Interior Gradient Estimates for Solutions to the Mean Curvature Equation

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Abstract
For a solution $u$ to the equation of prescribed mean curvature, we impose suitable conditions on $u$ so as to explicitly determine the gradient estimates at an interior point in terms merely of the prescribed mean curvature and the distance from this point to the boundary. We emphasize that a distinction between these results and the usual estimates for solutions of equations of prescribed mean curvature is in that no global bound is imposed on $|u|$.

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Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Let $H(x, u(x))$ be a given Lipschitz-continuous function in $\mathbb{R}^n$. We consider solutions to the equation of surfaces of prescribed mean curvature

$$\text{div} \, Tu = nH(x, u(x)) \quad \text{in} \quad \Omega,$$

where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}},$$

with $\nu$ being the outward pointing unit normal of $\partial \Omega$.

The main purpose of this paper is to impose suitable conditions so as to explicitly determine gradient estimates for solutions $u$ at an interior point in terms merely of the mean curvature $H$ and the distance from this point to the
boundary. In case $H$ is constant, in \cite{3}, \cite{17}, \cite{18}, \cite{19}, such gradient estimates are proved to exist in a subset of $\Omega$ which includes a ball whose radius $R$ is greater than a number $R_n$ depending only on $H$ and $n$, $0 < R_n < \frac{1}{H}$. The value of $R_n$ was calculated explicitly in a manner indicated in those works. For $H$ which is not constant and satisfies some monotonicity conditions, \cite{4} shows that such a type of gradient estimates exists for any domain without any restriction on its size. In Theorem 4.2 of \cite{9}, such gradient estimates are shown to exist for a subset of so called “extremal domains” for Lipschitz-continuous $H$; the size of this subset was not explicitly estimated. We note that in all these works no feasible ways have ever been indicated to explicitly obtain gradient estimates which were shown to exist.

We recall that \cite{1}, \cite{16} (or later \cite{24}, \cite{25}) obtained for any solutions of the equation (1) and any point $y' \in \Omega$ the estimate

$$|Du(y')| \leq C_1 \exp\{C_2 \sup_{\Omega} \frac{(u - u(y'))}{d}\},$$

where $d = \text{dist}(y', \partial \Omega)$ and where $C_1 = C_1(n, d \sup_{\Omega} |H|, d^2 \sup_{\Omega} |DH|)$, $C_2 = C_2(n, d \sup_{\Omega} |H|, d^2 \sup_{\Omega} |DH|)$. In contrast to this result and some others in \cite{5} \cite{6} \cite{7} \cite{22}, our new results require no global bound imposed on $|u|$.

The main part of this work is devoted to proving the following results.

**Main Theorem I**  Let $u$ be a solution of equation (1) in a domain $\Omega \subset \mathbb{R}^n$. Suppose that $H(x, u(x)) = H(x) > 1$ in the ball $B_R$, $B_R \subset \Omega$, centered at some point $\tilde{x}$ and of radius $R$. Let $R_0 = \text{dist}(\tilde{x}, \partial \Omega)$.

Suppose

$$\Delta u \leq nH(\sqrt{1 + |Du|^2})^3 \text{ in } B_{R_0}.$$  \hspace{1cm} (2)

Let $x_0 \in B_{\frac{R_0}{2}}$ be a point at which $u(x_0) = \sup_{B_{\frac{R_0}{2}}} u$. Suppose in the hemisphere of $B_{R_0}$ with its boundary orthogonal to the radius through $x_0$, we have

$$|D^2u(x)| \leq \Lambda |Du(x)|^3 H,$$  \hspace{1cm} (3)

for some constant $\Lambda$ depending only on $n$; moreover, at each point $x$ in this closed hemisphere the gradient vector $Du(x)$ makes an angle less than $\frac{\pi}{2}$ with the radius through $x_0$. Setting

$$\beta_R = \sup_{B_R} |Du| \text{ and } \beta_R = \inf_{B_R} |Du|,$$

suppose

$$\beta_{B_{\frac{R_0}{2}}} \neq 0 \text{ and } \beta_{R_0} \geq (R_0) \frac{1 - (1/2n)}{\frac{3}{2} - (1/2n)};$$  \hspace{1cm} (4)
Furthermore, suppose there exists a constant $\mu$, $\frac{1}{2} < \mu < 1$, together with two constants $\Lambda_1$ and $\Lambda_2$ which are invariant under rescaling and determined completely by $n$, $R_0$ and $H$, with $H = H(\bar{x})$, such that

$$\frac{\bar{\beta}_{R_0/2}}{\beta_{R_0/2}} \geq \Lambda_1 \beta_{R_0/2},$$

(5)

and

$$\frac{\bar{\beta}_{R_0/2}}{\beta_{R_0/2}} (HR_0)^\mu \leq \Lambda_2 (\bar{\beta}_{R_0/2})^{1-\mu}.$$

(6)

Then, setting $H_{R_0}^- = \inf_{B_{R_0}} H(x)$, we have

$$\left(\frac{\bar{\beta}_{R_0/2}}{H_{R_0}^- R_0}\right)^2 \leq \frac{2\Theta}{(H_{R_0}^- R_0)},$$

(7)

with

$$\Theta \leq \begin{cases} \left(2^{7-(4q_\mu-6)\mu}C^*\frac{C^*}{(\Lambda_1)^2q_\mu-3}(\bar{\beta}_{R_0/2})^3(H_{R_0}^-)^{q_\mu+1}(R_0)^{q_\mu+1}, \right. & \text{if } 2(\bar{\beta}_{R_0/2})R_0 \leq 1, \\ \left. \frac{(2q_\mu-6)\mu C^*}{C^*} \frac{(\Lambda_2)^4q_\mu-6}{(\Lambda_1)^2q_\mu-3}(\bar{\beta}_{R_0/2})^3(H_{R_0}^-)^{q_\mu+1}(R_0)^{q_\mu+1}, \right) & \text{if } 2(\bar{\beta}_{R_0/2})R_0 > 1. \end{cases}$$

(8)

where $q_\mu$ is chosen to be so large that

$$(2q_\mu - 3)\mu \geq q_\mu + 1,$$

(9)

and where $C^*$ and $C^{**}$ are positive constants depending merely on $n$. In particular, if $\beta_{R_0/2} \geq 1$, then

$$\Theta \leq \max\left(\frac{(2)^{7-(4q_\mu-6)\mu}C^* (\Lambda_2)^4q_\mu-3}{C^{**} \Lambda_1} (H_{R_0}^-)^{q_\mu+1}(R_0)^{q_\mu+1}, \frac{(2)^{(4q_\mu-6)\mu}C^* (\Lambda_2)^{C^{**}q_\mu-6}}{(\Lambda_1)^2q_\mu-3}(H_{R_0}^-)^{q_\mu+1}(R_0)^{q_\mu+1}\right).$$

In Main Theorem I and throughout this paper, we shall denote by $B_s$, $s > 0$, a ball centered at a point $\bar{x}$ in $\Omega$, ($\bar{x}$ being fixed throughout this paper). And we denote by $B_s(x)$, $s > 0$, $x \in \Omega$, a ball centered at $x$ and of radius $s$.

In Section 2, we will give a proof of Main Theorem which closely parallels that devised in Section 2.2 of Kontrat’ev-Landis[14] for uniformly elliptic equations in divergence form, with important and indispensable modifications made in each stage essentially for handling difficulties caused by nonlinearity of the equation (1). In Step 1, the condition (2) on $\Delta u$ is derived, which facilitates a Moser-type iteration in Step 4 via an integral inequality (38) derived in Step 2. A suitable choice of the function undergoing Moser-type iteration is made in Step 5, through which estimates on $\sup_{B_{R/2}} u$, $0 < R \leq R_0$ is reduced to that of its $L^1$-norm over $B_{R/2}$. The modified version of Poincaré’s inequality
given in Section 1 further reduces the estimate of \( \sup_{B_R} u \) into that of the \( L^1 \)-norm of \( |Du| \) over \( B_R \). An estimate of the \( L^1 \)-norm of \( |Du| \) is given in Step 6, with the assumption that \( \beta_R \equiv \inf_{B_R} |Du| \neq 0 \).

A number \( \hat{C} \) shows up due to the nonlinearity of the integral inequality (38) inherited from that of the equation (1). In order to make this number \( \hat{C} \) bounded, we restrict the domain where Moser-type iteration carried through to be inside a hemisphere which fulfills the requirement made in Main Theorem I. Some technical and lengthy work are done in Step 7 for estimating this number \( \hat{C} \). Discussions in Step 7.2 will clarify our motivation for imposing those conditions on \( u \) over a hemisphere.

A combination of these results gives us a quadratic inequality \( a(y_*)^2 + b y_* + c \geq 0 \) of the variable

\[
y_* = \frac{\sup_{B_R} u - \inf_{B_R} u}{(\inf_{B_R} |Du|)R},
\]

with \( b < 0 \) and \( c > 0 \); the assumption \( H > 0 \) is crucial for \( b \) to be negative. From this inequality with \( b \) and \( c \) being respectively positive and negative, together with the fact that \( y_* \) is bounded below by the number 2, we finally arrive at estimates of \( |Du| \) under some conditions on the growth of \( \beta_{R_0/2} \) and \( \beta_{R_0/2} \), as formulated in (5) and (6) in Main Theorem I. A proof of Main Theorem I is completed in the end of Step 8.

For Main Theorem I to be of use, we shall find out some useful criteria to identify situations where the conditions listed in Main Theorem I are fulfilled. The first three Lemmas in the following will be concerned respectively about (2), (3) and the distribution of the direction of the gradient vector, which will be proved respectively in 3.3, 3.4, and 3.5, with considerations given respectively to the case \( n = 2 \). The last four Lemmas on identifying situations where (5) and (6) are fulfilled will be obtained in 3.1 and 3.2. All the results are derived by means of simple techniques in elementary calculus or middle school algebra; however, interesting phenomena are revealed through these elementary considerations.

Firstly, regarding (2), the following result will be shown in 3.3.

**Lemma 0.1** Suppose \( n = 2 \). Let \( u \) and \( R_0 \) be as indicated in Main Theorem I. Let \( S_{R_0} \) be the graph described by \( u|_{B_{R_0}} \). Condition (2) is fulfilled in each of the following situations:

1. the Gauss curvature \( K \) is nonnegative throughout \( S_{R_0} \),
2. at each point \( x \) on \( S_{R_0} \) where \( K \) is negative, denoting the direction of the gradient vector \( Du(x) \) as that the positive \( y \)-axis, we have \( u_{yy} < 0 \),
3. at each point \( x \) on \( S_{R_0} \) where \( K \) is negative and where (2) fails to hold, a coordinate system exists for which the following four inequalities are satisfied
simultaneously, namely
\[ u_{xx}(x) \geq 0, \quad u_{yy} < 0, \quad (1 - \frac{(u_y)^2}{(u_x)^2}) \frac{1}{1 + (u_y)^2} \leq \frac{1}{2}, \]
and
\[ (1 - \frac{(u_x)^2}{(u_y)^2}) \frac{\sqrt{|K|}}{\sqrt{1 + (u_y)^2}} \leq H. \]

(We may note that the last two conditions are satisfied, in particular if \( u_y \geq 2 \) and \( |K| \leq H^2 \).)

Secondly, as concerns (3), we show in 3.4 the following

**Lemma 0.2** Suppose \( n = 2 \). Let \( u \) and \( R_0 \) be as indicated in Main Theorem I. Let \( S_{R_0}^* \) be the graph of \( u \) on a hemisheare of \( B_R \). Then, condition (3) will be fulfilled in either of the following situations:

(a) the Gauss curvature \( K \) is positive throughout \( S_{R_0}^* \);
(b) at each point at which \( K \) is negative, we have \( |K||Du(x)| \) bounded by a number depending only on \( H \) and \( \text{dist}(x, \partial \Omega) \).

We note that Gauss curvature of the type indicated in (b) of Lemma 0.2 are proved in [11], [13], [14], [21] for the case where \( H = 0 \), that is, for minimal surfaces. For our present setting, we have \( H > 0 \) and the only related result known to the author is in [23], where an estimate for \( |K| \) in terms of \( H \) and \( R_0 \) are obtained; however, an estimate of the type in (b) of Lemma 0.2 cannot hold in general as the simplest case of a sphere indicates. It will be interesting to find out situations where such a type of Gauss curvature estimates realizes.

As of the condition on distribution of the directions of the gradient vector in Main Theorem I, the only situation verified to fulfill it is that where \( K > 0 \) throughout the graph \( S_{R_0}^* \) introduced in Lemma 0.2. Indeed, the following result will be shown in 3.5

**Lemma 0.3** Suppose \( n = 2 \) and \( K > 0 \) throughout the graph of \( u|_{\Omega} \) over a subset \( \Omega^* \) of \( \Omega \), \( u \) being a solution to (1) with \( H(x, u(x)) > 0 \) in \( \Omega \). If \( |Du| \neq 0 \) in \( \Omega^* \), then the set \( \{ \frac{Du}{|Du|}(x) : x \in \Omega^* \} \) is included in a half-space of the \( \mathbb{R}^2 \) domain.

As of (5), the following result will be shown in 3.1.

**Lemma 0.4** Suppose, for \( \gamma \geq 0 \), there exists a constant \( \Lambda_\gamma \), invariant under rescaling and determined completely completely by \( n \) and \( H \), such that
\[ |D^2u(x)| \geq \Lambda_\gamma H_R |Du(x)|^{1+\gamma}, \] (10)
for each point \( x \in B_R \) and for some \( R \leq \text{dist}(x, \partial \Omega) \). Then, for some constant \( C_\gamma^* \) depending only on \( n, H \) and \( \gamma \), we have

\[
\frac{\tilde{\beta}_R}{\beta_R} > (2)^{C_\gamma^* R (H_R^-)} (\beta_R)^\gamma,
\]

if we set \( \tilde{\beta}_R = \sup_{B_R} |Du|, \beta_R = \inf_{B_R} |Du|, \) and \( H_R^- = \inf_{B_R} H \). Therefore, in this case (5) holds with

\[
\Lambda_1 = C_\gamma^* (H_R^-) \frac{2R}{R_0}.
\]  

(11)

As of (6), we first derive from (9) in 3.1.1 the following

**Lemma 0.5** Setting

\[
C_\gamma = \left(1 - \left(\frac{1}{2}\right)^\gamma\right) \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j,
\]

and \( H^+_R = \sup_{B_R} H(x) \), we have

\[
|Du(x_1)| \leq \left(\frac{C_\gamma}{C_\gamma^* (H_R^+)} R_1\right)^\gamma,
\]

in case (8) holds at some point with distance from the point \( x_1 \) no less than \( R_1 \).

From Lemma 0.5, we obtain in 3.1.2 the following

**Lemma 0.6** For \( \frac{1}{2} < \mu < 1 \), we let \( S_\mu \) be the subset of \( B_{R_0} \) at each point in which (10) holds with some \( \gamma \geq (2 - \mu) \) and some constant \( \Lambda_\mu \) determined completely by \( n \) and \( H \). We assume that either the set \( S_\mu \) is empty or some component of \( S_\mu \) meets the boundary \( \partial B_{R_0} \) of \( B_{R_0} \), with \( B_{R_0} \setminus S_\mu \) including the center \( \tilde{x} \) of the ball \( B_{R_0} \) and being star-shaped. Then (6) holds with some \( \Lambda_2 \).

We notice that in the previous three Lemmas, the function \( u \) is not required to satisfy (1). However, \( u \) is so required in the next result derived in 3.2. This result suggests us some ways to find out situations where (9) and the assumptions made in Lemma 6 on \( u \) are fulfilled.

**Lemma 0.7** Suppose \( u \) satisfies (1) with \( H > 0 \) at some point \( x \). Suppose \( u_{xx} \geq 0 \). Then

\[
u_{xx} \geq \frac{\sqrt{1 + |Du|^2 H} - \sqrt{(1 + |Du|^2)^3 H^2 - (1 + (u_y)^2)(1 + (u_x)^2)(1 + |Du|^2)K}}{1 + (u_y)^2}.
\]
If we suppose that
\[ K = \frac{u_{xx}u_{yy} - (u_{xy})^2}{((1 + |Du|^2))^2} > 0 \]
at \( x \), then
\[ u_{xx} \geq \frac{1 + (u_x)^2}{1 + (u_y)^2} (1 + |Du|^2) \sqrt{|K|}. \] (14)

We finally arrive at

**Main Theorem II**  Suppose \( n = 2 \) and \( u \) satisfies (1) with \( H(x, u(x)) = \text{constant} > 1 \) in \( \Omega \). For \( R_0 \) as before, suppose (4) holds; furthermore, suppose that over a hemisphere \( B_{R_0}^* \) of \( B_{R_0} \) and its complement \( B_{R_0} \setminus B_{R_0}^* \), the graph of \( u \) has respectively nonnegative and nonpositive Gauss curvature \( K \). Furthermore, suppose that \( K \) is bounded away from zero in some connected subset of \( B_{R_0/2} \cap B_{R_0}^* \) with nonempty interior, with star-shaped complement and meeting \( \partial B_{R_0/2} \). Then the requirements on \( u \) made in Main Theorem I are all fulfilled. Thus, the estimate on \( |Du| \) given in Main Theorem is satisfied in such a situation.

1  A Modified Version of Poincaré’s Inequality

In the next section, we shall, motivated by Section 2.2 of [14], divide the proof of the First Growth Lemma into nine steps. In Step 5 of [14], a so-called *generalized Friedrich’s inequality* is applied without being given a proof. Accordingly, we shall place a special emphasis on a derivation of a modified version of the *Poincaré’s inequality* in its precise formulation below. We note that Step 5 and Step 7.1 in Section 2 are where we indicate the manner in which our derivation of Main Theorem I relies on the ratio of the measure of level sets to the measure of the whole ball.

We remark that an inequality of this type is indicated to hold, for example, in Section 1.1.11 of [20] and Lemma 4.4.4 of [26], for a class of domains with much less restrictions than that of convexity imposed below. However, in results in [20] and [26] the constant we are concerned about is not given explicitly.

**Lemma 1.1 (a modified version of Poincaré’s inequality)** Suppose \( w \in W^{1,p}(\Omega) \) for some \( p \geq 1 \) and convex \( \Omega \), with
\[ |\{x : x \in \Omega, \ w(x) \leq 0\}| \geq \alpha_1|\Omega| \] (15)

If \( p > 1 \), then
\[ \|w\|_p \leq (1 - (1 - \alpha_1)^{\frac{p-1}{p}})^{-1} \left( \frac{\omega_p}{|\Omega|} \right)^{\frac{1}{p}} (\text{diam } \Omega)^n \|Dw\|_p. \] (16)
If \( p = 1 \) and if we have, in addition,
\[
| \{ x : x \in \Omega, w(x) \leq 0 \} | \geq \alpha_2|\Omega|,
\]
then
\[
\| w \|_1 \leq \max(\alpha_1, \alpha_2) \left( \left( \frac{1}{\alpha_1} \right)^{1-\frac{1}{p}} + \left( \frac{1}{\alpha_2} \right)^{1-\frac{1}{p}} \right) \left( \frac{\omega_n}{|\Omega|} \right)^{1-\frac{1}{p}} (\text{diam } \Omega)^n \| Dw \|_1. \tag{17}
\]

Proof of Lemma 1.1. Let us set
\[
C_{0,\Omega} = \left( \frac{\omega_n}{|\Omega|} \right)^{1-\frac{1}{p}} (\text{diam } \Omega)^n,
\]
and
\[
w_A = \frac{1}{|A|} \int_A w \, dx,
\]
for measurable subsets \( A \) of \( \Omega \). The well known version of Poincaré inequality (cf. [6], page 164) says that, for \( p \geq 1 \) and \( A \) convex,
\[
\| w - w_A \|_p \leq C_{0,\Omega} \| Dw \|_p.
\]
Hence
\[
\| w \|_p \leq \| w - w_A \|_p + \| w_A \|_p \tag{18}
\]
\[
\leq C_{0,\Omega} \| Dw \|_p + \| w_A \|_p.
\]
Let us set
\[
E^+ = \{ x : x \in \Omega, w(x) > 0 \}, \quad \text{and} \quad E^- = \{ x : x \in \Omega, w(x) \leq 0 \}.
\]
In case \( p > 1 \), Hölder continuity and (15) yield
\[
\| w_A \|_p = \frac{1}{|\Omega|^{p-1}} (\int_{E^+} w \, dx)^p \leq \frac{1}{|\Omega|^{p-1}} (\int_{E^+} w \, dx)^p
\]
\[
\leq \frac{|E^+|^{p-1}}{|\Omega|^{p-1}} (\int_{E^+} w^p \, dx) \leq (1 - \alpha_1)^{1-p} \int_{E^+} w^p \, dx,
\]
which and (18) imply (16).

For \( p = 1 \), we have
\[
\int_{\Omega} (w_{E^+} - w) \, dx + \int_{\Omega} (w - w_{E^-}) \, dx
\]
\[
= \int_{\Omega} w_{E^+} \, dx - \int_{\Omega} w_{E^-} \, dx
\]
\[
= \frac{|\Omega|}{|E^+|} \int_{E^+} w \, dx - \frac{|\Omega|}{|E^-|} \int_{E^-} w \, dx
\]
\[
\geq \min(\frac{|\Omega|}{|E^+|}, \frac{|\Omega|}{|E^-|}) (\int_{E^+} w \, dx - \int_{E^-} w \, dx)
\]
\[
= \min(\frac{|\Omega|}{|E^+|}, \frac{|\Omega|}{|E^-|}) \int_{\Omega} w \, dx.
\]
On the other hand
\[
\int_{\Omega} (w_{E^+} - w) dx + \int_{\Omega} (w - w_{E^-}) dx \\
\leq \int_{E^+} (w_{E^+} - w) dx + \int_{E^-} (w - w_{E^-}) dx \\
\leq \int_{E^+} |w_{E^+} - w| dx + \int_{E^-} |w - w_{E^-}| dx.
\]

Although \(E^+\) and \(E^-\) are not necessarily convex, \(u\) is defined on the convex set \(\Omega\) containing \(E^+\) and \(E^-\). For the respective convex hulls \(\tilde{E}^+\) and \(\tilde{E}^-\) of \(E^+\) and \(E^-\), we have
\[
\Omega \supset \tilde{E}^+ \supset E^+ \quad \text{and} \quad \Omega \supset \tilde{E}^- \supset E^-.
\]

And then the proof of Lemma 7.16 of [6] yield that
\[
|w(x) - w_{E^+}(x)| \leq \frac{(\text{diam } \tilde{E}^+)^n}{n|E^+|} \int_{\tilde{E}^+} |x - y|^{1-n}|Dw(y)| dy,
\]
and
\[
|w(x) - w_{E^-}(x)| \leq \frac{(\text{diam } \tilde{E}^-)^n}{n|E^-|} \int_{\tilde{E}^-} |x - y|^{1-n}|Dw(y)| dy,
\]
which yield
\[
\int_{E^+} |w(x) - w_{E^+}(x)| dx \leq \left(\frac{\omega_n}{|E^+|}\right)^{1-\frac{n}{2}} (\text{diam } \Omega)^n \int_{\tilde{E}^+} |Dw| dx
\]
and
\[
\int_{E^-} |w_{E^-}(x) - w(x)| dx \leq \left(\frac{\omega_n}{|E^-|}\right)^{1-\frac{n}{2}} (\text{diam } \Omega)^n \int_{\tilde{E}^-} |Dw| dx.
\]

Inserting the last two inequalities into (20) and then using (19), we readily obtain (17). ♦

## 2 A Proof of Main Theorem I

Set, as before,
\[
Tw = \frac{|Dw|}{\sqrt{1 + |Dw|^2}},
\]
for \(w \in C^2(\Omega)\).

**Step 1.** Let \(u\) be a solution of equation (0.1). Let \(f(t)\) be a function in \(C^2(\inf B_R u, \sup B_R u)\) satisfying the conditions
\[
f(t) \geq 1, \quad f'(t) \geq 1, \quad \text{and} \quad f(t)f''(t) \geq (f'(t))^2.
\]
If \( \phi(t), \ t > 0, \) is a twice differentiable function such that
\[
\phi \geq 0, \quad \phi'(t)f(t)f'(t) \geq 1, \quad \text{for} \quad t \in (\inf_{B_R} u, \sup_{B_R} u), \tag{23}
\]
and
\[
\phi'(t) \geq t|\phi''(t)|, \quad \text{for} \quad t \geq 1. \tag{24}
\]
Then, setting
\[
g(x) = \phi'(f(u))f'(u(x)), \tag{25}
\]
and
\[
h(x) = \phi'(f(u))f''(u) + \phi''(f(u))(f'(u))^2, \tag{26}
\]
we have
\[
(\phi(f(u))_x, = g(x)u_x, \tag{27}
\]
\[
(\phi(f(u))_x, = g(x)u_x, \tag{28}
\]
and, by (23) and (24),
\[
g(x) \geq 1, \quad h(x) \geq 0. \tag{29}
\]
Moreover, we have
\[
\text{div} \, T\phi(f(u)) + \frac{g^3 \Delta u - \Delta (\phi(f(u)))}{(1 + g^2 |Du|^2)^\frac{3}{2}} = \frac{g^3(1 + |Du|^2)^\frac{3}{2}}{(1 + g^2 |Du|^2)^\frac{3}{2}} H(x, u(x)). \tag{30}
\]
This identity is verified by direct substitution. In fact, for \( Tw \) defined in (21), we have
\[
\text{div} \, T w = \frac{(1 + |Dw|^2)\Delta w - \sum_{i,j=1}^n D_i w D_j w D_{ij} w}{(1 + |Dw|^2)^\frac{3}{2}}, \tag{31}
\]
Hence (1) can be written equivalently as
\[
nH(x, u(x)) = \frac{(1 + |Du|^2)\Delta u - \sum_{i,j=1}^n D_i u D_j u D_{ij} u}{(1 + |Du|^2)^\frac{3}{2}} \tag{32}
\]
\[
= \frac{1}{(1 + |Du|^2)^\frac{3}{2}} \left( \Delta u - \frac{1}{(1 + |Du|^2)^\frac{3}{2}} \sum_{i,j=1}^n D_i u D_j u D_{ij} u \right).
\]
On the other hand, inserting (27) and (28) into (31), we obtain
\[
\text{div} \, T\phi(f(u)) = \frac{(1 + g^2 |Du|^2)(g \Delta u + h|Du|^2)}{(1 + g^2 |Du|^2)^\frac{3}{2}} \tag{33}
\]
\[
- \frac{g^2 \sum_{i,j=1}^n D_i u D_j u (gD_{ij} u + hD_i u D_j u)}{(1 + g^2 |Du|^2)^\frac{3}{2}} \tag{33}
\]
\[
= \frac{g(1 + g^2 |Du|^2)\Delta u - g^3 \sum_{i,j=1}^n D_i u D_j u D_{ij} u}{(1 + g^2 |Du|^2)^\frac{3}{2}} + \frac{h|Du|^2}{(1 + g^2 |Du|^2)^\frac{3}{2}}\]
\[ \begin{align*}
\frac{(g + g^3 |Du|^2) \Delta u}{(1 + g^2 |Du|^2)^{\frac{3}{2}}} + \frac{h|Du|^2}{(1 + g^2 |Du|^2)^{\frac{3}{2}}} - \frac{g^3}{(1 + g^2 |Du|^2)^{\frac{3}{2}}} \left( (1 + |Du|^2) \Delta u - n(1 + |Du|^2)^{\frac{3}{2}} H \right) \\
= \frac{g - g^3}{(1 + g^2 |Du|^2)^{\frac{3}{2}}} \Delta u + \frac{h|Du|^2}{(1 + g^2 |Du|^2)^{\frac{3}{2}}} + n^2 \frac{g^3(1 + |Du|^2)^{\frac{3}{2}}}{(1 + g^2 |Du|^2)^{\frac{3}{2}}} H(x, u(x)),
\end{align*} \]

where (32) is used to derive the third equality. Using (28), we obtain (30).

Setting
\[ z(x) = \phi(f(u(x))), \]

from (33) and (29), we immediately obtain the following result.

**Lemma 2.1** If \( f \) is defined at a point \( x_0 \in \Omega \) where
\[ H(x_0, u(x_0)) \geq 0 \tag{34} \]
and where (2) holds, then we have
\[ \text{div} Tz(x_0) \geq 0. \tag{35} \]

**Step 2.** Suppose henceforth in this section that (34) and (2) hold throughout \( B_R, 0 < R \leq \text{dist}(\tilde{x}, \partial \Omega) \). Let \( A, A \subset \overline{\Omega} \), be a domain which has nonempty intersection with \( B_{\frac{R}{3}} \) and has
\[ \frac{\partial z}{\partial n} \leq 0, \tag{36} \]
along \( \partial A \cap B_R \), where \( n \) is the unit outward normal with respect to \( A \cap B_R \). Let \( \frac{R}{3} < r_a < r_b < \frac{2R}{3} \). Using (35), we shall show that
\[ \int_{A \cap B_{r_b}} \frac{z \eta^{|Dz|}}{\sqrt{1 + |Dz|^2}} dx \leq \frac{16n^2}{(r_b - r_a)^2} \left( \int_{A \cap B_{r_b}} \frac{z^{|Dz|}}{\sqrt{1 + |Dz|^2}} dx \right). \tag{37} \]

For this, let \( \eta \in C^2(B_R) \) be a cut-off function in \( B_R \) such that
\[ 0 \leq \eta \leq 1, \quad \eta = 1 \text{ in } B_{r_a}, \quad \eta = 0 \text{ in } B_R \setminus B_{r_b}, \]
and
\[ |D\eta| \leq \frac{2}{r_b - r_a}. \]

By (35), we have \( \int_{A \cap B_{r_b}} \eta^2 z^{\frac{1}{n}} \text{div} Tz dx \geq 0 \), which and (36) yield, after an integration by parts,
\[ \frac{1}{n} \int_{A \cap B_{r_b}} \eta^2 z^{\frac{1}{n}} \frac{|Dz|^2}{\sqrt{1 + |Dz|^2}} dx + \int_{A \cap B_{r_b}} 2\eta^{\frac{1}{2}} \frac{(D\eta, Dz)}{\sqrt{1 + |Dz|^2}} dx \leq 0, \]
and hence
\[
\int_{A \cap B_{r_{\eta}}} \eta |D\eta|^2 \sqrt{1 + |D\eta|^2} \, dx \leq 8n^2 \int_{A \cap B_{r_{\eta}}} z \sqrt{1 + |Dz|^2} \, dx,
\]
from which and the choice of the function \(\eta\), (37) follows.

**Step 3.** We shall apply *Sobolev embedding theorem* in the following formulation. Namely, if \(n \geq 2\) and \(v \in W^{1,1}\), then
\[
\|v\|_{L^\infty((n-1)}} \leq C_* \|v\|_{W^{1,1}}, \tag{38}
\]
where \(C_*\) is a constant determined by \(n\). For example, we can take
\[
C_* = \frac{1}{n\sqrt{\pi}} \left( \frac{n! \Gamma(n/2)}{2\Gamma(n)\Gamma(1)} \right)^{\frac{1}{n}} = \frac{1}{n\sqrt{\pi}} \left( \frac{n \Gamma(n/2)}{2\Gamma(n/2)} \right)^{\frac{1}{n}}, \tag{39}
\]
(cf. [6], page 158).

**Step 4. (a Moser-type iteration)** We will perform a Moser-type iteration in Step 4.1, by means of (37) and (38). First of all, we choose in Step 4.0 some nonnegative functions \(\varphi_\ell(t), t > 0, \ell \in \mathbb{N} \cup \{0\}\), which satisfy (24) and, for \(f\) so chosen to satisfy (22) and the inequality (41) below, satisfy also the second condition in (23).

**Step 4.0.** Set
\[
z_0 = f(u(x)),
\]
for some function satisfying the conditions in (22). Then, set
\[
p = \frac{n}{n - 1}
\]
and set
\[
\varphi_\ell(t) = t^{\left(\frac{n}{n + 1}\right)^{\ell + 1}} p^\ell, \quad \ell \in \mathbb{N} \cup \{0\}. \tag{40}
\]
The first condition in (23) is satisfied and we have
\[
(\varphi_\ell)'(t) = \frac{np}{n + 1} t^{\left(\frac{n}{n + 1}\right)^{\ell + 1}} (-1)^{\ell + 1} (\frac{n^2}{n^2 - 1})^{\ell + 1} p^\ell,
\]
and
\[
(\varphi_\ell)''(t) = \left(\frac{n^2}{n^2 - 1}\right)^{\ell + 1} (-1)^{\ell + 1} (\frac{n^2}{n^2 - 1})^{\ell + 1} (\frac{n^2}{n^2 - 1})^{\ell + 1} p^\ell.
\]
Hence
\[
(\varphi_\ell)'(t) > (\varphi_0)'(t), \quad \text{if } \ell > 0
\]
and

$$|(\phi_t)''(t)| < t(\phi_t)'(t), \quad \text{if } t \geq 1.$$ 

Hence, (24) is satisfied, and to satisfy the second condition in (23), it suffices to choose \(f\) in such a way that

$$(\phi_0)'(f(t))f'(t) \geq 1, \quad \text{for } t \geq 1$$

that is

$$\left(\frac{n}{n+1}\right)(f(t))^\frac{n}{n+1}|f'(t)| \geq 1 \quad \text{for } t \geq 1. \quad (41)$$

We will make a choice of such a function in Step 5.1.3 below.

**Step 4.1.** Setting in (38)

$$v = (z_0)^{\left(\frac{n}{n+1}\right)^\ell\ell^p^f^r}, \quad \ell = 0, 1, \ldots,$$

we obtain

$$\int_{A \cap B_{r_0}} (z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1} dx \leq (C_*)^p \left( \int_{A \cap B_{r_0}} \left( \frac{(z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1} |D(z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1}|^2 dx \right)^{\frac{1}{2}} \right)^p$$

$$\leq \frac{4n(C_*)^p}{r_b - r_a} \left( \int_{A \cap B_{r_0}} \left( \frac{(z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1}}{1 + |D(z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1}|^2 dx \right)^{\frac{1}{2}} \right)^p$$

$$+ \left( \int_{A \cap B_{r_0}} (z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1} dx \right)^{\frac{n}{n+1}} \left( \omega_n \frac{n}{n+1} (2r_a)^{\frac{n}{n+1}} \right)^{\frac{n}{n+1}},$$

by (37) and Hölder’s inequality. Hence, setting, for measurable sets \(E \subset B_R\),

$$\hat{C}_{\ell,E}^* = \frac{\int_E (z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1} \left( 1 + |D(z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1}|^2 dx \right)^{\frac{1}{2}}}{\inf_{E} \sqrt{1 + |D(z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1}|^2}}, \quad (42)$$

and

$$\hat{C}_{\ell,E}^{**} = \left( \int_E (z_0)^{\left(\frac{n}{n+1}\right)^{\ell+1}p^f^r^1} dx \right)^{\frac{1}{n+1}}.$$ 

\(\Box\)
we have
\[ \int_{A \cap B_{r_a}} (z_0) \frac{np}{n+1} d^k x \leq (C_*)^p \left[ \left( \frac{4n}{r_b - r_a} \right) \left( \int_{A \cap B_{r_a}} (z_0)^{\frac{np}{n+1}} d^k x \right)^{\frac{1}{2}} (C_{\ell, A \cap B_{r_a}})^{\frac{1}{2}} \right]^p \]
\[ + \left( \int_{A \cap B_{r_a}} (z_0)^{\frac{np}{n+1}} d^k x \right)^{\frac{n}{n+1}} \left( \frac{1}{2} \right) (2r_a)^{\frac{n}{n+1}} \left( \frac{n}{n+1} \right) p \]
\[ \leq (C_*)^p \left( \int_{A \cap B_{r_a}} (z_0)^{\frac{np}{n+1}} d^k x \right)^{\frac{n}{n+1}} \cdot \left[ \left( \frac{4n}{r_b - r_a} \right) \left( C_{\ell, A \cap B_{r_a}}^{**} \right)^{\frac{1}{2}} \left( C_{\ell, A \cap B_{r_a}}^* \right)^{\frac{1}{2}} + \left( \frac{1}{2} \right) (2r_a)^{\frac{n}{n+1}} \right]^p. \]

For \( k \in \mathbb{N} \), we set
\[ R_{(\ell)} = \frac{R}{3} \left( 1 + \frac{1}{\Lambda_0 R (2^{\ell+1})} \right), \quad (44) \]
\[ \ell = k, k-1, \ldots, 0, \quad \text{and set} \]
\[ r_b = R_{(\ell)}, \quad r_a = R_{(\ell+1)} \]
successively. Choose \( A_{\ell} \), \( \ell = k, k-1, \ldots, 0 \), with \( A_1 \supseteq A_2 \supseteq \cdots \supseteq A_k \), such that \( A_{\ell} \cap \partial B_R \) includes the point \( x_0 \) at which
\[ u(x_0) = \sup_{B_{\frac{R}{2}}} u, \]
\( A_{\ell} \cap B_{\frac{R}{2}} \) is nonempty and (36) holds along \( \partial A_{\ell} \cap B_R \), for \( \ell = k, k-1, \ldots, 0 \). We find, with the help of the facts that
\[ \frac{1}{n+1} < \frac{1}{2} < \frac{n}{n+1}, \quad \text{for} \quad n \geq 2, \quad \text{and} \quad z_0 > 1, \]
that, letting
\[ \hat{C}_\ell = \hat{C}_{\ell, A_{\ell} \cap B_{R_{(\ell)}}}, \quad \text{and} \quad \hat{C}_\ell^{**} = \hat{C}_{\ell, A_{\ell} \cap B_{R_{(\ell)}}}^{**}, \quad (45) \]
there holds
\[ \int_{A_{\ell} \cap B_{\frac{R}{2}}} (z_0)^{\frac{np}{n+1} k+1} d^k x \leq 2^k (C_*)^p \sum_{\ell=0}^{k} \left( \int_{B_{\frac{R}{2}}} z_0 d^k x \right)^{\frac{np}{n+1} k+1} \]
\[ \cdot \left[ \left( \frac{48n \Lambda_0 R}{R} \right) \sum_{\ell=0}^{k} \left( \frac{p^{\ell+1}}{2^{(n+1)\ell'}} \left( \hat{C}_{k_\ell}^{**} \hat{C}_{k_{\ell-1}}^* \right)^{ \frac{2}{p^{\ell+1} k+1} } (\hat{C}_{k_{\ell-1}} \hat{C}_{k_{\ell-2}}^*)^{ \frac{2}{p^{\ell+1} k+1} } \cdots (\hat{C}_{k_0} \hat{C}_{k_{-1}}^*)^{ \frac{2}{p^{\ell+1} k+1} } \right) \right]^k \]
\[ + \left( \frac{1}{2} \right) (2r_a)^{\frac{n}{n+1}} \sum_{\ell=0}^{k} \left( \frac{np}{n+1} \right)^k \]
\[ + \frac{48n \Lambda_0 R}{R} \left( \frac{1}{2} \right)^{\frac{n}{n+1}} \sum_{\ell=0}^{k} \left( \frac{p^{\ell+1}}{2^{(n+1)\ell'}} \left( \hat{C}_{k_\ell} \hat{C}_{k_{\ell-1}}^* \right)^{ \frac{2}{p^{\ell+1} k+1} } (\hat{C}_{k_{\ell-1}} \hat{C}_{k_{\ell-2}}^*)^{ \frac{2}{p^{\ell+1} k+1} } \cdots (\hat{C}_{k_0} \hat{C}_{k_{-1}}^*)^{ \frac{2}{p^{\ell+1} k+1} } \right) \]
Hence, if a positive number $\hat{C}$ exists such that
\[
\left(\left(\hat{C}_{k-1}^* \hat{C}_k^*\right)^{\frac{1}{n+1}} \cdots \left(\hat{C}_0^* \hat{C}_1^*\right)^{\frac{1}{n+1}}\right) < \hat{C},
\]
for $k$ sufficiently large, then
\[
\sup_{B_{\frac{2}{3}R}} z_0 \leq (C_*)^{n(n+1)} \left(2\hat{C} + \left(\frac{2R}{3}\right)^n \frac{\omega_n}{n}\right)^{\frac{1}{n+1}} \int_{B_{\frac{2}{3}R}} z_0 \, dx.
\]

**Step 5.** We shall apply the modified version of Poincaré inequality derived in Section 1 to estimate the $L^1$-norm of a function $z$ with a suitably chosen function $f$ satisfying conditions (22) and (26) and hence obtain estimates of the integrals on the right hand side of (47).

**Step 5.1.** Let us set
\[
M = \sup_{B_R} u, \quad m = \inf_{B_R} u,
\]
and
\[
M_* = \sup_{B_{\frac{2}{3}R}} u, \quad m_* = \inf_{B_{\frac{2}{3}R}} u.
\]
Then there exists a number $\tilde{u}$,
\[
m_* \leq \tilde{u} \leq M_*,
\]
for which the sets
\[
E^+ = \{x : x \in B_{\frac{2}{3}R}, u(x) \geq \tilde{u}\} \quad \text{and} \quad E^- = \{x : x \in B_{\frac{2}{3}R}, u(x) \leq \tilde{u}\}
\]
have respectively
\[
|E^+| = \frac{1}{2} |B_{\frac{2}{3}R}| = \frac{1}{2^{n+1}} |B_{\frac{2}{3}R}|,
\]
and
\[
|E^-| = \frac{1}{2} |B_{\frac{2}{3}R}| = \frac{1}{2^{n+1}} |B_{\frac{2}{3}R}|.
\]
Let us set
\[
\tilde{f}_0(t) = \left(\frac{n+1}{n}\right)^{C_0} \left(\frac{n+1}{n}\right)^{-\frac{1}{n+1}} (t - m + \tilde{C})
\]
where $\tilde{C}, \tilde{\hat{C}} \geq 1$, is a constant to be determined in **Step 7.1.2** below, and $C_0$ is so chosen that we have
\[
\left(\frac{n+1}{n}\right)^{C_0} > \left(\frac{1}{\hat{C}}\right)^{\frac{1}{n+1}}.
\]
Then it is easy to see that the conditions (22) and (41) are fulfilled for $f(t) = f_0(t)$. Let us set

$$z_0(x) = f_0(u),$$

and set

$$z^*_0 = z_0 - \left(\frac{n+1}{n}\right)c_0(\frac{n+1}{n}) (\bar{u} - m + \bar{C}).$$

The sets $E^-$ and $E^+$ are included respectively in the sets of points in $B_{\frac{2}{3}R}$ where $z^*_0 \leq 0$ and $z^*_0 \geq 0$.

In view of (50), and Lemma 1.1, we find that

$$\int_{B_{\frac{2}{3}R}} z^*_0 dx \leq (2)^{2n+1+(1-\frac{1}{n})(n+1)} \left(\frac{2R}{3}\right) \int_{B_{\frac{2}{3}R}} |Dz_0| dx,$$

$$\leq (2)^{\frac{2n+1}{n}} \left(\frac{2R}{3}\right) \int_{B_{\frac{2}{3}R}} |Dz_0| dx.$$

This yields

$$\int_{B_{\frac{2}{3}R}} z_0 dx < (2)^{\frac{2n+1}{n}} \left(\frac{2R}{3}\right) \int_{B_{\frac{2}{3}R}} |Dz_0| dx,$$

$$+ \left(\frac{n+1}{n}\right)c_0(\frac{n+1}{n}) \omega_n \left(\frac{2R}{3}\right)^n (\bar{u} - m + \bar{C}).$$

**Step 5.2.** We shall evaluate the integral on the right hand side of (2.43) in Step 6 below. Instead of treating directly the gradient of the function $z_0$, we shall evaluate the $L^1$-norm of the gradient of the function

$$\tilde{z}(x) = \tilde{f}(u(x)),$$

where

$$\tilde{f}(t) = \left(\frac{n+1}{n}\right)c_0(\frac{n+1}{n}) (M - m) \ln(\frac{M - m}{M - t}).$$

It is obvious that

$$|D\tilde{z}| \geq |Dz_0|.$$  

The resultant estimate will depend on $n$, the radius $R$ and the value of

$$C_{1,R} = \frac{M - m}{\beta R R}.$$  

**Step 6. (an estimate of the $L^1$-norm of $|Du|$)** We choose a cut-off function $\tilde{\eta} \in C^2(B_R)$ such that

$$0 \leq \tilde{\eta} \leq 1, \quad \tilde{\eta} = 1 \text{ in } B_{\frac{2}{3}R}, \quad \tilde{\eta} = 0 \text{ in } B_R \setminus B_{\frac{2}{3}R},$$
and
\[ |D\tilde{n}| \leq \frac{12}{R}. \]  

(57)

Note that
\[ \tilde{f}'(t) = \left( \frac{n + 1}{n} \right) c_0 \left( \frac{n + 1}{n} \right) (M - m) \frac{1}{M - t}, \]  

(58)

\[ \tilde{f}''(t) = \left( \frac{n + 1}{n} \right) c_0 \left( \frac{n + 1}{n} \right) (M - m) \frac{1}{(M - t)^2}, \]  

(59)

and
\[ (\tilde{f}'(u)) u_{x_k} = (\tilde{z}_0)_{x_k}, \quad k = 1, \ldots, n. \]  

(60)

From (58) and (59), we have
\[ \tilde{f}''(u) = \frac{1}{\left( \frac{n + 1}{n} \right) c_0 \left( \frac{n + 1}{n} \right)(M - m)} (\tilde{f}'(u))^2 \]  

(61)

Noticing also that
\[ \tilde{f}'(u) \geq \left( \frac{n + 1}{n} \right) c_0 \left( \frac{n + 1}{n} \right) C_{1,R}, \]  

(62)

where \( C_{1,R} \) is given in (55).

We have
\[ \int_{B_{\frac{5}{6}R}} \tilde{n}^2 (\tilde{f}'(u)) \text{div} T\tilde{z} dx \geq \int_{B_{\frac{5}{6}R}} \tilde{n}^2 (\tilde{f}'(u)) H dx, \]

or equivalently,
\[ \int_{B_{\frac{5}{6}R}} \tilde{n}^2 (\tilde{f}''(u)) \frac{\langle Du, D\tilde{z} \rangle}{\sqrt{1 + |D\tilde{z}|^2}} dx + 2 \int_{B_{\frac{5}{6}R}} \tilde{n} (\tilde{f}'(u)) \frac{\langle D\tilde{z}, D\tilde{n} \rangle}{\sqrt{1 + |D\tilde{z}|^2}} dx \]
\[ \leq - \int_{B_{\frac{5}{6}R}} \tilde{n}^2 (\tilde{f}'(u)) H dx. \]

Hence, by (60), (61) and Young’s inequality,
\[ \int_{B_{\frac{5}{6}R}} \tilde{n}^2 (\tilde{f}'(u)) \frac{|D\tilde{z}|^2}{\sqrt{1 + |D\tilde{z}|^2}} dx \leq 8 \left( \frac{n + 1}{n} \right) c_0 \left( \frac{n + 1}{n} \right) (M - m) \]
\[ \cdot \left[ \left( \frac{n + 1}{n} \right) c_0 \left( \frac{n + 1}{n} \right)(M - m) \int_{B_{\frac{5}{6}R}} (\tilde{f}'(u)) \frac{|D\tilde{n}|^2}{\sqrt{1 + |D\tilde{z}|^2}} dx - 2 \int_{B_{\frac{5}{6}R}} \tilde{n}^2 (\tilde{f}'(u)) H dx \right]. \]
From this and (55), we obtain
\[
\int_{B_{\hat{R}}^{n}} \bar{\eta}(\bar{f}'(u)) \frac{|D\bar{z}|^2}{\sqrt{1 + |D\bar{z}|^2}} \, dx
\leq 8 \left( \frac{n + 1}{n} \right)^{2C_0 \left( \frac{n + 1}{n} \right)} (M - m)^2 \int_{B_{\hat{R}}^{n}} \left( \frac{M - m}{M - u} \right) \frac{|D\bar{\eta}|^2}{\sqrt{1 + |D\bar{\eta}|^2}} \, dx
\]
\[
-2 \left( \frac{n + 1}{n} \right) C_0 \left( \frac{n + 1}{n} \right) (M - m) \int_{B_{\hat{R}}^{n}} \bar{\eta}^2(\bar{f}'(u)) H \, dx.
\]
This, together with (55), (56), (57) and (62), yields
\[
\int_{B_{\hat{R}}^{n}} \frac{|D\bar{z}|^2}{\sqrt{1 + |D\bar{z}|^2}} \, dx
\leq 8 \left( \frac{n + 1}{n} \right)^{2C_0 \left( \frac{n + 1}{n} \right)} (M - m) \frac{144\beta_R}{\sqrt{1 + (\beta_R)^2}} \frac{(C_1R)^2}{6} \frac{\omega_n}{\omega_n} \left( \frac{5R}{6} \right)^{n - 2}
\]
\[
-2 \left( \frac{n + 1}{n} \right) C_0 \left( \frac{n + 1}{n} \right) (M - m) H_R \left( \frac{n + 1}{n} \right) C_0 \left( \frac{n + 1}{n} \right) \left( \frac{2R}{3} \right)^n,
\]
with \(H_R = \inf_{B_R} H\). Since, by (54), \(\frac{|D\bar{z}|^2}{\sqrt{1 + |D\bar{z}|^2}} \geq \frac{\beta_R}{\sqrt{1 + (\beta_R)^2}} |D\bar{z}|\), we obtain
\[
\int_{B_{\hat{R}}^{n}} |D\bar{z}| \, dx \leq \int_{B_{\hat{R}}^{n}} |D\bar{z}| \, dx \leq 8(144)(C_1R)^2 \frac{\omega_n}{\omega_n} \left( \frac{5R}{6} \right)^n - 2H_R \omega_n \left( \frac{2R}{3} \right)^n.
\]

**Step 7. (an estimate of the number \(\hat{C}\))** We now proceed to seek a number \(C\) satisfying (56) with \(z = z_0\) in (42), (43), and (45). For this, adopting the notations designated in (44), we choose the cut-off functions \(\bar{\eta}_{\ell} \in C^2(B_R)\) such that
\[
0 \leq \bar{\eta}_{\ell} \leq 1,
\]
\[
\bar{\eta}_{\ell} = 1 \quad \text{in} \quad B_{R_{\ell}}(0),
\]
\[
\bar{\eta}_{\ell} = 0 \quad \text{in} \quad B_{R} \setminus B_{R_{\ell}},
\]
and

$$|D\tilde{\eta}_\ell| \leq \frac{2 \max((\frac{n}{n+1})^{\ell+1}p^\ell, 12)\Lambda_{0,R}}{R},$$

(67)

where we set

$$R_{\ell,0} = R(t) + \frac{R}{\max((\frac{n}{n+1})^{\ell+1}p^\ell(\frac{3n-1}{n}), 12)\Lambda_{0,R}}.$$

Set for $\phi_\ell(t)$ given in (40),

$$\tilde{z}_{\ell,0} = \phi_\ell(\tilde{z}) = (\tilde{z})^{(\frac{n}{n+1})^{(\ell+1)p^\ell}},$$

we have

$$\int_{A_\ell \cap B_{R_{\ell,0}}} (\tilde{\eta}_\ell)^2 (\tilde{z}_{\ell,0})^{2\frac{n-1}{n}} \text{div} T\tilde{z}_{\ell,0} dx \geq 0,$$

or equivalently

$$\int_{A_\ell \cap B_{R_{\ell,0}}} (\tilde{\eta}_\ell)^2 (\tilde{z}_{\ell,0})^{\frac{n-1}{n}} \frac{|D\tilde{z}_{\ell,0}|^2}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} dx + 2 \int_{A_\ell \cap B_{R_{\ell,0}}} \tilde{\eta}_\ell (\tilde{z}_{\ell,0})^{\frac{n-1}{n}} \langle D\tilde{\eta}_\ell, D\tilde{z}_{\ell,0} \rangle \sqrt{1 + |D\tilde{z}_{\ell,0}|^2} dx \leq 0.$$

Hence, by Young’s inequality,

$$\frac{1}{2} \int_{A_\ell \cap B_{R_{\ell,0}}} (\tilde{\eta}_\ell)^2 (\tilde{z}_{\ell,0})^{\frac{n-1}{n}} \frac{|D\tilde{z}_{\ell,0}|^2}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} dx \leq 2 \int_{A_\ell \cap B_{R_{\ell,0}}} (\tilde{z}_{\ell,0})^{\frac{3n-1}{n}} \frac{|D\tilde{\eta}_\ell|^2}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} dx.$$

This, (64), (65), (66) and (67) yield

$$\frac{(\beta_R)^2}{1 + (\beta_R)^2} \int_{A_\ell \cap B_{2R}} (\tilde{z}_{\ell,0})^{\frac{n-1}{n}} \sqrt{1 + |D\tilde{z}_{\ell,0}|^2} dx \leq \frac{(8)(\max((\frac{n}{n+1})^{\ell+1}p^\ell, 12)\Lambda_{0,R})^2}{(R)^2} \int_{A_\ell \cap B_{R_{\ell,0}}} (\tilde{z}_{\ell,0})^{\frac{3n-1}{n}} \sqrt{1 + |D\tilde{z}_{\ell,0}|^2} dx. \quad (68)$$

**Step 7.1.** To estimate the integral on the right hand side of (68), we set

$$A_{\ell,0} = A_\ell \cap B_{R_{\ell,0}}$$
and use (13) in Lemma 1.1, in a manner analogously to that in Step 5 to obtain

\[
\int_{A_\ell \cap B_{R_\ell,0}} (\tilde{z}_{\ell,0})^{\frac{3n-1}{n}} dx \leq 2 \left( \frac{2 \omega_n}{|A_{\ell,0}|} \right)^{1-\frac{1}{n}} \|diam \ A_{\ell,0}\|^n \int_{A_\ell \cap B_{R_\ell,0}} |D((\tilde{z}_{\ell,0})^{\frac{3n-1}{n}})| dx
\]

\[
+ \left( \frac{n+1}{n} \right)^{\frac{3n}{2}} \left( \left( u_{\ell,0} - m \right) + \tilde{C} \right) \left( \frac{n}{n+1} \right)^{\ell+1} \left( \frac{3n-1}{n} \right)^{\ell+1} |A_{\ell,0}|,
\]

where \(u_{\ell,0}\) is the value which satisfies

\[
m_\infty \leq u_{\ell,0} \leq M_\infty,
\]

\[
|\{x : x \in A_{\ell,0}, \ u(x) \geq u_{\ell,0}\}| \geq \frac{|A_{\ell,0}|}{2}, \quad |\{x : x \in A_{\ell,0}, \ u(x) \leq u_{\ell,0}\}| \leq \frac{|A_{\ell,0}|}{2},
\]

with

\[
M_\infty = \sup_{A_\infty} u, \quad m_\infty = \inf_{A_\infty} u, \quad \text{and} \quad A_\infty = \bigcup_{\ell=0}^\infty A_\ell.
\]

**Step 7.1.1.** To estimate the integral on the right hand side of (69), we set

\[
\tilde{z}_{\ell,0} = (\tilde{z}_{\ell,0})^{\frac{3n-1}{n}} = \hat{\phi}_{\ell}(\tilde{z}_0),
\]

where

\[
\hat{\phi}_{\ell}(t) = t^{\left( \frac{n}{n+1} \right)^{\ell+1} \left( \frac{3n-1}{n} \right)^{\ell+1}}, \quad \ell = 0, 1, \ldots, \quad \text{and} \quad t \geq 0.
\]

Setting

\[
R_{\ell,s} = R_\ell + \frac{R}{\max((\frac{n}{n+1})^{\ell+1} \rho^f, 12) \Lambda_{0,R}} + \frac{s(\min(1, R))^2}{4((\frac{n}{n+1})^{\ell+1} \rho^f (\frac{3n-1}{n}))^2 \Lambda_{0,R}}, \quad s = 1, 2, \ldots,
\]

we choose cut-off functions \(\tilde{\eta}_{\ell,s} \in C^2(B_R), \ s = 0, 1, 2, \ldots, \) such that

\[
0 \leq \tilde{\eta}_{\ell,s} \leq 1, \quad \tilde{\eta}_{\ell,s} = 1 \text{ in } B_{R_{\ell,s}}, \quad \tilde{\eta}_{\ell,s} = 0 \text{ in } B_R \setminus B_{R_{\ell,s} + \rho_{0,R}}, \quad (71)
\]

and

\[
|D\tilde{\eta}_{\ell,s}| \leq \frac{12}{\Lambda_{0,R} R}. \quad (72)
\]

Set

\[
A_{\ell,s}^* = A_\ell \cap B_{R_{\ell,s}}, \quad s = 1, 2, \ldots,
\]

and let

\[
\tilde{f}_\ell(u) = (M_\ell - m_\ell) \ln \left( \frac{1}{M_\ell - u} \right),
\]

in \(A_\ell\), where

\[
M_\ell = \sup_{A_\ell} u, \quad m_\ell = \inf_{A_\ell} u,
\]
with
\[ \tilde{A}_\ell = A_\ell \cap B_{R_\ell, s_\ell + (R/(\Lambda_0, R))} \quad \text{and} \quad s_\ell = \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right). \] (73)

We have
\[ \int_{A_{\ell,1}} (\tilde{\eta}_{\ell,1})^2 ((\hat{\phi}_\ell)'(\tilde{z}_0)) \tilde{f}_\ell(u) \text{div} T\tilde{z}_{\ell,0} \, dx \geq 0, \]
or, using (36),
\[ \int_{A_{\ell,1}} (\tilde{\eta}_{\ell,1})^2 ((\hat{\phi}_\ell)'(\tilde{z}_0)) \tilde{f}_\ell''(u) \frac{\langle Du, D\tilde{z}_{\ell,0} \rangle}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx \]
\[ + \int_{A_{\ell,1}} (\tilde{\eta}_{\ell,1})^2 \tilde{f}_\ell'(u) ((\hat{\phi}_\ell)''(\tilde{z}_0)) \frac{\langle D\tilde{z}_{0}, D\tilde{z}_{\ell,0} \rangle}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx \]
\[ + 2 \int_{A_{\ell,1}} \tilde{\eta}_{\ell,1} ((\hat{\phi}_\ell)'(\tilde{z}_0)) \tilde{f}_\ell''(u) \frac{\langle D\tilde{\eta}_{\ell,1}, D\tilde{z}_{\ell,0} \rangle}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx \]
\[ \leq \int_{A_{\ell,1} \cap \partial B_{R_{\ell,1}}} (\tilde{\eta}_{\ell,1})^2 ((\hat{\phi}_\ell)'(\tilde{z}_0)) \tilde{f}_\ell(u). \] (74)

Taking into account that
\[ \left( \frac{n+1}{n} \right)^{C_0 \left( \frac{n+1}{n} \right)} ((\hat{\phi}_\ell)'(\tilde{z}_0)) u_{x_k} = (\tilde{z}_{\ell,0})_{x_k}, \quad k = 1, 2, \ldots, n, \]
\[ (\hat{\phi}_\ell)'(t) = \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) t^{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)-1} \]
and
\[ (\hat{\phi}_\ell)''(t) = \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) - 1 \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) t^{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)-2} \]
for \( t \geq 0 \) and \( n \geq 2 \), we obtain from (71), (72) and (74) that
\[ \int_{A_{\ell,1}} \left( \frac{n}{n+1} \right)^{C_0 \left( \frac{n+1}{n} \right)} + \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) - 1 \right) \tilde{f}_\ell'(u) (\tilde{\eta}_{\ell,1})^2 \frac{|D\tilde{z}_{\ell,0}|^2}{\sqrt{1+|D\tilde{z}_{\ell,0}|^2}} \, dx \]
\[ \leq 2 \int_{A_{\ell,1}} ((\hat{\phi}_\ell)'(\tilde{z}_0)) \tilde{f}_\ell'(u) \frac{|D\tilde{\eta}_{\ell,1}| \cdot |D\tilde{z}_{\ell,0}|}{\sqrt{1+|D\tilde{z}_{\ell,0}|^2}} \, dx + \int_{A_{\ell,1} \cap \partial B_{R_{\ell,1}}} (\tilde{\eta}_{\ell,1})^2 ((\hat{\phi}_\ell)'(\tilde{z}_0)) \tilde{f}_\ell'(u). \]

We have \( \frac{1}{M_{\ell-m}} = \frac{M-m}{(M-m)} \), and thus
\[ \left( \frac{n}{n+1} \right)^{C_0 \left( \frac{n+1}{n} \right)} + \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) - 1 \right) \tilde{z}_0 \]
\[ \geq \frac{M-m}{M_{\ell-m}} + \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) - 1 \]
\[ \geq (C_{\ell,s})^{-1}, \]
where
\[ (\max(M-m,1) + \tilde{C}). \]
where we set

\[ C_{\ell,s} = \left( \frac{n+1}{n} \right) C_0 \left( \frac{n+1}{n} \right) \left[ \max(M-m,1) + \bar{C} \right] \left( \frac{M-m}{M_\ell-m_\ell} \right) \]

for \( s = 1, 2, \cdots \). Inserting this into (75) and using Young's inequality, we obtain

\[
\begin{align*}
\frac{1}{2C_{\ell,1}} \int_{A_{\ell,1}} \frac{|D\tilde{z}_{\ell,0}|^2}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx & \leq C_{\ell,2} C_{\ell,1} \int_{A_{\ell,1}} \frac{2((\hat{\phi}_\ell)'(\hat{z}_0))^2|D\tilde{\eta}_{\ell,1}|^2}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx + \int_{A_{\ell} \cap \partial B_{\ell,1}} C_{2,\ell}((\hat{\phi}_\ell)'(\hat{z}_0)),
\end{align*}
\]

with

\[ C_{2,\ell} = \sup_{A_{\ell,\ell}} \left( \frac{M_\ell - m_\ell}{M_\ell - u(x)} \right). \] (76)

This, (72) and (75) yield

\[
\begin{align*}
\int_{A_{\ell,0}} \frac{|D\tilde{z}_{\ell,0}|}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx & \leq \int_{A_{\ell} \cap \partial B_{\ell,1}} C_{2,\ell}((\hat{\phi}_\ell)'(\hat{z}_0)) \, dx \quad \text{(77)}
\end{align*}
\]

\[
\begin{align*}
&\quad + 4 \left( \frac{n+1}{n} \right) C_0 \left( \frac{n+1}{n} \right) C_{2,\ell}(C_{\ell,1})^2 \left( \frac{12 \Lambda_{0,R}}{R} \right)^2 \int_{A_{\ell,1}} \frac{((\hat{\phi}_\ell)'(\hat{z}_0))^2}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx \\
\leq & \left( \frac{n+1}{n} \right) C_0 \left( \frac{n+1}{n} \right) C_{2,\ell}(C_{\ell,1})^2 \left( \frac{24 \Lambda_{0,R}}{R^2} \right)^2 \int_{A_{\ell,1}} \frac{1}{\sqrt{1 + (\beta_\ell^*)^2}} ((\hat{\phi}_\ell)'(\hat{z}_0)) \, dx \\
&\quad + \int_{A_{\ell} \cap \partial B_{\ell,1}} C_{2,\ell}((\hat{\phi}_\ell)'(\hat{z}_0)),
\end{align*}
\]

with

\[ \beta_\ell^* = \inf_{A_{\ell}} |Du|. \] (78)

Since

\[
\begin{align*}
(C_{\ell,1})^2 & = \frac{(\max(M-m,1) + \bar{C})^2}{\left( \frac{M-m}{M_\ell-m_\ell} \right) \left( \frac{\bar{C}}{\max(M-m,1)} \right) + \left( \frac{n}{n+1} \right) \ell + 1 p \left( \frac{2n-1}{n} \right)^2} \\
& = \frac{(\max(M-m,1) + \bar{C})^2}{\left( \frac{M-m}{M_\ell-m_\ell} \right) \left( \frac{\bar{C}}{\max(M-m,1)} \right) + \left( \frac{n}{n+1} \right) \ell + 1 p \left( \frac{2n-1}{n} \right)^2} \\
& \leq \frac{(\max(M-m,1))^2 (1 + \frac{2\bar{C}}{\max(M-m,1)})}{\left( \frac{n}{n+1} \right) \ell + 1 p \left( \frac{2n-1}{n} \right)^2} + 2 \left( \frac{M-m}{M_\ell-m_\ell} \right) \left( \frac{n}{n+1} \right) \ell + 1 p \left( \frac{2n-1}{n} \right)^2 \left( \frac{\bar{C}}{\max(M-m,1)} \right) \\
& \leq \frac{(\max(M-m,1))^2 (1 + \frac{2\bar{C}}{\max(M-m,1)})}{\left( \frac{n}{n+1} \right) \ell + 1 p \left( \frac{2n-1}{n} \right)^2} \left( \frac{\bar{C}}{\max(M-m,1)} \right) \\
& \leq \frac{(\max(M-m,1))^2 (1 + \frac{2\bar{C}}{\max(M-m,1)})}{\left( \frac{n}{n+1} \right) \ell + 1 p \left( \frac{2n-1}{n} \right)^2} \left( \frac{\bar{C}}{\max(M-m,1)} \right)
\end{align*}
\]
\[ + (\max(M - m, 1))(M_\ell - m_\ell)\min\left(\frac{M - m}{M_\ell - m_\ell}, \frac{M_\ell - m_\ell}{M - m}\right) \]

we have

\[ (C_{\ell,1})^2 \leq \frac{(\max(M - m, 1))(M_\ell - m_\ell)}{\left[\left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)}\right]^{\frac{2}{3}}} (C_{\ell,1}^* + C_{\ell,2}^*) \] (79)

where we set

\[ C_{\ell,s}^* = \left(1 + \frac{2\tilde{C}}{\max(M - m, 1)}\right)\left[\left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)} - s + 1\right]^{-\frac{1}{2}} \cdot \min\left(\frac{M - m}{M_\ell - m_\ell}, \left[\left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)} - s + 1\right]\left(\frac{\max(M - m, 1)}{\tilde{C}}\right)\right) \]

\[ \leq \left(1 + \frac{2\tilde{C}}{\max(M - m, 1)}\right)\left(\frac{M - m}{M_\ell - m_\ell}\right)\left(\frac{\tilde{C}}{\max(M - m, 1)}\right)^{-\frac{1}{2}} \] (80)

and

\[ C_{\ell,s}^{**} = \min\left(\frac{(M - m)(\max(M - m, 1))}{\left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)} - s + 1\left(\frac{M_\ell - m_\ell}{M - m}\right)}\right) \]

\[ \leq \min\left(1, \left(\frac{\tilde{C}}{\max(M - m, 1)}\right)^{\frac{1}{2}}\right), \] (81)

for \(s = 1, 2, \ldots\). Inserting this into (77), we obtain

\[ \int_{A_{\ell,0}} \frac{\tilde{z}_{\ell,0}}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx \leq \left(\frac{n+1}{n}\right)C_0^{(n+1)}C_{2,\ell}C_{3,\ell} \sqrt{1 + \left(\frac{\beta^*}{\beta^*_\ell}\right)^2} \left(\frac{24\Lambda_{0,R}^2}{R}\right) \]

\[ \cdot \left(\frac{M_\ell - m_\ell}{\left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)}\right)^{\frac{1}{2}}\right) (C_{\ell,1}^* + C_{\ell,1}^{**}) \cdot \int_{A_{\ell,0}} \left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)}dx \]

\[ + \int_{A_{\ell,0}\cap \partial B_{\ell,1}} C_{2,\ell}(\tilde{\phi}_\ell)'(\tilde{z}_0), \] (82)

with \( C_{3,\ell} = \frac{M_\ell - m_\ell}{\beta^*_R^2} \). In view of the fact that \( \frac{|D\tilde{z}_{\ell,0}|^2}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \geq \frac{\beta^*_\ell}{\sqrt{1 + |\beta^*_\ell|^2}} |Du| \), we find

\[ \int_{A_{\ell,0}} |D\tilde{z}_{\ell,0}| \, dx \leq \left(\frac{24\Lambda_{0,R}^2}{R}\right) \left(\frac{n+1}{n}\right)C_0^{(n+1)}C_{2,\ell}C_{3,\ell} \left(\frac{M_\ell - m_\ell}{\left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)}\right)^{\frac{1}{2}}\right) \]

\[ \cdot (C_{\ell,1}^* + C_{\ell,1}^{**}) \cdot \int_{A_{\ell,0}} \left(\frac{n}{n+1}\right)^{\ell+1}p^{\left(\frac{3n-1}{n}\right)}dx \]

\[ + \int_{A_{\ell,0}\cap \partial B_{\ell,1}} C_{2,\ell}(\tilde{\phi}_\ell)'(\tilde{z}_0). \]
Then, by (69), we have

\[ \int_{A_{\ell,0}} \frac{\tilde{z}_{\ell,0}}{1 + |D\tilde{z}_{\ell,0}|^2} \, dx \leq \frac{1}{\sqrt{1 + (\beta R)^2 \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell - 1}} \cdot \frac{1}{\left( \frac{n+1}{n} \right)^{C_0(n+1)} (m_\ell - m + \tilde{C})^{(n+1) p^{\ell+1} - 1}} \int_{A_{\ell,0}} \frac{3^{n-1}}{n} \, dx. \]  

(83)

and

\[ C_4 = \left( 2^{2} \cdot \sup_{\ell,s} \left( \frac{\omega_n}{|A_{\ell,s}|} \right)^{1-\frac{1}{n}} (\text{diam } A_{\ell,s})^n \right) R^{-f_k}, \]  

(85)

where \( m_\infty \leq u_{\ell,s} \leq M_\infty, s = 1, 2, \ldots, \) are the numbers which satisfy the conditions

\[ \{ x : x \in A_{\ell,s}, u(x) \geq u_{\ell,s} \} = \frac{1}{2} |A_{\ell,s}|, \quad \{ x : x \in A_{\ell,s}, u(x) \leq u_{\ell,s} \} = \frac{1}{2} |A_{\ell,s}|. \]

The reason for such a choice of the power of \( R \) in (84) and (85) is due to our specification of the set \( A_{\ell} \) made in Step 7.2 below and will be clarified in Step 7.2.5. We have

\[ \int_{A_{\ell,0}} \frac{(\tilde{z}_{\ell,0})^{\frac{3n-1}{n}}}{1 + |D\tilde{z}_{\ell,0}|^2} \, dx \leq \frac{1}{\sqrt{1 + (\beta R)^2 \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell - 1}} \cdot \frac{1}{\left( \frac{n+1}{n} \right)^{C_0(n+1)} (m_\ell - m + \tilde{C})^{(n+1) p^{\ell+1} - 1}} \cdot \int_{A_{\ell,0}} \frac{3^{n-1}}{n} \, dx. \]  

(87)

Inserting (70) and (84) into this, we have

\[ \int_{A_{\ell,0}} \frac{(\tilde{z}_{\ell,0})^{\frac{n+1}{n}} p^{\ell+1} (\frac{2n-1}{n})}{1 + |D\tilde{z}_{\ell,0}|^2} \, dx = \int_{A_{\ell,0}} \frac{\tilde{z}_{\ell,0}}{1 + |D\tilde{z}_{\ell,0}|^2} \, dx. \]  

Step 7.1.2. Set

\[ \tilde{C} = (\max(M - m, 1))^6(144)(\sup_{\ell} C_{2,\ell}) \max\left( (\sup_{\ell} 1, C_{3,\ell}) C_4 (A_{\ell,0})^2 \right) R^{-f_k}, \]  

(84)

and

\[ \tilde{C}_{\ell,s} = ((u_{\ell,s} - m) + \tilde{C}), \]  

(86)
\[
\begin{align*}
&\leq \frac{(\frac{n+1}{n}) C_0 \left(\frac{n+1}{n}\right)^{\ell+1} \left(\frac{2n-1}{n}\right) + C(C_{\ell,1} + C_{\ell,1}^{**}) \int_{A_{\ell,1}} (\tilde{z}_0) \left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1} dx \\
&\quad + \frac{\int_{A_{\ell,1} \cap \partial B_{R_{\ell,s}}} (\tilde{z}_\ell,0) \left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1}}{\sqrt{1 + (\frac{n+1}{n}) C_0 \left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1}}} \\
&\quad + \frac{\left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1} \sup_{\ell,s} \tilde{C}_{\ell,s} \left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1}}{\sqrt{1 + (\frac{n+1}{n}) C_0 \left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1}}} |A_{\ell,0}|, (88)
\end{align*}
\]

Successively use (82) with \( \tilde{n}_{\ell,1} \) replaced by \( \tilde{n}_{\ell,s} \), the domain of integration replaced by \( B_{R_{\ell,s}} \), and the factor \( C_{\ell,1}^{**} \) on the right hand side replaced by \( C_{\ell,s}^{**} \), \( s = 2, 3, \ldots \). Insert the result into (84) with the integrand on the left hand side replaced by that on the right hand side of (84), with the domain of integration on the left hand side replaced by \( A_{\ell,s} \) and the domain of integration on the right hand side replaced by \( A_{\ell,s+1} \). Afterwards insert the result into the right hand side of (88) with the number \( \tilde{C}_{\ell,s-1} \), \( C_{\ell,s}^{**} \) and \( |A_{\ell,s-1}| \) replaced respectively by \( \tilde{C}_{\ell,s} \), \( C_{\ell,s+1}^{**} \) and \( |A_{\ell,s}| \), successively. Iterate in this manner so that the exponent of \( \tilde{z}_0 \) in the integrand decreases in number to become one less at a time. The number of terms on the right hand side, however, increases so that the terms which need to be further iterated become twice as many as before at a time and two extra terms are produced, one involving integration along \( A_{\ell} \cap \partial B_{R_{\ell,s}} \), the other involving the numbers \( \tilde{C}_{\ell,s-1} \) and \( |A_{\ell,s-1}| \). By the choice of \( \tilde{C} \) and \( \tilde{C}_{\ell,s} \), each term produced in the process of iteration is bounded above by one of the following numbers

\[
C_5 \frac{\left(\frac{n}{n+1}\right)^{\ell+1} p^{(\frac{2n-1}{n})}}{\sqrt{1 + (\beta R)^2}},
\]

\[
C_5 \frac{\left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1} \sup_{\ell,s} \tilde{C}_{\ell,s} \left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1}}{\sqrt{1 + (\beta R)^2}},
\]

and

\[
\frac{\left(\frac{n}{n+1}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1} \sup_{\ell,s} \sup_{A_{\ell} \cap \partial B_{R_{\ell,s}}} \left(\frac{n+1}{n}\right)^{\ell+1} p^{(\frac{2n-1}{n}) - 1}}{\sqrt{1 + (\beta R)^2}},
\]

where the set \( \tilde{A}_{\ell} \) is given in (73), and we let \( C_5 = C_{5a} C_{5b} C_{5c} \), with

\[
C_{5a} = \left(\frac{\tilde{C}}{\max(M - m, 1)}\right)^{\frac{M - m}{m\ell - m^2} (\max(M - m, 1))^2 + 1}, \quad C_{5b} = (C_4)^{C_{4} + 1},
\]

\[
C_{5c} = \left[1 + \left(\frac{\max(M - m, 1)}{2C}\right)\frac{M - m}{C}\right]^{\frac{\tilde{C}}{m\ell - m^2 + 1}},
\]
We finally obtain
\[
\int_{B_{\frac{3}{2}R}} \frac{\tilde{z}_{\ell,0}}{\sqrt{1 + |D\tilde{z}_{\ell,0}|^2}} \, dx \leq \frac{1}{(n+1)^{\ell+1}p^\ell} \left[ \left( \frac{n}{n+1} \right)^{\ell+1} \frac{n}{n+1} \right].
\]

\[
\left[ \left( \frac{n+1}{n} \right)^{C_0(\frac{n+1}{n})} \tilde{C}_\ell \right] \left( \frac{n}{n+1} \right)^{\ell+1} \frac{3n-1}{n} \right] \cdot \max(\sup|A_\ell|, \sup |A_\ell \cap \partial B_{\ell,s}|)
\]
\[
\sqrt{1 + (\beta R)^2},
\]

where
\[
\tilde{C}_\ell = \max(\sup \tilde{C}_{\ell,s}, \sup_s \sup_{A_\ell \cap \partial B_{\ell,s}} (u - m) + \tilde{C}).
\] (89)

Inserting this into (68) and recall (70) and (42), we obtain
\[
\hat{C}_{\ell}^* = \inf_{A_\ell \subset B_{R(\ell)}} \int_{A_\ell \cap B_{R(\ell)}} \frac{1}{\sqrt{1 + |D(z_0)|^{\ell+1}p^\ell}} \left[ \left( \frac{n}{n+1} \right)^{\ell+1} \frac{3n-1}{n} \right] \cdot \max(\sup|A_\ell|, \sup |A_\ell \cap \partial B_{\ell,s}|)
\]
\[
\frac{1}{1 + (\beta R)^2} C_0(\hat{C}_{\ell,a} \cdot \hat{C}_{\ell,b}) C_{\ell,c} R^{-2},
\]

where
\[
C_6 = 8A_{0,R}(\max(\left( \frac{n}{n+1} \right)^{\ell+1}p^\ell, 12)) \left( \frac{3n-1}{n} \right) C_5,
\] (91)
\[
\hat{C}_{\ell,a}^* = \frac{p^\ell(3n-1)}{n} \left( \frac{n}{n+1} \right)^{\ell+1}, \quad \hat{C}_{\ell,b}^* = \frac{\left( \frac{n+1}{n} \right)^{C_0(\frac{n+1}{n})} \tilde{C}_\ell \left( \frac{n}{n+1} \right)^{\ell+1} \frac{3n-1}{n} \right] \cdot \max(\sup|A_\ell|, \sup |A_\ell \cap \partial B_{\ell,s}|)
\]
\[
[(\frac{n+1}{n})^{C_0(\frac{n+1}{n})}(m_{\ell} - m + \tilde{C})]^{2(\frac{n}{n+1})^{\ell+1}p^\ell - 2},
\]

and
\[
C_{\ell,c}^* = \max(|\tilde{A}_\ell|, \sup_s |A_\ell \cap \partial B_{\ell,s}|).
\] (92)

**Step 7.2.** To gain useful estimates of $C_{2,\ell}$, $C_{3,\ell}$, $\tilde{C}_\ell$ and $C_4$, we have to choose $A_\ell$ carefully and impose some additional condition on the function $u$.

**Step 7.2.1.** Recalling (48), let $x_0 \in B_{\frac{3}{2}R}$ be a point at which
\[
u(x_0) = M_*,
\]
with $M_*$ given in (48). Let
\[ S_\ell = B_{R_\ell}(x_0) \cup B_\mathbb{R}^\perp, \quad \text{with} \quad R_\ell = \frac{R}{[(\frac{n}{n+1})^{\ell+1} p^{(\frac{3n-1}{n})}]} \cdot \] and let $P_\ell$ be the $n$-dimensional plane passing through $S_\ell$, which divides $B_R$ into two components. Let $\hat{A}_\ell$ be that component in which the point $x_0$ lies. Firstly, we make the additional assumption that, for $x \in \hat{A}_\ell$, (3) holds for some constant $\Lambda$ depending only on $n$. The condition (3) yields
\[ \left( \frac{1}{|Du|^2} \right)' \geq -C_8 H, \] for some constant $C_8$ depending only on $n$. Let
\[ \Lambda_{0,R} = \max(1, 8C_8 |Du(x_0)|^2 (H_R^+) R), \] with $H_R^+ = \sup_{B_R} H(x)$ and
\[ R_* = \frac{R}{(\frac{n}{n+1})^{\ell+1} p^{(\frac{3n-1}{n})} \Lambda_{0,R}}. \]
Let
\[ S_* = B_{R_*}(x_0) \cup B_\mathbb{R}^\perp, \] and $P_*^\ell$ be the $n$-dimensional plane passing through $S_\ell$. We impose the additional condition that at each point $x \in P_*^\ell$, the vector $Du(x)$ makes an angle less than $\frac{\pi}{2}$ with the vector $Du(x_0)$. Under this condition, we set $A_*^\ell$ be one of the two components of $B_R$ divided by $P_*^\ell$ which is contained in $\hat{A}_\ell$. We note that this last additional assumption makes (36) be satisfied along $\partial A_*^\ell \cup B_R$ (which is the plane $P_\ell$).

**Step 7.2.2.** From (94) follows
\[ \frac{1}{|Du(x_2)|^2} - \frac{1}{|Du(x_1)|^2} \geq -C_8 H_R^+ \text{dist}(x_1, x_2), \] for points $x_1$ and $x_2$ in $\hat{A}_*^\ell$. Thus, if $\text{dist}(x_1, x_2)$ is so small that
\[ C_8 H_R^+ \text{dist}(x_1, x_2) \leq \frac{1}{|Du(x_1)|^2}, \] we obtain
\[ |Du(x_2)| \leq \frac{1}{\sqrt{|Du(x_1)|^2 - C_8 H_R^+ \text{dist}(x_1, x_2)}}. \]
Hence, if \( \text{dist}(x_1, x_2) \leq \frac{3}{4} C_8 H_R^+ |Du(x_1)|^2 \), then

\[
\frac{|Du(x_2)|}{|Du(x_1)|} \leq \frac{1}{\sqrt{1 - (C_8 H_R^+ |Du(x_1)|^2 \text{dist}(x_1, x_2))}} \leq 2.
\]

Thus, if \( \text{dist}(x_1, x_2) \leq \frac{3}{8} C_8 H_R^+ |Du(x_1)|^2 \), then

\[
\text{dist}(x_1, x_2) \leq \frac{3}{4} C_8 H_R^+ |Du(x_2)|^2,
\]

and we also have \( |Du(x_1)| \leq 2 |Du(x_2)| \).

**Step 7.2.3.** Since \( \text{dist}(x, x_0) \leq \frac{3}{8} C_8 H_R^+ |Du(x_0)|^2 \) for \( x \in \tilde{A}_\ell \), the estimate in Step 7.2.2 yield that

\[
\frac{1}{2} \leq \frac{|Du(x)|}{|Du(x_0)|} \leq 2,
\]

for \( x \in \tilde{A}_\ell \). Hence

\[
\beta_*^\ell \leq |Du(x)| \leq 16 \beta_*^\ell,
\]

for \( x \in \tilde{A}_\ell \).

Moreover, since \( \frac{1}{2^\ell} \leq \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) \), for all \( \ell \), we have

\[
\text{dist}(x, x_0) \leq \frac{3R}{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) \Lambda_{0,R}},
\]

for \( x \in \tilde{A}_\ell \). This and (96) yield

\[
u(x_0) - m \leq \frac{48 \beta_*^\ell R}{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) \Lambda_{0,R}},
\]

and

\[
\sup_{A_{\ell,s}} (u - m) \leq u(x_0) + \frac{16 s \beta_*^\ell R}{\left( \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) \right)^2 \Lambda_{0,R}} - m,
\]

from this follows that, for \( s \leq s_\ell \) given in (73),

\[
\sup_{A_{\ell,s}} (u - m) \leq m \leq \frac{64 \beta_*^\ell R}{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) \Lambda_{0,R}} - m.
\]

Hence, for the number \( \tilde{C}_\ell \) given in (89), we have

\[
\tilde{C}_\ell \leq (m - m + \tilde{C}) + \frac{64 \beta_*^\ell R}{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right) \Lambda_{0,R}}.
\]

(98)
Step 7.2.4. Since

\[ M - m \geq M_\ell - \sup_{A_{\ell,s}} u \geq \beta_\ell^* \frac{R}{\Lambda_{0,R}}, \]

and since \( \tilde{C} \geq M - m \) by (85), we have

\[ \frac{64\beta_\ell^* R}{\Lambda_{0,R} \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)} \leq \frac{64}{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)} (M - m) \leq \frac{64}{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)} \tilde{C}. \]

This and (98) yield that

\[ \tilde{C}_\ell \leq (m_\ell - m + \tilde{C}) \left( 1 + \frac{64}{s_\ell} \right)^{s_\ell} \leq e^{64} (m_\ell - m + \tilde{C})^{s_\ell}. \] (99)

Step 7.2.5. The relations \( \frac{1 - \cos \theta}{\sin \theta} \to 0 \) as \( \theta \to 0 \) and \( \frac{1 - \cos \theta}{(\sin \theta)^2} \to \frac{1}{2} \) as \( \theta \to 0 \) yield the existence of the numbers \( C_9 \) and \( C_{10} \) for which

\[ C_{10} |\text{diam } S_\ell \cup B_{R(\ell)}|^2 \geq \text{dist}(x_0, P_\ell) \geq C_9 |\text{diam } S_\ell \cup B_{R(\ell)}|^2, \]

for all \( \ell \). Hence

\[ \frac{(\text{diam } A_{\ell,s})^n}{|A_{\ell,s}|^{1 - \frac{2}{n}}} \leq C_{11} \left( \frac{R/(\Lambda_{0,R}s_\ell)}{R/(\Lambda_{0,R}s_\ell)^{\ell+1}(\frac{n}{n+1})} \right)^\frac{2}{n} \frac{1}{\left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)} \]

\[ = C_{11} \left( \frac{R}{\Lambda_{0,R} \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)} \right)^\frac{1}{n} \]

\[ = C_{11} \left( \frac{R}{\Lambda_{0,R} \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)} \right)^\frac{1}{n}, \]

where \( C_{11} = (C_9)^{n} (C_{10})^{\frac{2}{n}} \). This yields that

\[ C_4 \leq 2^{2 - \frac{1}{n}} \omega_1^{1 - \frac{1}{n}} C_{11} \left( \frac{1}{\Lambda_{0,R} \left( \frac{n}{n+1} \right)^{\ell+1} p^\ell \left( \frac{3n-1}{n} \right)} \right)^\frac{1}{n} \leq 2^{2 - \frac{1}{n}} \omega_1^{1 - \frac{1}{n}} C_{11} \left( \frac{1}{\Lambda_{0,R}} \right)^\frac{1}{n}. \] (100)

Step 7.2.6. We also note that our choice of \( A_{\ell,s} \) and \( A_\ell \) provides us with the estimates

\[ C_{2,\ell} \leq 6, \] (101)
and

\[ C_{3,\ell} \leq \frac{6}{A_{0, R}}. \]  

(102)

**Step 7.3.** Recalling (42) and using the fact that \(1 - \frac{2n}{n+1} = \frac{1-n}{n+1}\), we obtain

\[
\hat{C}_{\ell}^{*} \leq \frac{1}{(\frac{n}{n+1})^{\ell+1} p^\ell} \left[ \left( \frac{n+1}{n} \right) C_{0}^{(\frac{n+1}{n})} (m_{\ell}-m+\hat{C}) \right]^{\left(\frac{n}{n+1}\right)\ell+1} p^\ell \left( |A_{\ell} \cup B_{R(\ell)}| \right)^{\frac{1-n}{n+1}} 
\]

\[
= \frac{1}{(\frac{n}{n+1})^{\ell+1} p^\ell} \left[ \left( \frac{n+1}{n} \right) C_{0}^{(\frac{n+1}{n})} (m_{\ell}-m+\hat{C}) \right]^{\left(\frac{n}{n+1}\right)\ell+1} p^\ell \left( |A_{\ell} \cup B_{R(\ell)}| \right)^{\frac{1-n}{n+1}} . 
\]

From this, (90), (91), (92) and (98), we obtain

\[
\hat{C}_{\ell}^{*} \hat{C}_{\ell}^{*} \leq \frac{1}{1 + (\beta R)^2} \left[ \left( \frac{n+1}{n} \right) C_{12} \hat{C}_{\ell}^{*} (|A_{\ell} \cup B_{R(\ell)}|) \right]^{\frac{1-n}{n+1}} R^{-2} 
\]

\[
\leq \frac{1}{1 + (\beta R)^2} (C_{12} n(\omega_n) \frac{2}{n+1} (\max(1, R))^{n-1} - \frac{n(n-1)}{n+1})^{-2} 
\]

\[
= \frac{1}{1 + (\beta R)^2} (C_{12} n(\omega_n) \frac{2}{n+1} (\max(1, R))^{n-1} - \frac{n(n-1)}{n+1})^{-2} , 
\]

where \(C_{12} = \left( \frac{n+1}{n} \right)^{2C_{0}(\frac{n+1}{n})} e^{64C_{0}}\). And hence, in (46),

\[
\left( (\hat{C}_{k}^{*} \hat{C}_{k}^{*}) \right)^{\frac{1}{b}} \left( (\hat{C}_{k-1}^{*} \hat{C}_{k-1}^{*}) \right)^{\frac{1}{b}} \ldots \left( (\hat{C}_{0}^{*} \hat{C}_{0}^{*}) \right)^{\frac{1}{b}} \left( (\hat{C}_{k}^{*} \hat{C}_{k}^{*}) \right)^{\frac{1}{b}} \leq \frac{1}{1 + (\beta R)^2} \left( C_{12} n(\omega_n) \frac{2}{n+1} R \right)^{\frac{1}{b}} \sum_{\ell=0}^{k} (\frac{n}{n+1})^{\ell} 
\]

\[
\leq \frac{(\hat{C})^{2}}{1 + (\beta R)^2} (C_{12}) \left( \frac{n+1}{n} \right) ^{2} (3n-1)^{2} n(\omega_n) \frac{2}{n+1} R \right)^{\frac{1}{b}} \sum_{\ell=0}^{k} (\frac{n}{n+1})^{\ell} 
\]

\[
= \frac{(\hat{C})^{2}}{1 + (\beta R)^2} (C_{12}) \left( \frac{n+1}{n} \right) ^{2} (3n-1)^{2} n(\omega_n) \frac{2}{n+1} R \right)^{\frac{1}{b}} \sum_{\ell=0}^{k} (\frac{n}{n+1})^{\ell} , 
\]

where we set \( b = \frac{np}{n+1} = \frac{n^2}{n^2-1} \). Since it is easy to see that \( \sum_{\ell=0}^{\infty} (\frac{n}{n+1})^{\ell} = n^2 \) we can, recalling (55), (85), (96), (4) and set in (47)

\[
\hat{C} = \left( C_{13} C_{1,R}^{*} (\sup_{\ell} C_{2,\ell}) (\sup_{\ell} C_{3,\ell}) \right) ^{\frac{(n+1)^2(n-1)}{n}} R^{\frac{n+1}{n+1} + \frac{1}{n+1}} \]  

(103)

\[
= \left( C_{13} C_{1,R}^{*} (\sup_{\ell} C_{2,\ell}) (\sup_{\ell} C_{3,\ell}) \right) ^{\frac{(n+1)^2(n-1)}{n}} R^{\frac{n^2-3n}{n+1}} , 
\]

where \( C_{1,R}^{*} = \max(M-m,1) \), and \( C_{13} = 6(144)C_{4}C_{12}(\frac{n}{n+1})^{2} n(\omega_n) \frac{2}{n+1} \), with an estimation of \( C_{4} \) given in (100).
Step 8. Combining (87) with (53), we have

$$
\sup_{x \in B_R} (u(x) - m + \bar{C}) \leq (C_s)^{n(n+1)} \left[ 2\hat{c} + \left( \omega_n \left( \frac{2R}{3} \right)^n \right)^{\frac{2}{n-1}} \right] \omega_n R^{n+1} \tag{104}
$$

$$
\left\{ C_{15}(C_{1,R})^2 \left( \frac{M-m}{R} \right) - (M-m)C_{16}H_R^{-1} \right\} + C_{17} \left( \bar{u} - m + \bar{C} \right),
$$

where

$$
C_{15} = 8(2)^{\frac{n^2+n+1}{n}} \left( \frac{2}{3} \right) \left( \frac{5}{6} \right)^n, \quad C_{16} = 2(2)^{\frac{n^2+n+1}{n}} \left( \frac{2}{3} \right)^{n+1}, \quad C_{17} = \left( \frac{2}{3} \right)^n.
$$

We note that, by (53), we have

$$
C_{15}(C_{1,R})^2 \left( \frac{M-m}{R} \right) - (M-m)C_{16}H_R^{-1} \geq 0. \tag{105}
$$

Step 8.1. Since $M - m \geq 2(\beta_R)R$, we have

$$
\frac{\max(M - m, 1)}{R} = \frac{M-m}{R} \min(1, \frac{1}{M-m, 1}) \leq \frac{M-m}{R} \{ \max(1, (\beta_R R)^{-1}) \}. \tag{106}
$$

Thus, setting $y_* = C_{1,R} = \frac{M-m}{\beta_R}$, from (0.5), (2.65), (2.84) and (2.86) we obtain that $y_*$ satisfies the inequality

$$
a_0(y_*)^3 + b_0(y_*)^2 + c_0y_* \geq 0, \tag{107}
$$

where

$$
a_0 = C_{15} \beta_R, \quad b_0 = -C_{16} \min(1, 2(\beta_R)R) \frac{H_R}{(C_{1,R})^2} < 0, \quad c_0 = C_{17} C_{18} (\beta_R)^{2-(1/(2n))} R^{(1/(2n))-1} \max(1, \frac{1}{2} (\beta_R R)^{-1})
$$

$$
\begin{cases}
\frac{1}{2} C_{17} C_{18} (\beta_R)^{2-(1/(2n))} R^{(1/(2n))-1} - 2, & \text{if } 2\beta_R R \leq 1, \\
\frac{1}{4} C_{17} C_{18} (\beta_R)^{2-(1/(2n))} R^{(1/(2n))-1}, & \text{if } 2\beta_R R > 1
\end{cases}
$$

with

$$
C_{18} = 12(144) (\sup \frac{C_{2,\ell}}{\ell}) (\sup \frac{C_{3,\ell}}{\ell}) C_4 \left( \frac{\Lambda_0}{\beta_R^2} \right)^{\frac{1}{2} + \frac{1}{4n}}.
$$

We note that an estimation of $C_{18}$ in terms of $n$ and $R$ is obtained by combining (100), (101) and (102).

Setting $\beta_{R_0/2} = \sup_{B_R} |Du|$, $R_0 = \mathrm{dist}(\bar{x}, \partial \Omega)$, $\tilde{H} = H(\bar{x})$ and $H_{R_0} = \inf_{B_R} H(x)$ as in the statement of Main Theorem 1, and setting

$$
\Theta = (\beta_{R_0/2})^2 \tilde{H} \frac{R_0}{2}, \tag{108}
$$
we have, for $R \leq \frac{1}{2}R_0$,  
\[(\tilde{\beta}_R)^2\tilde{H}R \leq \Theta.\]  
(109)

We aim at deriving an estimate of $\Theta$. For this, we impose the assumption that  
\[H_R^- \geq 1,
\]
and observe, in view of (106), (107), (108) and (109), that $y_*$ satisfies the inequality
\[a_1(y_*)^3 + b_1(y_*)^2 + c_0y_* \geq 0,
\]
(110)
where
\[
a_1 = H_R^+(\max(1, \frac{1}{2}(\beta_R R)^{-1})^3 a_0 = \begin{cases}
\frac{1}{2}C_{15}H_R^-\beta_R \Theta & \text{if } 2\beta_R R \leq 1, \\
C_{15}H_R^-\beta_R & \text{if } 2\beta_R R > 1
\end{cases}
\]
\[
b_1 = H_R^-(\max(1, \frac{1}{2}(\beta_R R)^{-1})^3 \min(1, 2(\beta_R)R)\left(\frac{(\tilde{\beta}_R)^2H_R^-}{\Theta}\right)^q b_0 \right)
\]
\[
= \begin{cases}
\frac{C_{16}}{(C_{19})^2} \left(\frac{(\tilde{\beta}_R)^2H_R^-}{\Theta}\right)^q, & \text{if } 2\beta_R R \leq 1 \\
\frac{C_{16}}{(C_{19})^2} \left(\frac{(\tilde{\beta}_R)^2H_R^-}{\Theta}\right)^q, & \text{if } 2\beta_R R > 1,
\end{cases}
\]
for some $q \in \mathbb{N}$. Set
\[
C_{19} = \max \left(1, \frac{5a_1c_0}{(b_1)^2}\right)^{2+(1/(2n))^{-1}}
\]
\[
= \max \left(1, \frac{5C_{15}C_{17}C_{18}}{(C_{16})^2} \left(\frac{(C_{19})^4}{(H_R^-)^{2q+3}} \frac{(\beta_R)^{4-(1/(2n))}}{\Theta^{2q+1+(1/(2n))}}\right)^{(2+(1/(2n)))^{-1}}
\]

After the rescaling  
\[\begin{cases}
x_i \rightarrow (C_{19})^{-1}x_i, & i = 1, 2, \ldots, n,
\end{cases}
\]
we see that $y^*$ satisfies the inequality
\[a_2(y^*)^3 + b_2(y^*)^2 + c_2y^* \geq 0,
\]
(111)
with
\[
a_2 = \begin{cases}
(C_{19})^4a_1 & \text{if } 2\beta_R R \leq 1 \\
C_{19}a_1 & \text{if } 2\beta_R R > 1,
\end{cases}
\]
\[
b_2 = \begin{cases}
-(C_{19})^4b_1 & \text{if } 2\beta_R R \leq 1, \\
(C_{19})^2b_1 & \text{if } 2\beta_R R > 1,
\end{cases}
\]
\[
c_2 = \begin{cases}
(C_{19})^2-(1/(2n))c_0 & \text{if } 2\beta_R R \leq 1, \\
(C_{19})^{1-(1/(2n))}c_0 & \text{if } 2\beta_R R > 1.
\end{cases}
\]

By the choice of $C_{19}$, we have
\[
(b_2)^2 - 4a_2c_2 = \begin{cases}
(C_{19})^8(b_1)^2 - 4(C_{19})^6-(1/(2n))a_1c_0 & \text{if } 2\beta_R R \leq 1, \\
(C_{19})^4(b_1)^2 - 4(C_{19})^2-(1/(2n))a_1c_0 & \text{if } 2\beta_R R > 1,
\end{cases}
\]
\[
> \begin{cases}
5(C_{19})^6-\frac{1}{2n}a_1c_0 - 4(C_{19})^6-\frac{1}{2n}a_1c_0 & \text{if } 2\beta_R R \leq 1, \\
5(C_{19})^4-\frac{1}{2n}a_1c_0 - 4(C_{19})^4-\frac{1}{2n}a_1c_0 & \text{if } 2\beta_R R > 1.
\end{cases}
\]
Hence, by (111)

\[ y^* \leq -\frac{b_2 - \sqrt{(b_2)^2 - 4a_2c_2}}{2a_2} \quad \text{or} \quad y^* \geq -\frac{b_2 - \sqrt{(b_2)^2 - 4a_2c_2}}{2a_2}. \]

Therefore, either

\[ y^* \leq -\frac{b_2}{2a_2} = \begin{cases} \frac{C_{16}}{16C_{15}(C_1 R)^2} (\frac{\tilde{\beta}_R)^2(H_R^-)^{y+1} R^{q+1}}{\Theta^q}, & \text{if } 2\beta_R R \leq 1, \\ \frac{C_{16}}{2C_{15}(C_1 R)^2} \frac{(\tilde{\beta}_R)^2(H_R^-)^{y+1} R^q}{\Theta^q(\tilde{\beta}_R)}, & \text{if } 2\beta_R R > 1, \end{cases} \] (113)

or, by (112),

\[ y^* \geq \frac{\sqrt{4a_2 c_2}}{2a_2} = \begin{cases} (C_{19})^{-1} - \frac{1}{4\pi} \sqrt{\frac{c_0}{a_1}}, & \text{if } 2\beta_R R \leq 1 \\ (C_{19})^{-1} - \frac{1}{4\pi} \sqrt{\frac{c_0}{a_1}}, & \text{if } 2\beta_R R > 1 \end{cases} \] (114)

\[ = \begin{cases} \min\left(\sqrt{\frac{c_0}{a_1}}, -\frac{b_1}{\sqrt{a_1}}\right), & \text{if } 2\beta_R R \leq 1, \\ \min\left(\sqrt{\frac{c_0}{a_1}}, -\frac{b_1}{\sqrt{a_1}} \left(\frac{b_1}{\sqrt{a_1}}\right)^\frac{1}{2}(1-\frac{1}{2n+2})\right), & \text{if } 2\beta_R R > 1, \end{cases} \]

However, \( C_{1,R} \geq 2 \), for all \( R \leq \text{dist}(\tilde{x}, \partial \Omega) \) and for all \( \tilde{x} \in \Omega \). This excludes the occurrence of (113) for \( R \) sufficiently small. The smoothness of the function \( u \) therefore implies that (114) holds for all \( R \leq \frac{1}{2} \text{dist}(\tilde{x}, \partial \Omega) \), which yields

\[ (C_{1,R})^3 \geq \begin{cases} \frac{C_{16}}{16C_{15}} (\frac{\tilde{\beta}_R)^2(H_R^-)^{y+1} R^{q+1}}{\Theta^q}, & \text{if } 2(\beta_R) R \leq 1, \\ \frac{C_{16}}{2C_{15}} \frac{(\tilde{\beta}_R)^2(H_R^-)^{y+1} R^q}{\Theta^q(\tilde{\beta}_R)}, & \text{if } 2(\beta_R) R > 1; \end{cases} \]

that is,

\[ \left( \frac{M - m}{(\tilde{\beta}_R) R} \right)^3 \left( \frac{\tilde{\beta}_R}{(\beta_R)} \right)^3 \geq \begin{cases} \frac{C_{16}}{16C_{15}} (\frac{\tilde{\beta}_R)^2(H_R^-)^{y+1} R^{q+1}}{\Theta^q}, & \text{if } 2(\beta_R) R \leq 1 \\ \frac{C_{16}}{2C_{15}} \frac{(\tilde{\beta}_R)^2(H_R^-)^{y+1} R^q}{\Theta^q(\tilde{\beta}_R)}, & \text{if } 2(\beta_R) R > 1, \end{cases} \]

or equivalently,

\[ \left( \frac{M - m}{(\tilde{\beta}_R) R} \right)^3 \geq \begin{cases} \frac{C_{16}}{16C_{15}} \frac{\beta(\tilde{\beta}_R)^2(H_R^-)^{y+1} R^{q+1}}{\Theta^q(\tilde{\beta}_R)}, & \text{if } 2(\beta_R) R \leq 1, \\ \frac{C_{16}}{2C_{15}} \frac{\tilde{\beta}_R(\tilde{\beta}_R)^2(H_R^-)^{y+1} R^q}{\Theta^q(\tilde{\beta}_R)}, & \text{if } 2(\beta_R) R > 1. \end{cases} \]
Since \( \frac{M-m}{(\beta_R)R} \leq \frac{2(\tilde{\beta}_R)R}{(\beta_R)R} = 2 \), we obtain

\[
8 \geq \begin{cases} 
\frac{C_{16}}{16 C_{15}} \frac{(\beta_R)^{2q-3}(H_R)^{q+1}R^{q+1}}{\Theta^q}, & \text{if } 2 \beta_R R \leq 1, \\
\frac{C_{16}}{2 C_{15}} \frac{(\beta_R)^{2q-3}(H_R)^{q+1}R^q}{\Theta^q}, & \text{if } 2 \beta_R R > 1 
\end{cases}
\]

hence,

\[
\frac{(\tilde{\beta}_R)^{2q-3}}{\Theta^q} \frac{\tilde{\beta}_R}{(\beta_R)^{2q-3}} \leq \begin{cases} 
\frac{C_{16}}{16 C_{15}} \frac{(\beta_R)^{3}(H_R)^{q+1}R^{q+1}}{\Theta^q}, & \text{if } 2 \beta_R R \leq 1, \\
\frac{C_{16}}{2 C_{15}} \frac{(\beta_R)^{3}(H_R)^{q+1}R^q}{\Theta^q}, & \text{if } 2 \beta_R R > 1 
\end{cases}
\]  

We observe that (115) can be written equivalently as

\[
\frac{(\beta_R)^{2q-3}}{\Theta^q} \frac{\tilde{\beta}_R}{(\beta_R)^{2q-3}} \leq \begin{cases} 
\frac{C_{16}}{16 C_{15}} \frac{1}{(\beta_R)^{3}(H_R)^{q+1}R^{q+1}}, & \text{if } 2 \beta_R R \leq 1, \\
\frac{C_{16}}{2 C_{15}} \frac{1}{(\beta_R)^{3}(H_R)^{q+1}R^q}, & \text{if } 2 \beta_R R > 1 
\end{cases}
\]  

Suppose that conditions are imposed on \( u \) to ensure the existence of a constant \( \mu \), \( \frac{1}{2} < \mu < 1 \), together with two constants \( \Lambda_1 \) and \( \Lambda_2 \) which are invariant under rescaling and determined completely by \( n, H \) and \( R_0 \), such that (5) and (6) hold. We obtain from (6) and (108) that

\[
\beta_{R_0/2} \geq \frac{(2\Theta^\frac{1}{2})^\mu}{\Lambda_2}.
\]

From (5) (116) and (117) we obtain that

\[
(2)^{(4q-6)\mu} \left( \frac{\Lambda_1}{\Lambda_2} \right)^{2q-3} \left( \frac{4q-6}{2} \right)^{q-\frac{1}{2}} \left( \frac{4q-6}{2} \right)^{q-\frac{1}{2}} \leq \begin{cases} 
\frac{C_{16}}{2 C_{15}} \frac{1}{(\beta_R)^{3}(H_R)^{q+1}R^{q+1}}, & \text{if } 2(\beta_{R_0/2})R_0 \leq 1, \\
\frac{C_{16}}{2 C_{15}} \frac{1}{(\beta_R)^{3}(H_R)^{q+1}R^q}, & \text{if } 2(\beta_{R_0/2})R_0 > 1. 
\end{cases}
\]

Choose \( q \) sufficiently large, say \( q = q_\mu \), so that (9) holds. We obtain (8) from (9) and (119) with \( C^* = C_{15} \) and \( C^{**} = C_{16} \). This completes our proof of Main Theorem I.

## 3 A Proof of Lemmas 0.1-0.7

We prove in this section Lemmas 0.1-0.7 formulated below Main Theorem I.

### 3.1 A Proof of Lemmas 0.4-0.6

We first proceed to seek conditions which assures us of the fulfillment (5) and (6) for suitable constants \( \Lambda_1 \) and \( \Lambda_2 \).
Suppose that, for $\gamma \geq 0$, there exists a constant $\Lambda_\gamma$, invariant under rescaling, such that
\[
|D^2u(x)| \geq \Lambda_\gamma H \overline{R}|Du(x)|^{1+\gamma},
\] (119)
for points $x \in B_R$.

In contrast to (94), we now have
\[
\left(\frac{1}{|Du|}\right)' \leq -C_{20}H \overline{R},
\]
for some constant $C_{20}$ which is invariant under rescaling. Thus, in contrast to the estimates in Step 7.2.2, we now obtain
\[
\frac{1}{|Du(x_2)|^\gamma} - \frac{1}{|Du(x_1)|^\gamma} \leq -C_{20}H \overline{R}\text{dist}(x_1, x_2),
\]
for points $x_1$ and $x_2$ in $B_R$. Thus, if $\text{dist}(x_1, x_2)$ is so small that
\[
C_{20}H \overline{R}\text{dist}(x_1, x_2) \leq \frac{1}{|Du(x_1)|^\gamma},
\]
we have
\[
|Du(x_2)|^\gamma \geq \frac{1}{\sqrt{\frac{1}{|Du(x_1)|^\gamma} - C_{20}H \overline{R}\text{dist}(x_1, x_2)}}.
\]
Hence, if
\[
\left(1 - \left(\frac{1}{2}\right)^\gamma\right)\frac{1}{C_{20}H \overline{R}|Du(x_1)|^\gamma} \leq \text{dist}(x_1, x_2) \leq \frac{1}{C_{20}H \overline{R}|Du(x_1)|^\gamma},
\]
then
\[
\frac{|Du(x_2)|}{|Du(x_1)|} \geq \left(\frac{1}{C_{20}H \overline{R}\text{dist}(x_1, x_2)|Du(x_1)|^\gamma}\right)^{\frac{1}{\gamma}} \geq 2.
\]
Thus, under the hypothesis (119) in $B_R$, we have
\[
\frac{\overline{\beta}_R}{\beta_R} > (2)^{C_{20}RH \overline{R}^\gamma(\beta_R)^\gamma} > C_{20}RH \overline{R}^\gamma(\beta_R)^\gamma,
\] (120)
for $0 \leq R \leq R_0$. Thus, if in (119) the constant $\Lambda_\gamma$ is determined completely by $n$ and $H$, we are allowed to take in (5) $\Lambda_1 = C_{20}H \overline{R}^\gamma R_0^2$, which is (11) and completes the proof of Lemma 0.4.

3.1.1. A Proof of Lemma 0.5. Setting $C_\gamma$ as in (11), we observe that if (119) were valid at all points in $B_{R_1}(x_1)$ with
\[
\frac{C_\gamma}{C_{20}H \overline{R}^\gamma |Du(x_1)|^\gamma} \leq R_1 \leq \text{dist}(x_1, \partial\Omega),
\]
the value of the gradient $|Du|$ would become unbounded at points with distance from the point $x_1$ greater than or equal to $\frac{C_\gamma}{C_\mathcal{Y}_1|Du(x_1)|^\gamma}$, and this would contradict the inner regularity of the function $u$. From this, we obtain (13) in case (119) holds at some point with distance from the point $x_1$ no less than $R_1$. This proves Lemma 0.5.

3.1.2. A proof of Lemma 0.6. Under the hypotheses on $u$ in Lemma 0.6, suppose that at $x^* \in B_{R_0/2}$, we have

$$\tilde{\beta}_{R_0/2} = |Du(x^*)|.$$

If $x^* \in B_{R_0/2} \setminus S_\mu$, $\frac{1}{2} < \mu < 1$, where we have

$$|D^2u(x)| \leq \Lambda_\mu H |Du(x)|^{2-\mu},$$

for some constant $\Lambda_\mu$ invariant under rescaling and determined completely by $n$ and $H$, then it is obvious that (6) holds with some $\Lambda_2$. On the other hand, if $x^* \in S_\mu$, then (13) indicates that (6) also holds with some $\Lambda_2, R_0/2 = \Lambda_2$.

3.2. A Proof of Lemma 0.7. We now proceed to identify situations where the condition (119) (i.e. (10) holds for suitable $\gamma$. For this, we restrict our attention to the case where $n = 2$. We have

$$2H = \frac{(1 + (u_y)^2)u_{xx} - 2u_xu_yu_{xy} + (1 + (u_x)^2)u_{yy}}{(\sqrt{1 + |Du|^2})^3}, \quad (121)$$

and

$$K = \frac{u_{xx}u_{yy} - (u_{xy})^2}{(1 + |Du|^2)^2}. \quad (122)$$

Hence

$$u_{yy} = \frac{K}{u_{xx}} (1 + |Du|^2)^2 + \frac{(u_{xy})^2}{u_{xx}}, \quad \text{if } u_{xx} \neq 0.$$

Thus, we are allowed to rewrite (121) as

$$(1 + (u_y)^2)(u_{xx})^2 - 2(\sqrt{1 + |Du|^2})^3 Hu_{xx} + (1 + (u_x)^2)(1 + |Du|^2)^2 K = -(u_{xy})^2. \quad (123)$$

We observe that, since $K \leq H^2$,

$$1 + |Du|^2)^2 H^2 \geq (1 + (u_y)^2)(1 + (u_x)^2)(1 + |Du|^2)^2 K.$$

Suppose

$$u_{xx} \geq 0. \quad (124)$$

This and (123) yield that

$$u_{xx} \geq \frac{(\sqrt{1 + |Du|^2})^3 H - \sqrt{(1 + |Du|^2) H^2 - (1 + (u_y)^2)(1 + (u_x)^2)(1 + |Du|^2)^2 K}}{1 + (u_y)^2}. \quad (125)$$
and

\[ u_{xx} \leq \frac{\sqrt{1+|Du|^2}^3 H + \sqrt{1+(Du)^2}^3 H^2 - (1+(u_y)^2)(1+(u_x)^2)(1+|Du|^2)K}{1 + (u_y)^2}. \]  \hspace{1cm} (126)

If we make the additional assumption that \( K \geq 0 \), then (126) yields (14).

### 3.3. A Proof of Lemma 0.1.

Choosing an arbitrary point \((x_0, y_0) \in B_{R_0}\), possibly after an orthonormal transformation, we have

\[ u_x u_y u_{xy} (x_0) = 0. \]  \hspace{1cm} (127)

Then, by (121), we have

\[ (1 + (u_x)^2) u_{xx} + (1 + (u_x)^2) u_{yy} = 2(1 + |Du|^2)^3 H. \]  \hspace{1cm} (128)

Assume without loss of generality that

\[ K < 0, \quad u_{xx} \geq 0, \quad \text{and} \quad u_{yy} \leq 0. \]  \hspace{1cm} (129)

This and (128) then yield

\[ \frac{|u_{xx}|}{|u_{yy}|} > \frac{1 + (u_x)^2}{1 + (u_y)^2}. \]  \hspace{1cm} (130)

Suppose first that we have also \(|u_y| \geq |u_x|\). Then we have \( \frac{1 + (u_x)^2}{1 + (u_y)^2} > \frac{(u_x)^2}{(u_y)^2} \). This and (130) yield

\[ u_{xx} (u_y)^2 \geq u_{yy} (u_x)^2. \]  \hspace{1cm} (131)

Inserting this into (128), we obtain \( \Delta u \leq 2(\sqrt{1 + |Du|^2}^3 H \). Hence, (2) is valid in the second case indicated in Lemma 0.1.

Suppose instead we have \(|u_y| < |u_x|\). Then, to find out conditions which assures us of the truth of (2), we may assume without loss of generality that (131) fails to hold, that is \(|u_{yy}| \geq \frac{(u_x)^2}{(u_y)^2} u_{xx}\). Thus, we have \( \Delta u \leq \left( 1 - \frac{(u_y)^2}{(u_x)^2} \right) u_{xx}\). Hence, by (125), we have

\[ \Delta u \leq 2 \left( 1 - \frac{(u_y)^2}{(u_x)^2} \right) (1 + |Du|^2)^3 H + \left( 1 - \frac{(u_y)^2}{(u_x)^2} \right) (1 + |Du|^2) \sqrt{|K|}. \]

Hence, for (2) to be true, it suffices to impose the conditions

\[ \left( 1 - \frac{(u_y)^2}{(u_x)^2} \right) \frac{1}{1 + (u_y)^2} \leq \frac{1}{2}, \quad \text{and} \quad \left( 1 - \frac{(u_y)^2}{(u_x)^2} \right) \frac{\sqrt{|K|}}{\sqrt{1 + (u_y)^2}} \leq H. \]

This proves Lemma 0.1.
3.4. A proof of Lemma 0.2.

By (123), we have
\[(u_{xy})^2 = -K(1 + |Du|^2)^2 + u_{xx}u_{yy}.\]

By (122), we may assume without loss of generality that (129) holds. And hence
\[|u_{xy}| \leq (1 + |Du|^2)\sqrt{|K|}, \quad \text{if } K < 0.
\]

This and (122) yield
\[
\Delta u = 2(\sqrt{1 + |Du|^2})^3H - (u_y)^2u_{xx} - (u_x)^2u_{yy} + 2u_xu_yu_{xy} \tag{132}
\]
\[
\leq 2(\sqrt{1 + |Du|^2})^3H + 2|u_x||u_y|(1 + |Du|^2)\sqrt{|K|} - (u_x)^2u_{yy}.
\]

Suppose \(\Delta u \geq 0\). Then \(|u_{yy}| \leq (1 + |Du|^2)\sqrt{|K|}\). This and (132) yield
\[
\Delta u \leq 2(\sqrt{1 + |Du|^2})^3H + 2|u_x||u_y|(1 + |Du|^2)\sqrt{|K|} + (u_x)^2(1 + |Du|^2)\sqrt{|K|}.
\]

Moreover, we have
\[|u_{xx}|u_{yy} \leq (1 + |Du|^2)^2|K|, \quad \text{if } K < 0.
\]

Hence,
\[
|u_{xx}| + |u_{yy}| = \sqrt{(\Delta u)^2 + 4|u_{xx}u_{yy}|} \tag{133}
\leq 2(\sqrt{1 + |Du|^2})^3H + 2|u_x||u_y|(1 + |Du|^2)\sqrt{|K|}
+ (u_x)^2(1 + |Du|^2)\sqrt{|K|} + (1 + |Du|^2)\sqrt{|K|}.
\]

Suppose instead that \(\Delta u \leq 0\). Then we have \(u_{xx} \leq (1 + |Du|^2)^2\sqrt{|K|}\) and thus
\[
-(1 + (u_x)^2)u_{yy} = (1 + (u_y)^2)u_{xx} - 2(\sqrt{1 + |Du|^2})^3H - 2u_xu_yu_{xy} \tag{134}
\leq (1 + (u_y)^2)(1 + |Du|^2)\sqrt{|K|} - 2(\sqrt{1 + |Du|^2})^3H
+ 2|u_x||u_y|(1 + |Du|^2)\sqrt{|K|}.
\]

We obtain Lemma 0.2 from (123) and (124).

3.5. A proof of Lemma 0.3.

We have
\[
\left(\frac{u_x}{\sqrt{1 + |Du|^2}}\right)_x = u_{xx} - \frac{u_x(u_{xx}u_{xx} + u_yu_{xy})}{(\sqrt{1 + |Du|^2})^3} \tag{135}
\leq \frac{(1 + (u_y)^2)u_{xx} - u_xu_yu_{xy}}{(\sqrt{1 + |Du|^2})^3}.
\]
Likewise,
\begin{align*}
\left( \begin{array}{c}
\frac{u_x}{\sqrt{1 + |Du|^2}} \\
\frac{u_y}{\sqrt{1 + |Du|^2}}
\end{array} \right)_y &= \frac{(1 + (u_y)^2)u_{xy} - u_x u_y u_{yy}}{(\sqrt{1 + |Du|^2})^3}, \\
\left( \begin{array}{c}
\frac{u_x}{\sqrt{1 + |Du|^2}} \\
\frac{u_y}{\sqrt{1 + |Du|^2}}
\end{array} \right)_x &= \frac{(1 + (u_x)^2)u_{xy} - u_x u_y u_{xx}}{(\sqrt{1 + |Du|^2})^3},
\end{align*}
(136)
and
\begin{align*}
\left( \begin{array}{c}
\frac{u_y}{\sqrt{1 + |Du|^2}} \\
\frac{u_x}{\sqrt{1 + |Du|^2}}
\end{array} \right)_y &= \frac{(1 + (u_x)^2)u_{yy} - u_x u_y u_{xy}}{(\sqrt{1 + |Du|^2})^3}.
\end{align*}
(138)

If, at some point \((x_0, y_0)\), we have
\[ u_x(x_0, y_0) = 0, \quad \text{or} \quad u_y(x_0, y_0) = 0, \]
(139)
and if we make the additional assumption that \(K > 0\), then the right hand side of (135) and (138) takes positive value at the point \((x_0, y_0)\) and the right hand side of (136) and (137) takes nonnegative value at the point \((x_0, y_0)\). Hence, denoting \(e_1 = (1, 0)\) and \(e_2 = (0, 1)\), we have
\[ \left. \left\langle D \left( \frac{\langle Du, e_1 \rangle}{\sqrt{1 + |Du|^2}} \right), e_1 \right\rangle \right|_{(x_0, y_0)} = \left. \left( \frac{u_x}{\sqrt{1 + |Du|^2}} \right) \right|_{(x_0, y_0)} > 0, \]
and
\[ \left. \left\langle D \left( \frac{\langle Du, e_2 \rangle}{\sqrt{1 + |Du|^2}} \right), e_1 \right\rangle \right|_{(x_0, y_0)} = \left. \left( \frac{u_y}{\sqrt{1 + |Du|^2}} \right) \right|_{(x_0, y_0)} \geq 0. \]
(140)

Under the assumption that \(K > 0\), we have \(\frac{\partial |Du|}{\partial r} > 0\); hence,
\[ \left. \left\langle D \left( \frac{\langle Du \rangle}{|Du|} \right), e_1 \right\rangle \right|_{(x_0, y_0)} > 0, \]
(141)
and
\[ \left. \left\langle D \left( \frac{\langle Du \rangle}{|Du|} \right), e_2 \right\rangle \right|_{(x_0, y_0)} \geq 0. \]
(142)

Therefore, for \(y^* \in \mathbb{R}\), each of the sets
\[ S^+_y = \{ \frac{Du}{|Du|}(x, y^*) : x > 0 \} \quad \text{and} \quad S^-_y = \{ \frac{Du}{|Du|}(x, y^*) : x < 0 \} \]
is included in one of the half-spaces
\[ \{ x \in \mathbb{R}, y \geq 0 \}, \quad \{ x \in \mathbb{R}, y \leq 0 \}. \]
Furthermore, we have
\[ u_y(x, y_0) = 0, \quad \text{for all } x \geq x_0, \quad \text{if } u_y(x_0, y_0) = 0. \] (143)

Likewise, since there also hold
\[ \left\langle D\left(\frac{\langle Du, e_1 \rangle}{\sqrt{1 + |Du|^2}}\right), e_2\right\rangle_{(x_0, y_0)} = \left(\frac{u_x}{\sqrt{1 + |Du|^2}}\right)_y(x_0, y_0) > 0, \] (144)
and
\[ \left\langle D\left(\frac{\langle Du, e_2 \rangle}{\sqrt{1 + |Du|^2}}\right), e_2\right\rangle_{(x_0, y_0)} = \left(\frac{u_y}{\sqrt{1 + |Du|^2}}\right)_y(x_0, y_0) \geq 0, \] (145)
we have also
\[ u_x(x, y_0) = 0, \quad \text{for all } y \geq y_0, \quad \text{if } u_x(x_0, y_0) = 0. \] (146)

We observe that if (135) and
\[ u_{xy}(x_0, y_0) \neq 0 \]
occur simultaneously, then the strict inequality takes place in (140), and so does in (142). This then tells us that each of the sets \( S_{y}^+ \) and \( S_{y}^- \) considered above is included in one of the quadrants
\[ \{ x \geq 0, y \geq 0 \}, \{ x \geq 0, y \leq 0 \}, \{ x \leq 0, y \leq 0 \}, \{ x \leq 0, y \geq 0 \}. \]
Thus, to prove Lemma 3, we proceed to characterize the situation where (135) and
\[ u_{xy}(x_0, y_0) = 0 \]
occur simultaneously. By (140) and (144), we know that at such a point \( Du \) must point into the outside of the ball \( B_{R_0} \) and the level curve passing through it is curved concave downward with respect to the vector \( \frac{Du}{|Du|} \); hence on such a level curve of the function \( u \) there are at most four vertices at each of which (135) holds. Were there four such vertices on a level curve, then a critical point of the function \( u \) would be enclosed inside the ball \( B_{R_0} \), contradicting to the fact that \( \beta_{R_0/2} > 0 \). Were there three such vertices on a level curve \( \Gamma \), then, by the smoothness of the function \( u \), neighboring level curves would also have three vertices and by (143), (146) and the assumption that only three vertices were on the level curve \( \Gamma \), we know that all these vertices on neighboring level curves would be lined up and would either have \( x_0 \) as the \( x \)-coordinates or have \( y_0 \) as the \( y \)-coordinate, which would yield finally that \( (x_0, y_0) \) is a critical point. This contradiction yields that there can at most be two such vertices on such each level curve of the function \( u \) and completes our proof of Lemma 0.3.
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