \frac{\partial}{\partial u_t} - 3u_x \frac{\partial}{\partial u_x} \\ &\quad - 8u_{tt} \frac{\partial}{\partial u_{tt}} - 6u_{tx} \frac{\partial}{\partial u_{tx}} - 4u_{xx} \frac{\partial}{\partial u_{xx}}.\end{aligned}$$

First of all we should show that the set of these four vector fields  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ , makes a Lie algebra construction, it is sufficient to show that  $[\mathbf{v}_i, \mathbf{v}_j]$  lies in the vector space constructed by  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ , where  $[\ , \ ]$  is the Lie bracket of vector fields. By a directly computation we have

$$\begin{aligned}[\mathbf{v}_1, \mathbf{v}_2] &= 0, & [\mathbf{v}_1, \mathbf{v}_3] &= 0, & [\mathbf{v}_1, \mathbf{v}_4] &= \mathbf{v}_1, \\ [\mathbf{v}_2, \mathbf{v}_3] &= \mathbf{v}_1, & [\mathbf{v}_2, \mathbf{v}_4] &= 3\mathbf{v}_2, & [\mathbf{v}_3, \mathbf{v}_4] &= -2\mathbf{v}_3,\end{aligned}$$

if we label the point parts of these vector fields by

$$\begin{aligned}X_1 &= \frac{\partial}{\partial x}, & X_2 &= \frac{\partial}{\partial t}, \\ X_3 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, & X_4 &= x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - 2u \frac{\partial}{\partial u},\end{aligned}$$

it is easy to see that the third prolongations of  $X_1, X_2, X_3, X_4$ , vanishes the KdV equation, (see [14], [15], for more details.)  $X_i^{(3)}(u_{xxx} + uu_x + u_t) = 0$ , for  $i = 1, 2, 3, 4$ . Where  $X_i^{(3)}$ , is the third prolongation of  $X_i$ , thus  $\{X_1, X_2, X_3, X_4\}$ , makes a set of four parameter symmetry group for KdV equation. It is necessary to say that the four vector fields  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ , which we had found are second prolongation of  $X_1, X_2, X_3, X_4$ , because in the incipience of the section, we decreased the order of equation to two.

### 3 Solutions and First Integral of KdV Equation

First integral(s) of an ordinary differential equation, which will be defined more precisely, is a function which its differential lies in the annihilator of the

distribution corresponds to the assumed ODE. If we have a first integral(s) of an ODE we can find its solution, specially for some differential equations which we are unable to find its solution with any model. For the beginning of this part, we have some analysis on solutions of equations then, we should have a view on some fundamental concepts in theory of differential equations. The reader may see [13] for more details. Now consider the KdV equation, let  $G$  be an  $n$ -dimensional Lie group with the Lie algebra  $\mathfrak{g}$ .  $G$  is called *solvable* if there exist a sequence of subgroups  $\{e\} \subset G_0 \subset G_1 \subset \dots \subset G_{n-1} \subset G_n = G$ , such that each  $G_i$  is a normal subgroup of  $G_{i+1}$ . This is equivalence to the requirement that the corresponding subalgebras of  $\mathfrak{g}$  satisfies  $[\mathfrak{g}_i, \mathfrak{g}_{i+1}] \subset \mathfrak{g}_i$ .

### 3.1 Bianchi Theorem

A theorem of Bianchi, [3], states that if an ordinary differential equation admits an  $n$ -dimensional solvable symmetry group, then its solution can be determined, by quadratures, from those to reduces the order of equation; see also [15].

**Theorem 3.1** *Let  $\Delta(x, u^{(m)}) = 0$  be an  $m$ th order ordinary differential equation. If  $\Delta$  admits a solvable  $n$ -parameter group of symmetries  $G$  such that for  $1 \leq i \leq n$  the orbits of  $G^{(i)}$  ( $i$ th prolongation of  $G$ ) have dimension  $i$ , then the general solution of  $\Delta$  can be found by quadratures from the general solution of an  $(m-n)$ th order differential equation  $\tilde{\Delta}(y, w^{(m-n)}) = 0$ . In particular, if  $\Delta$  admits an  $m$ -parameter solvable group of symmetries, then the general solution to  $\Delta$  can be found by quadratures alone.*

The Bianchi theorem is extended on an arbitrary distribution, which theorem (3.1) is its special case, [13].

By a change of variables as  $y = x - ct$  and  $v = u$ , the reduced equation is  $v''' + vv' - cv' = 0$ , where  $v''' = d^3v/dy^3$  and  $v' = dv/dy$ . This can be immediately integrated once,  $v'' + v^2/2 - cv = c_1$ , another integration reduced equation to

$$\frac{1}{2}v'^2 + \frac{1}{6}v^3 - \frac{1}{2}cv^2 - c_1v - c_2 = 0, \quad (3)$$

where  $c_1$  and  $c_2$  are constants. The general solution can be written in terms of elliptic function,  $u = \mathcal{P}(x - ct + \varepsilon)$ ,  $\varepsilon$  being an arbitrary phase shift. If  $u \rightarrow 0$  sufficiently rapidly as  $|x| \rightarrow \infty$ , then  $c_1 = c_2 = 0$  in equation (3), this equation has the solution  $v = 3c \operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x - ct) + \varepsilon\right]$ , provided the wave speed  $c$  is positive. This produce the celebrated "one soliton" solutions

$$u(x, t) = 3c \operatorname{sech}^2\left[\frac{1}{2}\sqrt{c}(x - ct) + \varepsilon\right],$$

to the KdV equation, which is called *Travelling Wave Solution*.

There is a lot kinds of solutions for Kortweg-de Vries, such as *Galilean-Invariant Solution* and *Scale-Invariant Solution*.

Consider the flows generated by the four symmetries,

$$\begin{aligned}\theta_1(s)(t, x, u) &= (t, x + s, u), \\ \theta_2(s)(t, x, u) &= (t + s, x, u), \\ \theta_3(s)(t, x, u) &= (t, x + ts, s + u), \\ \theta_4(s)(t, x, u) &= (te^{3s}, xe^s, ue^{-2s}),\end{aligned}$$

according to the flows, we can find some special solutions for KdV equation. For example if  $H(x, t)$  be a solution of KdV, then is so  $u(t, x) = \delta^2 H(\delta^3 t + \alpha, \delta x + \beta + \gamma \delta t) - \lambda$ , where  $\alpha, \beta, \gamma, \delta$  and  $\lambda$  are arbitrary constants. We can see that

$$\begin{aligned}u(t, x) &= \frac{12\gamma^2}{(\beta t + \gamma x + \alpha)^2} - \frac{\beta}{\gamma}, \\ u(t, x) &= -12 \tanh(x^2) + 8, \\ u(t, x) &= -12\gamma^2 \tanh(\beta t + \gamma x + \alpha)^2 + 8\gamma^2 - \frac{\beta}{\gamma},\end{aligned}$$

are another solutions which are constructed with the symmetries and flows.

If  $u$  depends only on  $x$ , i.e.,  $u = y(x)$ , then KdV equation reduced to

$$y''' + yy' = 0, \quad (4)$$

and  $X_1 = \partial/\partial x$ ,  $X_2 = x\partial/\partial x - 2y\partial/\partial y$ , are two symmetries for equation (4), and  $x = v(u)$ ,  $y = u$ , is a change of variable corresponds to  $X_1$  for equation (4). With these new variables and by substituting  $v' = \eta(u)$  the equation (4) changes to

$$u\eta^4 + 3\eta'^2 - \eta\eta'' = 0, \quad (5)$$

where  $\eta' = d\eta/du$ . This new equation (5) has the following three parameter symmetry group

$$\tilde{X}_1 = \eta^3 \frac{\partial}{\partial \eta}, \quad \tilde{X}_2 = \eta^3 u \frac{\partial}{\partial \eta}, \quad \tilde{X}_3 = 2u \frac{\partial}{\partial u} - 3\eta \frac{\partial}{\partial \eta},$$

corresponding to  $\tilde{X}_3$ , we have the change of variable

$$t = \eta(u)u^{\frac{3}{2}}, \quad s(t) = \frac{1}{2} \ln(u),$$

by substituting  $s' = \xi(t)$ , equation (5) reduces to

$$4t^4 \xi^3 + 3\xi - 10t\xi^2 + 12t^2 \xi^3 + t\xi' = 0, \quad (6)$$

where  $\xi' = d\xi/dt$ . This equation has one parameter symmetry group

$$\bar{X} = t(3 + t^2)\frac{\partial}{\partial t} - 3(1 + t^2)\xi\frac{\partial}{\partial \xi}.$$

The solution of the equation (6) is the general solution of KdV equation.

### 3.2 Frobenius Theorem

The Frobenius theorem is one of the most important theorem in theory of differential equations, which one of its results guarantees that an ODE has first integral(s).

**Definition 3.2** *Suppose  $M$  is an  $n$ -dimensional manifold and  $p \in M$ . A choice of  $k$ -dimensional linear subspace  $D_p \subset T_pM$  is called a  $k$ -dimensional tangent distribution or a  $k$ -dimensional distribution.  $D_p$  is called smooth if  $D = \bigcup_{p \in M} D_p \subset TM$ , is a smooth subbundle of  $TM$ .*

Here  $T_pM$  and  $TM$  are tangent space on  $M$  in point  $p$ , and  $TM$  is the tangent bundle on  $M$ .

**Lemma 3.3** [12] *Let  $M$  be a smooth  $n$ -manifold, and let  $D \subset TM$  be a  $k$ -dimensional distribution. Then  $D$  is smooth if and only if each point  $p \in M$  has a neighborhood  $U$  on which there are smooth 1-forms  $\omega^1, \dots, \omega^{n-k}$  such that for each  $q \in U$ ,*

$$D_q = \ker \omega^1|_q \cap \dots \cap \ker \omega^{n-k}|_q.$$

More precisely, if we denote the annihilator of  $D$  by

$$\text{Ann}(D) = \{\omega \in \Omega^1(M) : \omega = 0 \text{ on } D\},$$

then for any  $\omega^i$  defined in lemma (3.3) we have  $\omega^i \in \text{Ann}(D)$ .

If  $D$  is a distribution generates by  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ , then  $D$  could be discussed by  $\{\omega^1, \dots, \omega^{n-k}\}$  too. We will show such a distribution as

$$D = \mathcal{F}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \mathcal{F}(\omega^1, \dots, \omega^{n-k}),$$

where  $n - k$  is codimension of  $D$ .

**Definition 3.4** *Suppose  $D = \mathcal{F}(\mathbf{v}_1, \dots, \mathbf{v}_k)$  is a  $k$ -dimensional distribution. The distribution  $D^{(1)}$  which is generates by the vector fields  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  and by all possible sorts of commutators  $[\mathbf{v}_i, \mathbf{v}_j]$  ( $i < j; i, j = 1, \dots, k$ ), is called the first derivative of  $D$ , i.e.,*

$$D^{(1)} = \mathcal{F}(\mathbf{v}_1, \dots, \mathbf{v}_k, [\mathbf{v}_1, \mathbf{v}_2], \dots, [\mathbf{v}_1, \mathbf{v}_k], \dots, [\mathbf{v}_{k-1}, \mathbf{v}_k]).$$

**Lemma 3.5** [13] *Let  $D = \mathcal{F}(\mathbf{v}_1, \dots, \mathbf{v}_k)$  is a distribution such that  $\mathcal{F}(\omega^1, \dots, \omega^{n-k})$ . Then  $D^{(1)} = D$  if and only if for  $i = 1, \dots, n - k$*

$$d\omega^i \wedge \omega^1 \wedge \dots \wedge \omega^{n-k} = 0.$$

**Definition 3.6** *A smooth distribution  $D$  on a smooth manifold  $M$  is called Completely Integrable Distribution or a CID, if all points of  $M$  contain in an integral manifold of  $D$ . (A submanifold  $N \subset M$  is called an integral manifold of  $D$  if  $T_p N \subset D$ .)*

Now we are ready to give the Frobenius theorem.

**Theorem 3.7** [12] *Let  $D$  is a smooth distribution on a smooth manifold  $M$ .  $D$  is CID if and only if  $D^{(1)} = D$ .*

Now by using the Frobenius theorem it will be shown that any first order ODE has first integral(s). This is the result of the Frobenius theorem and its following corollary, before we have a necessary definition.

**Definition 3.8** *Let  $D$  is a smooth distribution on smooth manifold  $M$ , a smooth function  $\varphi \in C^\infty(M)$  is called a first integral for  $D$  if  $d\varphi \in \text{Ann}(D)$ . Here  $C^\infty(M)$  is the set of all smooth real valued function on manifold  $M$ . Another definition for a CID, is that  $D$  is CID if and only if there exist  $n - k$  first functional independent integrals  $\varphi_1, \dots, \varphi_{n-k}$  such that  $D = \mathcal{F}(d\varphi_1, \dots, d\varphi_{n-k})$ ,*

so if a distribution is CID it means that it has first integral(s).

**Corollary 3.9** *Suppose  $D$  is a smooth distribution such that  $D = \mathcal{F}(\omega)$ . Then  $D$  is CID if and only if*

$$\omega \wedge d\omega = 0. \quad (7)$$

Now consider a first order ODE

$$y' = \frac{dy}{dx} = f(x, y), \quad (8)$$

it's clear that the equation (8) obtained by taking the 1-form  $\omega = dy - f(x, y)dx$ , to zero, so  $\omega$  satisfies the equation (7), thus the distribution corresponds to the equation (8) has first integral(s).

### 3.3 Symmetries of Distribution

**Definition 3.10** *A symmetry of a distribution is the transformation of the manifold  $M$  that maps distribution into itself. In other words, a diffeomorphism  $F : M \rightarrow M$  is a symmetry of a distribution  $D$  if  $F_*(D_p) = D_{F(p)}$ , for all  $p \in M$ . Here  $F_*$ , is push forward of  $F$ .*

Suppose  $\mathbf{v}$  is an smooth vector field on manifold  $M$ , and  $\theta_t$  be its flow, then we know  $\theta_t : M \rightarrow M$  induces a diffeomorphism on  $M$ .

**Definition 3.11**  $\theta_t$  which has defined above is called an infinitesimal symmetry or a symmetry of a distribution  $D$  if  $\theta_t$  along the vector field  $\mathbf{v}$  consists of symmetries of  $D$ . i.e,  $\theta_*(D_p) = D_{\theta_t(p)}$ , for all  $p \in M$  and  $t$ .

Denote  $\text{Sym}(D)$  the set of all infinitesimal symmetries of distribution  $D$ .

**Theorem 3.12** [13] Let  $D$  be a distribution and  $\mathcal{X}(D)$  denotes the set of all vector fields on  $D$ , then the following statements are equivalent.

- i)  $\mathbf{v} \in \text{Sym}(D)$ .
- ii)  $\forall \mathbf{w} \in \mathcal{X}(D) \Rightarrow [\mathbf{v}, \mathbf{w}] \in \mathcal{X}(D)$ .
- iii)  $\forall \omega \in \text{Ann}(D) \Rightarrow \mathcal{L}_{\mathbf{v}}\omega \in \text{Ann}(D)$ .

In this theorem  $\mathcal{L}_{\mathbf{v}}\omega$ , denotes the Lie derivative of  $\omega$  with respect to  $\mathbf{v}$ .

**Corollary 3.13**  $\text{Sym}(D)$  has a real Lie algebra structure with respect to the commutator of the vector fields.

### 3.4 Distribution with a Commutative Symmetry Algebra

Let  $\mathfrak{g}$  be a commutative symmetry Lie algebra which its dimension is equal to the dimension of  $\mathcal{F}(\omega^1, \dots, \omega^{n-k})$  is equal to  $k$ . Let  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  be a basis of  $\mathfrak{g}$  and let  $D$  is a CID. Then form the matrix

$$\mathbf{Z} = \begin{pmatrix} \omega^1(\mathbf{v}_1) & \cdots & \omega^1(\mathbf{v}_k) \\ \vdots & \vdots & \vdots \\ \omega^k(\mathbf{v}_1) & \cdots & \omega^k(\mathbf{v}_k) \end{pmatrix},$$

because of independence of  $\omega^i$ 's, then  $\mathbf{Z}^{-1}$  is exist, now we are going to construct a new basis  $\bar{\omega}^1, \dots, \bar{\omega}^k$  for  $\text{Ann}(D)$ . These basis are constructed by the following relation

$$\begin{pmatrix} \bar{\omega}^1 \\ \vdots \\ \bar{\omega}^k \end{pmatrix} = \mathbf{Z}^{-1} \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^k \end{pmatrix}. \quad (9)$$

It is possible to see that  $\bar{\omega}^i$ 's are closed, so the functions  $\varphi_i(p) = \int_{\alpha} \bar{\omega}^i$ , are called *first integrals* of  $D$ . Here  $\alpha$  is a path from the fix point  $p_0$  to a point

$p \in M$ . Because of closeness of  $\bar{\omega}^i$ 's, the integration is independent from choice of  $\alpha$ .

Consider a completely integrable distribution  $D = \mathcal{F} = (\omega)$  with a symmetry  $\mathbf{v}$ , then according to the relation (9), the 1-form  $\bar{\omega} = \frac{1}{\omega(\mathbf{v})}\omega$ , is closed and the function

$$\varphi = \int_{\alpha} \frac{\omega}{\omega(\mathbf{v})}, \quad (10)$$

is a first integral of  $D$ .

For another example, consider an ODE in the form of equation (8), we know that the corresponding 1-form is  $\omega = dy - f(x, y)dx$ , suppose that  $\mathbf{v} = a(x, y).\partial/\partial x + b(x, y).\partial/\partial y$ , is a symmetry of the above distribution, that is the symmetry of the equation (8),  $\mathbf{Z} = b(x, y) - f(x, y)a(x, y)$  and the differential 1-form  $\bar{\omega} = (dy - f(x, y)dx)/(b(x, y) - f(x, y)a(x, y))$ , is closed, and the function

$$\varphi = \int_{\alpha} \bar{\omega},$$

is a first integral of the equation (8). The function  $\mathbf{Z}^{-1}$  is called an *integrating factor* for the equation (8).

### 3.5 First Integral of KdV Equation

After we reduced the equation to an ODE, we can obtain its first integral due to equation (10). The equation (3) could be written as 1-form

$$\omega = dv - \left( \sqrt{cv^2 + 2c_1v + 2c_2 - v^3/3} \right) dy,$$

such that its independent variable  $y$ , does not enter explicitly, thus it is obvious that  $\mathbf{v} = \partial/\partial y$ , is a symmetry for the equation (3), consequently the closed 1-form  $\bar{\omega} = -\omega/\sqrt{cv^2 + 2c_1v + 2c_2 - v^3/3}$ , is constructed and the function

$$\varphi = \int_{\alpha} \left( dy - \frac{dv}{\sqrt{cv^2 + 2c_1v + 2c_2 - v^3/3}} \right),$$

which is a line integral on any arbitrary path  $\alpha$  is the first integral ( $d\varphi \in \text{Ann}(D)$  where  $D = \mathcal{F}(\omega) = \mathcal{F}(\mathbf{v})$ ) for KdV equation, and the set  $\{u(t, x) : \varphi = 0\}$ , is the set of all solutions for the equation.

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