Prime and Irreducible Ideals
in Subtraction Algebras

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Abstract

The notion of prime and irreducible ideals in subtraction algebras is introduced. Characterizations of a prime ideal are given. Extension property of a prime ideal is established. Conditions for an ideal to be an irreducible ideal are given.

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1 Introduction

B. M. Schein [4] considered systems of the form \((\Phi; \circ, \setminus)\), where \(\Phi\) is a set of functions closed under the composition “\(\circ\)” of functions (and hence \((\Phi; \circ)\) is a function semigroup) and the set theoretic subtraction “\(\setminus\)” (and hence \((\Phi; \setminus)\) is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [5] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [2] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results.
In this paper, we investigate some properties of ideals in subtraction algebras. We give a condition for a nonempty subset to be an ideal. We introduce the notion of prime and irreducible ideals of a subtraction algebra, and we give a characterization of a prime ideal. We also provide a condition for an ideal to be a prime/irreducible ideal.

2 Preliminaries

By a subtraction algebra we mean an algebra \((X; -)\) with a single binary operation “−” that satisfies the following identities: for any \(x, y, z \in X\),

(S1) \(x - (y - x) = x\);

(S2) \(x - (x - y) = y - (y - x)\);

(S3) \((x - y) - z = (x - z) - y\).

The last identity permits us to omit parentheses in expressions of the form \((x - y) - z\). The subtraction determines an order relation on \(X\): \(a \leq b \iff a - b = 0\), where 0 = \(a - a\) is an element that does not depend on the choice of \(a \in X\). The ordered set \((X; \leq)\) is a semi-Boolean algebra in the sense of \([1]\), that is, it is a meet semilattice with zero 0 in which every interval \([0, a]\) is a Boolean algebra with respect to the induced order. Here \(a \land b = a - (a - b)\); the complement of an element \(b \in [0, a]\) is \(a - b\); and if \(b, c \in [0, a]\), then

\[
    b \lor c = (b' \land c')' = a - ((a - b) \land (a - c)) = a - ((a - b) - ((a - b) - (a - c))).
\]

In a subtraction algebra, the following are true (see \([2]\)):

(p1) \((x - y) - y = x - y\).

(p2) \(x - 0 = x\) and \(0 - x = 0\).

(p3) \((x - y) - x = 0\).

(p4) \(x - (x - y) \leq y\).

(p5) \((x - y) - (y - x) = x - y\).

(p6) \(x - (x - (x - y)) = x - y\).

(p7) \((x - y) - (z - y) \leq x - z\).

(p8) \(x \leq y\) if and only if \(x = y - w\) for some \(w \in X\).

(p9) \(x \leq y\) implies \(x - z \leq y - z\) and \(z - y \leq z - x\) for all \(z \in X\).

(p10) \(x, y \leq z\) implies \(x - y = x \land (z - y)\).
3 Main Results.

Definition 3.1 (Jun et al. [2]). A nonempty subset $A$ of a subtraction algebra $X$ is called an ideal of $X$ if it satisfies

(I1) $0 \in A$

(I2) $y \in A$ and $x - y \in A$ imply $x \in A$ for all $x, y \in X$.

Lemma 3.2. Let $A$ be an ideal of a subtraction algebra $X$. If $x \leq y$ and $y \in A$, then $x \in A$.

Proof. It is straightforward. \hfill \Box

Lemma 3.3. In a subtraction algebra $X$, the following inequality is valid.

$$(x \wedge y) - (x \wedge z) \leq x \wedge (y - z).$$

Proof. For any $x, y, z \in X$, we have

$$(x \wedge y) - (x \wedge z) = (x - (x - y)) - (x - (x - z))$$

$$\leq (x - z) - (x - y) \leq y - z. \tag{3.1}$$

On the other hand,

$$(x \wedge y) - (x \wedge z) \leq x \wedge y \leq x. \tag{3.2}$$

Combining (3.1) and (3.2), we have $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$. \hfill \Box

Theorem 3.4. Let $A$ be an ideal of a subtraction algebra $X$. For any $w \in X$, the set

$$A_w^\wedge := \{ x \in X \mid w \wedge x \in A \}$$

is an ideal of $X$ containing $A$.

Proof. Since $w \wedge 0 = w - (w - 0) = w - w = 0 \in A$, we have $0 \in A_w^\wedge$. Let $x, y \in X$ be such that $y \in A_w^\wedge$ and $x - y \in A_w^\wedge$. Then $w \wedge y \in A$ and $w \wedge (x - y) \in A$. Since $(w \wedge x) - (w \wedge y) \leq w \wedge (x - y)$ by Lemma 3.3, it follows from Lemma 3.2 and (I2) that $w \wedge x \in A$, that is, $x \in A_w^\wedge$. Hence $A_w^\wedge$ is an ideal of $X$. Now let $x \in A$. Since $w \wedge x \leq x$ by (p4), we have $w \wedge x \in A$ by Lemma 3.3. Therefore $x \in A_w^\wedge$, and so $A \subseteq A_w^\wedge$. This completes the proof. \hfill \Box

Theorem 3.5. Let $A$ be a nonempty subset of a subtraction algebra $X$ such that

(i) $x \in A$ and $y \leq x$ imply $y \in A$. 


(ii) For \(x, y \in A\), there exists \(z \in A\) such that \(x \leq z\) and \(y \leq z\).

Then \(A\) is an ideal of \(X\).

Proof. Since \(A\) is nonempty, we have \(0 \in A\) by (i) and (p2). Let \(x, y \in X\) be such that \(y \in A\) and \(x - y \in A\). Then, by (ii), there exists \(z \in A\) such that \(y \leq z\) and \(x - y \leq z\). It follows from (p2) and [2, Lemma 3.10] that

\[
x - z = (x - z) - 0 = (x - z) - (y - z) = (x - y) - z = 0
\]

so that \(x \leq z\). Since \(z \in A\), it follows from (i) that \(x \in A\). Hence \(A\) is an ideal of \(X\). \(\square\)

Definition 3.6. Let \(X\) be a subtraction algebra. A prime ideal of \(X\) is defined to be an ideal \(P\) of \(X\) such that if \(x \land y \in P\) then \(x \in P\) or \(y \in P\).

Theorem 3.7. Let \(P\) be an ideal of a subtraction algebra \(X\). Then the following are equivalent.

(i) \(P\) is a prime ideal of \(X\).

(ii) For any ideals \(A\) and \(B\) of \(X\), \(A \land B \subset P\) implies \(A \subset P\) or \(B \subset P\), where \(A \land B := \{a \land b \mid a \in A, b \in B\}\).

Proof. Suppose that \(P\) is a prime ideal of \(X\) such that \(A \land B \subset P\), where \(A\) and \(B\) are ideals of \(X\). Assume that \(A \not\subset P\) and \(B \not\subset P\). Then there exist \(x \in A \setminus P\) and \(y \in B \setminus P\), and so \(x \land y \in A \land B \subset P\). Since \(P\) is prime, it follows that \(x \in P\) or \(y \in P\), which is a contradiction. Consequently, \(A \land B \subset P\) implies \(A \subset P\) or \(B \subset P\). Conversely assume that for any ideals \(A\) and \(B\) of \(X\), \(A \land B \subset P\) implies \(A \subset P\) or \(B \subset P\). Let \(x, y \in X\) be such that \(x \land y \in P\). Note from [2, Theorem 3.4] that 

\[
(x) := \{a \in X \mid a \leq x\}
\]

and 

\[
(y) := \{b \in X \mid b \leq y\}
\]

are ideals of \(X\). Let \(a \in (x)\) and \(b \in (y)\). Then \(a \leq x\) and \(b \leq y\). It follows from \(a \land b \leq a, b\) that \(a \land b \leq x\) and \(a \land b \leq y\) so that \(a \land b \leq x \land y\). Since \(P\) is an ideal and \(x \land y \in P\), by Lemma 3.2 we get \(a \land b \in P\). Therefore \((x) \land (y) \subset P\), which implies that \((x) \subset P\) or \((y) \subset P\) by hypothesis. In particular, \(x \in P\) or \(y \in P\), and thus \(P\) is a prime ideal of \(X\). \(\square\)

Theorem 3.8. Every prime ideal of a subtraction algebra is a maximal ideal, that is, every prime ideal is not contained in any other proper ideal.

Proof. Let \(P\) be a prime ideal of a subtraction algebra \(X\). Suppose that \(P\) is not maximal. Then there exists a proper ideal \(A\) of \(X\) such that \(P \subset A\) and \(P \neq A\). Let \(y \in X\) and consider \(x \in A \setminus P\). Then

\[
x \land (y - x) = x - (x - (y - x)) = x - x = 0 \in P,
\]

and so \(y - x \in P\) because \(P\) is a prime ideal and \(x \notin P\). It follows from (I2) that \(y \in A\), that is, \(X = A\), which contradicts the assumption that \(A\) is proper. Hence \(P\) is a maximal ideal of \(X\). \(\square\)
Proposition 3.9. If $A$ is a maximal ideal of a subtraction algebra $X$, then $x - y \in A$ or $y - x \in A$ for all $x, y \in X$.

Proof. Let $x, y \in A$. Then $x - y \in A$ since $A$ is an ideal. Assume that $x \in A$ and $y \notin X \setminus A$. Since $x - y \leq x$, it follows from Lemma 3.3 that $x - y \in A$. Similarly, if $x \in X \setminus A$ and $y \in A$, then $y - x \in A$. Let $x, y \in X \setminus A$ and assume that $y - x \notin A$. Then the set

$$Q := \{ z \in X \mid z - (y - x) \in A \}$$

is the least ideal of $X$ containing $A$ and $y - x$ (see [2, Theorem 3.11]). Since $y - x \notin A$, we have $A \neq Q$, and so $Q = X$ because $A$ is maximal. Therefore $x - y \in Q$, that is, $(x - y) - (y - x) \in A$. Using (p2), (p3), (S3), and [2, Lemma 3.10], we get

$$x - y = (x - y) - 0 = (x - y) - ((y - x) - y) = (x - (y - x)) - y = (x - y) - (y - x) \in A.$$ 

This completes the proof.

Corollary 3.10. If $A$ is a prime ideal of a subtraction algebra $X$, then $x - y \in A$ or $y - x \in A$ for all $x, y \in X$.

Now we consider the converse of Corollary 3.10.

Theorem 3.11. Let $A$ be an ideal of a subtraction algebra $X$ such that $x - y \in A$ or $y - x \in A$ for all $x, y \in X$. Then $A$ is a prime ideal of $X$.

Proof. Let $A$ be an ideal of $X$ such that $x - y \in A$ or $y - x \in A$ for all $x, y \in X$. Assume that $x \land y \in A$. If $x - y \in A$, then $x - (x - y) = x \land y \in A$ and so $x \in A$ by (I2). If $y - x \in A$, then $y - (y - x) = x - (x - y) = x \land y \in A$. It follows from (I2) that $y \in A$. Hence $A$ is a prime ideal of $X$.

Hence we know that the notion of prime ideals and maximal ideals in a subtraction algebra coincide, and we restate it as a theorem.

Theorem 3.12. Let $A$ be an ideal of a subtraction algebra $X$. Then the following are equivalent.

(i) $A$ is a prime ideal.

(ii) $A$ is a maximal ideal.

(iii) $x - y \in A$ or $y - x \in A$ for all $x, y \in X$.

Using Theorem 3.12, we can establish the extension property of prime ideals.

Corollary 3.13. Let $A$ and $B$ be ideals of a subtraction algebra $X$ such that $A \subset B$. If $A$ is a prime ideal of $X$, then so is $B$. 
Definition 3.14. An ideal $A$ of a subtraction algebra $X$ is said to be irreducible if for any ideals $C$ and $D$ of $X$, $A = C \cap D$ implies $A = C$ or $A = D$.

Theorem 3.15. Let $A$ be an ideal of a subtraction algebra $X$ and let $w \in X \setminus A$. Then there exists an irreducible ideal $M$ of $X$ such that $A \subset M$ and $w \notin M$.

Proof. Let $\mathcal{A} := \{I \mid I$ is an ideal of $X$, $A \subset I$, $w \notin I\}$. Note that any chain of elements in $\mathcal{A}$ has an upper bound. Thus, by Zorn’s Lemma, there exists a maximal element $M$ in $\mathcal{A}$. Then $A \subset M$ and $w \notin M$. Let $C$ and $D$ be ideals of $X$ such that $M = C \cap D$. Assume that $M \neq C$ and $M \neq D$. By the maximality of $M$, we have $w \in C$ and $w \in D$, that is, $w \in C \cap D$. Hence $M \neq C \cap D$, a contradiction. Therefore $M$ is an irreducible ideal of $X$.

Theorem 3.16. Let $A$ be an ideal of a subtraction algebra $X$. Assume that for any $x, y \in X \setminus A$, there exists $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$. Then $A$ is an irreducible ideal of $X$.

Proof. Suppose that $A$ is not an irreducible ideal of $X$. Then there are two ideals $C$ and $D$ of $X$ such that $A = C \cap D$, $A \neq C$, and $A \neq D$. Let $x \in C \setminus A$ and $y \in D \setminus A$. Using the assumption, there exists $z \in X \setminus A$ such that $z \leq x$ and $z \leq y$. Since $x \in C$ and $y \in D$, it follows from Lemma 3.2 that $z \in C \cap D = A$, which is a contradiction. Hence $A$ is an irreducible ideal of $X$.

References


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