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Oscillation Theorem for Perturbed Nonlinear Differential Equations

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Abstract. Sufficient conditions for the oscillation of the nonlinear second order differential equation (a(t)x')'+Q(t,x')=P(t,x,x') are established.Where the coefficients are continuous and a(t) is non negative.The result of this paper generalizes the results given in [8].

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1. INTRODUCTION

This paper concerns the oscillation of solutions of the perturbed second order nonlinear differential equation

$$(a(t)x')' + Q(t,x) = P(t,x,x') \tag{1}$$

where $a : [T_0,\infty) \to \mathbb{R}, Q : [T_0,\infty) \times \mathbb{R} \to \mathbb{R}$, and $P : [T_0,\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and $a(t) > 0$. Throughout the paper, we shall restrict our attention only to the solutions of the differentiable equation (1) which exist on some ray of the form $[T_0,\infty)$.

In this paper we give more general integral criteria to the oscillation of (1), which contain the results in [1] as particular cases.

A solution of (1) is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. If all solutions of (1) are oscillatory, (1) is called oscillatory. The oscillatory behavior of solutions of second order ordinary differential equation including the existence of oscillatory and nonoscillatory solutions) has been the subject of intensive investigations.This probleme has received the attention of many authors. Many criteria have been found which involve the average behavior of the integral of the alternating coefficient. Among numerous papers dealing with this subject we refer in particular to [1,3,4,5,6,7,8,9,10,11,12,13,14,15,16,19,20].

Note that in 1996, Wong and Agarwal [16] established oscillation criteria for the more general equation

$$\left(r(t)\,|y'|^{\alpha-1}\,y'\right)' + Q(t,y) = P(t,y,y')$$

These authors [17,18] and Hong [6] also considered some special case of this equation by using the technique which is an extension of the methods used in the works of Greaf and Spikes [7] and Kwang and Wong [9] for the differential equations.

2. MAIN RESULTS

Assume that there exist continuous functions $p, q : [T_0, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$,
such that

$$xf(x) > 0 \quad \text{for} \quad x \neq 0 \tag{2}$$

$$f'(x) \geq 0 \quad \text{for} \quad x \neq 0 \tag{3}$$

$$\frac{Q(t,x)}{f(x)} \geq q(t) \quad \text{and} \quad \frac{P(t,x,x')}{f(x)} \leq p(t) \quad \text{for } x \neq 0 \tag{4}$$

and let R be a positive continuously differentiable function on the interval $[T, \infty)$ such that R' is nonnegative and decreasing on $[T, \infty)$ such that

$$\lim_{t\to\infty} \frac{1}{R(t)} \int_{T_0}^{t} \frac{1}{a(s)} ds = \infty \tag{5}$$

Furthermore, we define for every $t \geq T_0$

$$Z(t) = R(t)[q(t) - p(t)]$$

Theorem 1. *Suppose (2),(3),(4),(5) hold and*

$$\int_{T_0}^{\infty} Z(s) ds = \infty \tag{6}$$

Then (1) is oscillatory.

Proof. Let x be a nonoscillatory solution on an interval $[T,\infty), T \geq T_0$ of the differentiable equation (1). Without loss of generality, this solution can be supposed such that $x(t) \neq 0$.We assume that $x(t)$ is positive on $[T,\infty)$ (the case $x(t) < 0$ can be treated similarly and will be omitted).

Then.

$$\begin{aligned}\left[\frac{a(t)x'(t)}{f[x(t)]}\right]' &= \frac{P[t,x'(t),x(t)]}{f[x(t)]} - \frac{Q[t,x(t)]}{f[x(t)]} - \frac{a(t)f'(x(t)[x'(t)]^2}{f^2[x(t)]} \\ &\leq -(q(t)-p(t))\end{aligned} \tag{7}$$

Multiplying (7) by $R(t)$ and integrating from T to t , we obtain

$$\begin{aligned}\frac{R(t)a(t)x'(t)}{f[x(t)]} &\leq C_0 - \int_T^t Z(s)ds \\ &\quad + \int_T^t R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds - \int_T^t R(s)\frac{a(s)f'(x(s)[x'(s)^2]}{f^2[x(s)]}ds\end{aligned} \tag{8}$$

where $C_0 = \frac{R(T)a(T)x'(T)}{f[x(T)]}$

We consider the following three cases for the behavior of $x'(t)$

Case 1 $x'(t) > 0$ for every $t \geq T$. By condition (6) we have

$$\frac{R(t)a(t)x'(t)}{f[x(t)]} \leq \int_T^t R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds. \tag{9}$$

Since $R'(t)$ is nonegative and decreasing on $[T,\infty)$ the last inequality implies

$$R(T)\frac{a(t)x'(t)}{f[x(t)]} \leq R'(T)\int_T^t \frac{a(s)x'(s)}{f[x(s)]}ds$$

We may use the Gronwall inequality to obtain $\frac{a(t)x'(t)}{f[x(t)]} = 0$. This clearly contradicts the fact that $x'(t) > 0$

Case 2 $x'(t)$ is oscillatory. Then there exists a sequence $(\alpha_n)_{n\in N} \to \infty$ in $[T,\infty)$ such that $x'(\alpha_n) = 0$. Choose N large enough so that

$$\int_{\alpha_N}^t Z(s)ds = \infty$$

Thus, there exists a positive constant K such that

$$\int_{\alpha_N}^t Z(s)ds \geq K \quad \text{for all} \quad t \geq \alpha_N$$

If we assume that α_N and α_{N+1} are two consecutive points in $[T,\infty)$ where $x'(t) < 0$ for $t \in]\alpha_N\ , \alpha_{N+1}[$.

From (8) we have

$$0 < \int_{\alpha_N}^{\alpha_{N+1}} Z(s)ds \leq \int_{\alpha_N}^{\alpha_{n+1}} R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds$$

which contradicts the fact that $x'(t)$ oscillates.

Case 3 $\mathbf{x'(t) < 0}$ on $[T, \infty)$.Multiplying inequality (7) by $R(t)$ and integrating by part we obtain

$$\frac{R(t)a(t)x'(t)}{f[x(t)]} \leq C_0 - \int_T^t R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'[x(s)]ds \tag{10}$$

Furthermore, we choose a $T_1 > T$ so that

$$-C_0 + \int_T^{T_1} R(s)\frac{a(s)x'^2(s)}{f^2[x(s)]}f'[x(s)]ds = C_1 > 0$$

and then for every $t \geq T_1$ we get

$$R(t)\frac{a(t)[x'(t)]^2}{(f[x(t)])^2}f'(x(t))\left\{-C_0 + \int_T^t R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'(x(s)ds\right\}^{-1}$$

$$\geq -\frac{x'(t)}{f[x(t)]}f'(x(t))$$

and hence, by integrating over $[T_1, t]$,we obtain for $t \geq T$

$$Log\frac{-C_0 + \int_T^t R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'[x(s)]ds}{C_1} \geq Log\frac{f[x(T)]}{f[x(t)]}$$

Thus,

$$-C_0 + \int_T^t R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'(x(s)ds \geq C_2\frac{1}{f[x(T)]} \quad \text{for all} \quad t \geq T$$

where $C_2 = C1\frac{1}{f(x(T_1))}$. So (10) yields

$$x'(t) \leq \frac{-C_2}{R(t)a(t)}$$

and consequently for $t \geq T_1$we have

$$x(t) \leq x(T_1) - C_2\int_{T_1}^t \frac{ds}{R(s)a(s)} \tag{11}$$

Since R is positive on $[T_1, \infty)$ and R' is nonnegative above on $[T_1, \infty)$; it follows that

$$x(t) \leq x(T_1) - C_2\frac{1}{R(t)}\int_{T_1}^t \frac{ds}{a(s)}$$

Therefore,we conclude from (5) that $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction. This completes the proof of the theorem. ∎

Theorem 2. *If the conditions (2),(3),(4),(5) hold, with*

$$\int_{T_0}^{t} Z(s)ds < \infty \tag{12}$$

$$\liminf_{t\to\infty} \int_{T}^{t} Z(s)ds \geq 0 \quad \textit{for all large } T \tag{13}$$

$$\lim_{t\to\infty} \frac{1}{R(t)} \int_{T_0}^{t} \frac{1}{a(s)} \int_{s}^{\infty} Z(s)ds = \infty \tag{14}$$

and

$$\int_{\epsilon}^{\infty} \frac{dy}{f(y)} < \infty \quad \textit{and} \quad \int_{-\epsilon}^{-\infty} \frac{dy}{f(y)} < \infty \quad \textit{for every} \ \ \epsilon > 0 \tag{15}$$

Moreover, let $R'(t)a(t)$ *be nonegative and decreasing on* $[T, \infty)$*. Then all solutions of (1) are oscillatory.*

Proof. **Proof** Let x be a nonoscillatory solution on an interval $[T,\infty)$ of the differentiable equation (1). We suppose, as in theorem 1, that x is positive on $[T, \infty)$.We consider the following three cases for the behavior of $x'(t)$.

case 1: $x'(t) > 0$. From (8) we have

$$0 \leq C_0 - \int_{T}^{t} Z(s)ds + \int_{T}^{t} R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds$$

Thus, for all $t \geq T$

$$\int_{T}^{t} Z(s)ds \leq R(T)\frac{a(T)x'(T)}{f[x(T)]} + \int_{T}^{t} R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds$$

By the Bonnet theorem, for any $t \geq T$, there exists a $\xi \in [T, t]$ so that

$$\begin{aligned}\int_{T}^{t} R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds &= R'(T)a(T)\int_{T}^{\xi} \frac{x'(s)}{f[x(s)]}ds = R'(T)a(T)\int_{x(T)}^{x(\xi)} \frac{dy}{f(y)} \\ &\leq R'(T)a(T)\int_{x(T)}^{\infty} \frac{dy}{f(y)}\end{aligned}$$

Hence, for all $t \geq T$

$$\int_{t}^{\infty} Z(s)ds \leq R(t)\frac{a(t)x'(t)}{f[x(t)]} + R'(t)a(t)M$$

where $M = \int_{x(T)}^{\infty} \frac{dy}{f(y)}$. So, for every $t \geq T$

$$\begin{aligned}\frac{1}{R(t)}\int_{T}^{t} \frac{1}{a(s)}\int_{s}^{\infty} Z(u)duds &\leq \frac{1}{R(t)}\int_{T}^{t} R(s)\frac{x'(s)}{f[x(s)]}ds + \frac{M}{R(t)}\int_{T}^{t} R'(s)ds \\ &\leq \int_{T}^{t} \frac{x'(s)}{f[x(s)]}ds + M \leq \int_{x(T)}^{\infty} \frac{dy}{f[y]} + M < \infty\end{aligned}$$

This contradicts condition (14).

Case 2: $x'(t)$ is oscillatory. Then there exists a sequence $(\alpha_n) \to \infty$ in $[T, \infty)$ such that $x'(\alpha_n) = 0$. If we choose N large enough so that

$$\lim_{t\to\infty} \inf \int_{\alpha_N}^{t} Z(s)ds \geq 0.$$

If we assume that α_N and α_{N+1} are two consecutive points in $[T, \infty)$ where $x'(t) < 0$ for $t \in]\alpha_N, \alpha_{N+1}[$. Then from (8) we have

$$0 \leq \int_{\alpha_N}^{\alpha_{N+1}} Z(s)ds \leq \int_{\alpha_N}^{\alpha_{n+1}} R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds < 0$$

which contradicts the fact that $x'(t)$ oscillates.

case 3 Because of condition (5), we can get a contradiction by the procedure of the proof of our theorem. ∎

Theorem 3. *Suppose (2),(4) hold and assume that there are constants K , a_1 such that*

$$f'(y) \geq K > 0 \tag{16}$$

and

$$\lim_{t\to\infty} \left[\int_T^t \frac{1}{R(s)}ds\right]^{-1} \int_T^t \frac{1}{R(s)} \int_T^s Z(u)du.ds = \infty \tag{18}$$

$$\lim_{t\to\infty} \int_T^t \frac{1}{sR(s)}ds = \infty, \tag{19}$$

$$a(t) \leq a_1 \tag{17}$$

$$\int_T^\infty \frac{R'^2(s)}{R^2(s)}ds < \infty, \tag{20}$$

Then (1) is oscillatory

Proof. Let x be a nonoscillatory solution on an interval $[T, \infty)$, of the differentiable equation (1) .Without loss of generality, this solution can be supposed such that $x(t) > 0$.for all $t \geq T$.(the case $x(t) < 0$ can be treated similarly and will be omitted).

From (7) we have

$$\frac{R(t)a(t)x'(t)}{f[x(t)]} + \int_T^t Z(s)ds - \int_T^t R'(s)\frac{a(s)x'(s)}{f[x(s)]}ds + \int_T^t KR(s)\frac{a(s)[x'(s)]^2}{(fx(s)])^2}ds \leq C_0 \tag{21}$$

difining for every $t \geq T$

$$w(t) = \frac{a(t)x'(t)}{f[x(t)]} > 0$$

$$g(t) = \left\{ \int_T^t \frac{ds}{R(s)} \right\}^{-1}$$

Therefore, for every $t \geq T$ we have

$$g(t) \int_T^t w(s)ds + g(t) \int_T^t \frac{1}{R(s)} \int_T^s Z(u)duds +$$
$$g(t) \int_T^t \frac{1}{R(s)} \int_T^s \{[\alpha(u)w(u) - \beta(u)]^2 - \beta^2(u)\} duds \leq C_0$$

where

$$\alpha^2(t) = \frac{KR(t)}{a(t)} \quad \text{and} \quad \beta^2(t) = \frac{1}{4K^2}\frac{R'^2(t)}{R^2(t)}a^2(t)$$

Let $y(t) = \alpha(t)w(t) - \beta(t)$ then we have

$$g(t) \left\{ \int_T^t \frac{y(s)}{\alpha(s)}ds + \int_T^t \frac{1}{R(s)} \int_T^s Z(u)du + \int_T^t \frac{1}{R(s)} \int_T^s y(u)^2 duds \right\}$$
$$\leq C_0 + g(t) \int_T^t \frac{1}{R(s)} \int_T^s \beta^2(u)duds$$

Now, from (19) the integral $\int_T^\infty \frac{R'^2(s)}{R^2(s)}ds$ is finite. In this case, and by conditions (18) and (20) we obtain

$$\lim_{t\to\infty} \left\{ g(t) \int_T^t \frac{y(s)}{\alpha(s)}ds + g(t) \int_T^t \frac{1}{R(s)} \int_T^s (y(u))^2 duds \right\} = -\infty$$

Defining

$$H(t) = \left| \int_T^t \frac{y(s)}{\alpha(s)}ds \right|$$

we may use the Schwarz inequality to obtain

$$H^2(t) \leq t \int_T^t \left[\frac{y(s)}{\alpha(s)}\right]^2 ds$$

since R is positive on $[T,\infty)$ and R' is nonnegative above $[T,\infty)$ and from (17) we have

$$H^{\ 2}(t) \leq C_1 t \int_T^t y^{\ 2}(s)ds$$

where $C_1 = \frac{a_1}{KR(T)}$. Thus, for $t \geq T$

$$-H(t)g(t) + g(t)C_1 \int_T^t \frac{H^{\ 2}(s)}{sR(s)}ds \leq g(t) \int_T^t \frac{y(s)}{\alpha(s)}ds + g(t) \int_T^t \frac{1}{R(s)} \int_T^s y^2(u)du.ds$$
$$\leq 0$$

It follows that

$$H^2(t) \geq C_1^2 \left[\int_T^t \frac{H^2(s)}{sR(s)}ds \right]^2 \quad \text{for all} \quad t \geq T$$

then

$$\frac{C_1^2}{tR(t)} \le \frac{\Psi'(t)}{\Psi^2(t)} \quad \text{for all} \quad t \ge T.$$

Where

$$\Psi(t) = \int_T^t \frac{H^2(s)}{sR(s)} ds \quad \text{for all} \;\; t \ge T$$

So for any $t \ge T$

$$C_1^2 \int_T^t \frac{1}{sR(s)} ds \le \int_T^t \frac{\Psi'(s)}{\Psi^2(s)} ds = \frac{1}{\Psi(T)} - \frac{1}{\Psi(t)} \le \frac{1}{\Psi(T)} < \infty$$

This contradicts condition (19).This completes the proof of the theorem. ∎

Theorem 4. *Let conditions (2),(4),(16), and (17) hold . If*

$$\liminf_{t\to\infty} \int_T^t Z(s)ds \; > -\lambda \; (\lambda > 0) \quad \textit{for all large } T \tag{23}$$

and

$$\limsup_{t\to\infty} \left[\int_T^t \frac{1}{R(s)} ds\right]^{-1} \int_T^t \frac{1}{R(s)} \int_T^s Z(u) du ds = \infty \;\; \textit{for all large } T \tag{24}$$

Then (1) is oscillatory.

Proof. Let x be a nonoscillatory solution on an interval $[T, \infty), T \ge T_0$ of the differentiable equation (1) .Without loss of generality, this solution can be supposed such that $x(t) > 0$. We consider the following three cases for the behavior of $x'(t)$.

Case 1. $x'(t)$ is oscillatory. Then there exists a sequence $(\alpha_n)_{n \in N} \to \infty$ in $[T, \infty)$ such that $x'(\alpha_n) = 0$

By (23) we conclude that for some positive constant M we have

$$\int_T^t \left\{ -R'(s) \frac{a(s)x'(s)}{f[x(s)]} ds + K.R(s) \frac{a(s)x'^2(s)}{f^2[x(s)]} \right\} ds \le M \quad \text{for every} \quad t \ge T$$

which give

$$\int_T^t \left[\alpha(s) \frac{a(s)x'(s)}{f[x(s)} - \beta(s)\right]^2 ds < N \tag{25}$$

where

$$N = M + \frac{a^2{}_1}{4K^2} \int_T^{\alpha_n} \frac{R'^2(s)}{R^2(s)} ds < \infty$$

By the Schwarz inequality , for $t \geq T$,we obtain

$$\left\{\int_T^t \frac{y(s)}{\alpha(s)}ds\right\}^2 \leq N\frac{a_1^2}{R(T)}\int_T^t \frac{1}{R(s)}ds \leq C^2[g(t)]^{-1}$$

This implies

$$\int_T^t \frac{y(s)}{\alpha(s)}ds \geq -\frac{C}{\sqrt{g(t)}} \text{ for all } t \geq T$$

and from (22) we have

$$g(t)\int_T^t \frac{1}{R(s)}\int_T^s Z(u)du \leq C_1 + C\sqrt{g(t)} \text{ for all } \ t \geq T$$

Since R is positive on $[T,\infty)$ and R' is nonnegative and bounded above on $[T,\infty)$, it follows that

$$R(t) \leq \eta t \quad \text{ for all large } \quad t$$

where $\eta > 0$ is a constant. This ensures that

$$\int_T^\infty \frac{ds}{R(s)} = \infty \tag{26}$$

Thus we have

$$\limsup_{t\to\infty} g(t)\int_T^t \frac{1}{R(s)}\int_T^s Z(u)du \leq C_1$$

This contradicts condition (24).

case 2 $x'(t) > 0$ for some $\mathrm{T}_1 \geq T$. From (21) it follows that for $t \geq T_1$

$$\int_T^t \left\{KR(s)\frac{a(s)x'^2(s)}{f^2[x(s)]} - R'(s)\frac{a(s)x'(s)}{f[x(s)]}\right\}ds \leq C_0 + \lambda$$

and we can proceed as above.

case 3 $x'(t) < 0$. From (21) it follows that

$$\begin{aligned}\frac{R(t)a(t)x'(t)}{f[x(t)]} \quad &\leq \quad C_0 - \int_T^t Z(s)ds \\ &\quad - \int_T^t \left\{R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'(x(s) - R'(s)\frac{a(s)x'(s)}{f[x(s)]}\right\}ds\end{aligned} \tag{27}$$

we distinguish two mutually exclusive cases where

$-\int_T^\infty \left\{R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'(x(s) - R'(s)\frac{a(s)x'(s)}{f[x(s)]}\right\}ds$ is finite or infinite

i) If $-\int_T^\infty \left\{R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2}f'(x(s) - R'(s)\frac{a(s)x'(s)}{f[x(s)]}\right\}ds$ is finite. In this case, from(10) it follows that (25) holds for $t \geq T$. Once again,we can complete the proof by the procedure of the proof of case 1

ii) If $-\int_T^\infty \left\{ R(s)\frac{a(s)[x'(s)]^2}{(f[x(s)])^2} f'(x(s) - R'(s)\frac{a(s)x'(s)}{f[x(s)]} \right\} ds$ is infinite. By Condition (23), and from (26) it follows that there exists a constant μ such that

$$-\frac{R(t)a(t)x'(t)}{f[x(t)]}$$

$$\geq \mu + \int_T^t \left(-\frac{R'(s)}{R(s)} + \frac{x'(s)f'(x(s))}{f[x(s)]} \right) \left(\frac{R(s)a(s)x'(s)}{f[x(s)]} \right) ds \;\; \text{for all } t \geq T$$

Put

$$G(t) = -\frac{R'(t)}{R(t)} + \frac{x'(t)f'(x(t))}{f[x(t)]} \leq 0$$

Furthermore, we choose a $T_1 \geq T$ so that

$$\mu + \int_T^{T_1} G(s) \left(\frac{R(s)a(s)x'(s)}{f[x(s)]} \right) ds = \mu_1 > 0$$

and then for every $t \geq T_1$ we have

$$\frac{R(t)a(t)x'(t)}{f[x(t)]} [G(t)] \left[\mu + \int_T^t G(s) \left(\frac{R(s)a(s)x'(s)}{f[x(s)]} \right) ds \right]^{-1} \geq -G(t)$$

and integrating from $T_1$to t, we obtain

$$Log \frac{\left[\mu + \int_T^t G(s) \left(\frac{R(s)a(s)x'(s)}{f[x(s)]} \right) ds \right]}{\mu_1} \geq Log \frac{R(t)f(x(T)}{R(T)f(x(t)}$$

Thus

$$\mu + \int_T^t (G(s)) \left(\frac{R(s)a(s)x'(s)}{f[x(s)]} \right) ds \geq \mu_1 \frac{R(t)f(x(T)}{R(T)f(x(t)}$$

The last inequality implies for $t \geq T_1$

$$x'(t) \leq -\eta \frac{1}{a(t)}$$

where $\eta = \frac{\mu_1 + f(x(T))}{R(T)} > 0$. And consequently for $t \geq T_1$

$$x(t) \leq x(T_1) - \eta \int_{T_1}^t \frac{1}{a(s)} ds \leq -\eta \frac{1}{a_1}(t - T_1)$$

Therefore, we conclude that $\lim_{t\to\infty} x(t) = -\infty$. This contradict the assumption that $x(t) > 0$.This completes the proof of the theorem. ∎

Theorem 5. *Suppose that conditions (2),(3),(4),(6) and (15) hold, and* $a(t) = 1$ *then all solutions of (1) are oscillatory.*

Proof. Let x be a nonoscillatory solution on an interval $[T,\infty), T \geq T_0$ of the differentiable equation (1). Without loss of generality, this solution can be supposed such that $x(t) > 0$.(the case $x(t) < 0$ can be treated similarly and will be omitted).

From (8) we have

$$\frac{R(t)x'(t)}{f[x(t)]} \leq C_0 - \int_T^t Z(s).ds + \int_T^t R'(s)\frac{x'(s)}{f[x(s)]}ds$$

By the Bonnet theorem, for a fixed $t \geq T$ and some $\xi \in [T,t]$ we have

$$\int_T^t R'(s)\frac{x'(s)}{f[x(s)]}ds = R'(T)\int_T^\xi \frac{x'(s)}{f[x(s)]}ds$$

$$= R'(T)\int_{x(T)}^{x(\xi)} \frac{dy}{f(y)} \leq R'(T)\int_{x(T)}^{\infty} \frac{dy}{f(y)} = C$$

Condition (6) implies that there exists $T_1 \geq T$,such that

$$\frac{R(t)x'(t)}{f[x(t)]} < 0 \text{ for every } t \geq T_1 \tag{28}$$

then (28) says that $x'(t)$ is negative on $[T_1,\infty)$. By condition (6) there exists T_2, $T_2 \geq T_1$ such that

$$\int_{T_1}^{T_2} Z(s) = 0 \text{ and } \int_{T_2}^{t} Z(s) \geq 0$$

Multiplying (1) by $R(t)$ and integrating by parts we obtain

$$\begin{aligned} R(t)x'(t) &\leq R(T_2)x'(T_2) + \int_{T_2}^t R'(s)x'(s)ds - \int_{T_2}^t f[x(s)]Z(s)ds \\ &\leq R(T_2)x'(T_2) - f(x(t)\int_{T_2}^t Z(s)ds + \int_{T_2}^t x'(s)f'(x(s)\int_{T_2}^t Z(u)du.ds \\ &\leq R(T_2)x'(T_2) \quad \text{for every } \; t \; \geq T_1 \end{aligned}$$

Thus

$$x(t) \leq R(T_2)x'(T_2)\int_{T_2}^t \frac{1}{R(s)}ds$$

Therefore, we conclude from (26) that $\lim_{t\to\infty} x(t) = -\infty$,a contradiction to positivity of $x(t)$. This completes the proof of the theorem. ∎

Remark 1. *If we let $R(t) = 1$ in our results we obtain respectively theorems 1,2,4, and 5 of J.R.Greaf, S.M.Rankin and P.W. Spikes [8].*

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