On Common Fixed Point Theorem
in Complete Metric Space

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Abstract
In this paper, we study concept compatible mapping of type (A) and proved a common fixed point theorems for four compatible type of (A) mappings satisfying a contractive condition.

1 Introduction
Sessa[1] introduced a concept of weakly commuting mappings and obtained some common fixed point theorems in complete metric space. In this way S. T. Patil [3] proved some common fixed point theorems for weakly commuting mappings satisfying a contractive conditions in complete metric space. In 1986 G. Jungck defined compatible mappings and discussed few common fixed point theorems in complete metric space. Also he showed weak commuting mappings are compatible mappings but converse need not hold. Again G. Jungck, P. P. Murthy and Y. J. Cho[4] introduced the new concept i.e. compatible type of (A) and proved some common fixed point theorems in complete metric spaces. Compatible type of (A) is more general than weakly commuting mappings and converse is not true.

So on this way we have proved a common fixed point theorems for four compatible mappings of type (A) satisfying a contractive condition in a complete metric space.

Definition 1.1 Self maps $S$ and $T$ of a metric space $(X,d)$ are said to be weakly commuting pair iff $d(STx,TSx) \leq d(Sx,Tx)$ for all $x \in X$. 
Clearly, commuting mappings are weakly commuting but converse is not true.

**Definition 1.2** Self maps $S$ and $T$ of a metric space $(X, d)$ are said to be compatible of type (A) if $\lim_{n \to \infty} d(TSx_n, SSx_n) = 0$ and $\lim_{n \to \infty} d(STx_n, TTx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t$, for some $t \in X$.

Clearly, weakly commuting mappings are compatible of type (A). Note that the implication is not reversible.

**Definition 1.3** A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

## 2 Main Results

**Theorem 2.1** Suppose $S, T$ and $I$ are three self mappings of a complete metric space $(X, d)$ into itself satisfying the conditions

(i) $S(X) \cup T(X) \subset I(X)$.

(ii) $d(Sx, Ty) \leq \alpha d(Ix, Iy) + \beta [d(Sx, Ix) + d(Ty, Iy)] + \gamma [d(Ix, Ty) + d(Iy, Sx)],$

for all $x, y \in X$ and $\alpha, \beta$ and $\gamma$ are non-negative reals such that $\alpha + 2\beta + 2\gamma < 1$.

(iii) one of $S, T$ and $I$ is continuous.

(iv) $(S, I)$ and $(T, I)$ are compatible of type (A).

Then $S, T$ and $I$ have a unique common fixed point.

**Proof:**

Let $x_0 \in X$ arbitrary. Construct a sequence $\{Ix_n\}$, as follows.

$$Ix_{2n+1} = Sx_{2n}; \quad Ix_{2n+2} = Tx_{2n+1}; \quad n = 0, 1, 2, \ldots$$  \hspace{1cm} (1)

From condition (ii) we have

$$d(Ix_{2n+1}, Ix_{2n+2}) = d(Sx_{2n}, Tx_{2n+1})$$

$$\leq \alpha d(Ix_{2n}, Ix_{2n+1}) + \beta [d(Sx_{2n}, Ix_{2n}) + d(Tx_{2n+1}, Ix_{2n+1})]$$

$$+ \gamma [d(Ix_{2n}, Tx_{2n+1}) + d(Ix_{2n+1}, Sx_{2n})]$$

$$= \alpha d(Ix_{2n}, Ix_{2n+1}) + \beta [d(Ix_{2n}, Ix_{2n}) + d(Ix_{2n+1}, Ix_{2n+1})]$$

$$+ \gamma [d(Ix_{2n}, Ix_{2n+2}) + d(Ix_{2n+1}, Ix_{2n+1})]$$

$$\leq (\alpha + \beta + \gamma) d(Ix_{2n}, Ix_{2n+1}) + (\beta + \gamma) d(Ix_{2n+1}, Ix_{2n+2}).$$
Therefore, we have
\[ d(Ix_{2n+1}, Ix_{2n+2}) \leq \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} d(Ix_{2n}, Ix_{2n+1}) \]
i.e.
\[ d(Ix_{2n+1}, Ix_{2n+2}) \leq h d(Ix_{2n}, Ix_{2n+1}), \]
where \( h = \frac{\alpha + \beta + \gamma}{1 - (\beta + \gamma)} < 1. \)
Similarly we can show that
\[ d(Ix_{2n+1}, Ix_{2n+2}) \leq h^{n+1} d(Ix_0, Ix_1). \]

For \( k > n, \) we have
\[ d(Ix_n, Ix_{n+k}) \leq \sum d(Ix_{n+i-1}, Ix_{n+i}) \leq h^{n+i-1} d(Ix_0, Ix_1) \leq \frac{h^n}{1 - h} d(Ix_0, Ix_1) \rightarrow 0 \text{ as } n \rightarrow \infty. \]

Hence \( \{Ix_n\} \) is a Cauchy sequence. Since \( X \) is complete metric space, there exist a \( z \in X \) such that \( Ix_n \rightarrow z. \) Then the subsequences of \( \{Ix_n\}, \{Sx_{2n}\} \) and \( \{Tx_{2n+1}\} \) also converges to \( z. \) i.e. \( Sx_{2n} \rightarrow z \) and \( Tx_{2n+1} \rightarrow z. \)
Suppose that \( I \) is continuous and the pair \( (S, I) \) is compatible type of \( (A). \) Then from condition (ii) we have
\[ d(ISx_{2n}, Tx_{2n+1}) \leq \alpha d(Sx_{2n}, Ix_{2n+1}) + \beta [d(Sx_{2n}, Iz) + d(Tx_{2n+1}, Ix_{2n+1})] + \gamma [d(Sx_{2n}, Tx_{2n+1}) + d(Ix_{n+1}, Sx_{2n})]. \]

Since \( I \) is continuous, \( I^2x_{2n} \rightarrow Iz \) as \( n \rightarrow \infty. \) The pair \( (S, I) \) is compatible of type \( (A), \) then \( Sx_{2n} \rightarrow Iz \) as \( n \rightarrow \infty. \)
Letting \( n \rightarrow \infty, \) we have
\[ d(Iz, z) \leq \alpha d(Iz, z) + \beta [d(Iz, Iz) + d(z, z)] + \gamma [d(Iz, z) + d(z, Iz)] = (\alpha + 2\gamma) d(Iz, z). \]
Hence \( d(Iz, z) = 0 \) and \( Iz = z, \) since \( \alpha + 2\gamma < 1. \)
Again we have
\[ d(Sz, Tx_{2n+1}) \leq \alpha d(Iz, Ix_{2n+1}) + \beta [d(Sz, Iz) + d(Tx_{2n+1}, Ix_{2n+1})] + \gamma [d(Iz, Tx_{2n+1}) + d(Ix_{2n+1}, Sz)]. \]
Letting $n \to \infty$ and using $Iz = z$, we have
\[
d(Sz, z) \leq \alpha d(z, z) + \beta [d(Sz, z) + d(z, z)]
+ \gamma [d(z, z) + d(z, Sz)]
= (\beta + \gamma)d(Sz, z).
\]
Hence $d(Sz, z) = 0$ and $Sz = z$, since $\beta + \gamma < 1$. So we have $Iz = Sz = z$.

By condition (ii), we have
\[
d(z, Tz) = d(Sz, Tz) \leq \alpha d(Iz, Iz) + \beta [d(Sz, Iz) + d(Tz, Iz)]
+ \gamma [d(Iz, Tz) + d(Iz, Sz)]
= (\beta + \gamma)d(Tz, z)
\]
$\Rightarrow d(z, Tz) = 0$ i.e. $z = Tz$, since $\beta + \gamma < 1$. Showing that $z$ is common fixed point of $S, T$ and $I$. Similarly we can show that $z$ ia common fixed point of $S, T$ and $I$ when the pair $(T, I)$ is compatible of type (A).

**Uniqueness:**
Let $z$ and $w$ be two common fixed point of $S, T$ and $I$. So we have $z = Sz = Tz = Iz$ and $w = Sw = Tw = Iw$.
\[
d(z, w) = d(Sz, Tw) \leq \alpha d(Iz, Tw) + \beta [d(Sz, Iz) + d(Tw, Iw)]
+ \gamma [d(Iz, Tw) + d(Iw, Sz)]
= \alpha d(z, w) + \beta [d(z, z) + d(w, w)]
+ \gamma [d(z, w) + d(w, z)]
= (\alpha + 2\gamma)d(z, w).
\]
Hence $d(z, w) = 0$ and $z = w$, since $\alpha + 2\gamma < 1$. Thus the common fixed point is unique.

**Theorem 2.2** Suppose $S, I, T$ and $J$ are four self mapping of complete metric space $(X, d)$ into itself satisfying the conditions

(i) $S(X) \subset J(X)$, $T(X) \subset I(X)$.

(ii) $d(Sx, Ty) \leq \alpha d(Ix, Jy) + \beta [d(Ix, Sx) + d(Jy, Ty)] + \gamma [d(Ix, Ty) + d(Jy, Sx)]$, for all $x, y \in X$ and $\alpha, \beta$ and $\gamma$ are non-negative reals such that $\alpha + 2\beta + 2\gamma < 1$.

(iii) one of $S, I, T$ and $J$ is continuous.

(iv) $(S, I)$ and $(T, J)$ are compatible of type (A).

Then $S, I, T$ and $J$ have a unique common fixed point.
Proof:
Let \( x_0 \in X \) be arbitrary. Choose a point \( x_1 \) in \( X \) such that \( Sx_0 = Jx_1 \). This can be done since \( S(X) \subset J(X) \). Let \( x_2 \) be a point in \( X \) such that \( Tx_1 =Ix_2 \). This can be done since \( T(X) \subset I(X) \). In general we can choose \( x_{2n}, x_{2n+1}, x_{2n+2}, \ldots \) such that \( Sx_{2n} = Jx_{2n+1} \) and \( Tx_{2n+1} =Ix_{2n+2} \). So that we obtain a sequence

\[
Sx_0, Tx_1, Sx_2, Tx_3, \ldots \tag{2}
\]

Using condition (ii), we have

\[
d(Sx_{2n}, Tx_{2n+1}) \leq \alpha d(Ix_{2n}, Jx_{2n+1}) + \beta[d(Ix_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1})] \\
+ \gamma [d(Ix_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, Sx_{2n})] \\
= \alpha d(Tx_{2n-1}, Sx_{2n}) + \beta [d(Tx_{2n-1}, x_{2n}) + d(Sx_{2n}, Tx_{2n})] \\
+ \gamma [d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n})] \\
\leq \alpha d(Tx_{2n-1}, Sx_{2n}) + \beta [d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n})] \\
+ \gamma [d(Tx_{2n-1}, Sx_{2n}) + d(Sx_{2n}, Tx_{2n})] \\
= (\alpha + \beta + \gamma)d(Tx_{2n-1}, Sx_{2n}) + (\beta + \gamma)d(Sx_{2n}, Tx_{2n})
\]

Hence

\[
d(Sx_{2n}, Tx_{2n+1}) \leq kd(Sx_{2n}, Tx_{2n-1}),
\]

where \( k = \frac{\alpha + \beta + \gamma}{1-(\beta + \gamma)} < 1 \). Similarly, we can show

\[
d(Sx_{2n}, Tx_{2n-1}) \leq kd(Sx_{2n-2}, Tx_{2n-1}).
\]

Therefore

\[
d(Sx_{2n}, Tx_{2n+1}) \leq k^2d(Sx_{2n-2}, Tx_{2n-1}) \\
\leq k^{2n}d(Sx_0, Tx_1),
\]

which implies that the sequence (1) is a Cauchy sequence and since \((X,d)\) is complete, so the sequence (1) has a limit point \( z \) in \( X \). Hence the subsequences \( \{Sx_{2n}\} = \{Jx_{2n-1}\} \) and \( \{Tx_{2n-1}\} = \{Ix_{2n}\} \) also converges to the point \( z \) in \( X \).

Suppose that the mapping \( I \) is continuous. Then \( Ix_{2n} \rightarrow Iz \) and \( ISx_{2n} \rightarrow Iz \) as \( n \rightarrow \infty \). Since the pair \((S, I)\) is compatible of type (A). We get \( SIx_{2n} \rightarrow Iz \) as \( n \rightarrow \infty \).

Now by (ii)

\[
d(SIx_{2n}, Tx_{2n+1}) \leq \alpha d(I^2x_{2n}, Jx_{2n+1}) + \beta [d(I^2x_{2n}, Sx_{2n}) + d(Jx_{2n+1}, Tx_{2n+1})] \\
+ \gamma [d(I^2x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, Sx_{2n})],
\]

where \( \alpha, \beta, \gamma > 0 \) and \( \alpha + \beta + \gamma < 1 \).
letting \( n \to \infty \), we get
\[
d(Iz, z) \leq \alpha d(Iz, z) + \beta [d(Iz, Iz) + d(z, z)] \\
+ \gamma [d(Iz, z) + d(z, Iz)] \\
= (\alpha + 2\gamma)d(Iz, z).
\]
This gives \( d(Iz, z) = 0 \), since \( 0 \leq \alpha + 2\gamma < 1 \). Hence \( Iz = z \).

Further
\[
d(Sz, Tx_{2n+1}) \leq \alpha d(Iz, Jx_{2n+1}) + \beta [d(Iz, Sz) + d(Jx_{2n+1}, Tx_{2n+1})] \\
+ \gamma [d(Iz, Tx_{2n+1}) + d(jx_{2n+1}, Sz)] \\
= (\alpha + 2\gamma)d(Iz, z)
\]
letting \( Jx_{2n+1}, Tx_{2n+1} \to z \) as \( n \to \infty \) and \( Iz = z \) we get
\[
d(Sz, z) \leq \alpha d(z, z) + \beta [d(z, Sz) + d(z, z)] \\
+ \gamma [d(z, z) + d(z, Sz)] \\
= (\beta + \gamma)d(Sz, z).
\]
Hence \( d(Sz, z) = 0 \) i.e. \( Sz = z \), since \( 0 \leq \beta + \gamma < 1 \). Thus \( Sz = Iz = z \). Since \( S(X) \subset J(X) \), there is a point \( z' \in X \) such that \( z = Sz = Jz' \).

Now by (ii)
\[
d(z, Tz') = d(Sz, Tz') \\
\leq \alpha d(Iz, Jz') + \beta [d(Iz, Sz) + d(Jz', Tz')] \\
+ \gamma [d(Iz, Tz') + d(Jz', Sz)] \\
= \alpha d(z, z) + \beta [d(z, z) + d(z, Tz')] \\
+ \gamma [d(z, Tz') + d(z, z)] \\
= (\beta + \gamma)d(z, Tz')
\]

hence \( d(z, Tz') = 0 \) i.e. \( Tz' = z = Jz' \), since \( 0 \leq \beta + \gamma < 1 \). Take \( y_n = z' \) for \( n \geq 1 \). Then \( Ty_n \to Tz' = z \) and \( Jy_n \to Jz' = z \) as \( n \to \infty \). Since the pair \( (T,J) \) is compatible of type (A), we get
\[
\lim_{n \to \infty} d(TJy_n, JJy_n) = 0,
\]
implies \( d(Tz, Jz) = 0 \), since \( Jy_n = z' \) for all \( n \geq 1 \). Hence \( Tz = Jz \).

Now
\[
d(z, Tz) = d(Sz, Tz) \leq \alpha d(Iz, Iz) + \beta [d(Iz, Sz) + d(Jz, Tz)] \\
+ \gamma [d(Iz, Tz) + d(Jz, Sz)] \\
= \alpha d(z, Tz) + \beta [d(z, z) + d(Tz, Tz)] \\
+ \gamma [d(z, Tz) + d(Tz, z)] \\
= (\alpha + 2\gamma)d(z, Tz).
\]
Since $\alpha + 2\gamma < 1$, we get $Tz = z$. Hence $z = Tz = Jz$, therefore $z$ is common fixed point of $S, I, T$ and $J$, when the continuity of $I$ is assumed.

Now suppose that $S$ is continuous. Then $S^2x_{2n} → Sz, SIx_{2n} → Sz$ as $n → ∞$. By condition (ii), we have

$$d(S^2x_{2n}, Tx_{2n+1}) ≤ αd(ISx_{2n}, Jx_{2n+1}) + β[d(ISx_{2n}, S^2x_{2n}) + d(Jx_{2n+1}, Tx_{2n+1})] + γ[d(ISx_{2n}, T_{2n+1}) + d(Jx_{2n+1}, S^2x_{2n})].$$

Letting $n → ∞$ and using the compatible of type (A) of the pair $(S, I)$, we get

$$d(Sz, z) ≤ αd(Sz, z) + β[d(Sz, S_{2n}) + d(z, z)] + γ[d(Sz, z) + d(z, Sz)] = (α + 2γ)d(Sz, z).$$

Since $α + 2γ < 1$, we get $Sz = z$. But $S(X) ⊂ J(X)$, there is a point $u ∈ X$ such that $z = Sz = Ju$.

Now by (ii)

$$d(S^2x_{2n}, Tu) ≤ αd(ISx_{2n}, Ju) + β[d(ISx_{2n}, S^2x_{2n}) + d(Ju, Tu)] + γ[d(ISx_{2n}, Tu) + d(Ju, S^2x_{2n})].$$

Letting $n → ∞$

$$d(z, Tu) = d(Sz, Tu)
\leq αd(z, z) + β[d(z, z) + d(z, Tu)] + γ[d(z, Tu) + d(z, z)] = (β + γ)d(z, Tu),$$

since $β + γ < 1$, we get $Tu = z$. Thus $z = Ju = Tu$.

Let $y_n = u$. Then $Ty_n → Tu = z$ and $Jy_n → Tu = z$. Since $(T, J)$ is compatible of type (A), we have

$$\lim_{n→∞} d(TJy_n, JJy_n) = 0,$$

this gives $TJu = JTu$ or $Tz = Jz$.

Further

$$d(Sx_{2n}, Tz) ≤ αd(Ix_{2n}, Jz) + β[d(Ix_{2n}, Sx_{2n}) + d(Jz, Tz)] + γ[d(Ix_{2n}, Tz) + d(Jz, Sx_{2n})].$$

Letting $n → ∞$, we get

$$d(z, Tz) ≤ αd(z, Tz) + β[d(z, z) + d(Tz, Tz)] + γ[d(z, Tz) + d(Tz, z)] = (α + 2γ)d(z, Tz).$$
Since $0 \leq \alpha + 2\gamma < 1$, we get $z = Tz$. Again we have $T(X) \subset I(X)$, there is a point $v \in X$ such that $z = Tz = Iv$. Now

$$d(Sv, z) = d(Sv, Tz) \leq \alpha d(Iv, Jz) + \beta[d(Iv, Sv) + d(Jz, Tz)]$$
$$+ \gamma[d(Iv, Tz) + d(Jz, Sv)]$$
$$= \alpha d(z, z) + \beta[d(z, Sv) + d(z, z)]$$
$$+ \gamma[d(z, Tz) + d(z, Sv)]$$
$$= (\beta + \gamma)d(z, Sv).$$

since $0 \leq \beta + \gamma < 1$, we get $Sv = z$.

Take $y_n = v$ then $Sy_n \to Sv = z$, $Iy_n \to Iv = z$. Since $(S, I)$ is compatible of type (A), we get

$$\lim_{n \to \infty} d(ISy_n, IIy_n) = 0.$$  

This implies that $SIv = ISv$ or $Sz = I\!z$. Thus we have $z = Sz = I\!z = Jz = Tz$. Hence $z$ is a common fixed point of $S, I, T$ and $J$, when $S$ is continuous. The proof is similar that $z$ is common fixed point of $S, I, T$ and $J$, when $T$ is continuous.

**Uniqueness:**

Let $z$ and $w$ be two common fixed point of $S, I, T$ and $J$. i.e. $z = Sz = I\!z = Tz = Jz$ and $w = Sw = Iw = Tw = Jw$. By condition (ii)

$$d(z, w) = d(Sz, Tw)$$
$$\leq \alpha d(Iz, Jw) + \beta[d(Iz, Sz) + d(Jw, Tw)]$$
$$+ \gamma[d(Iz, Tw) + d(Jw, Sz)]$$
$$= \alpha d(z, w) + \beta[d(z, z) + d(w, w)]$$
$$+ \gamma[d(z, w) + d(w, z)]$$
$$= (\alpha + 2\gamma)d(z, w),$$

since $\alpha + 2\gamma < 1$, we have $z = w$. hence the proof.

**References**


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