Arithmetic Neighbourhoods of Numbers

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Abstract

Let \( K \) be a ring and let \( A \) be a subset of \( K \). We say that a map \( f : A \rightarrow K \) is arithmetic if it satisfies the following conditions: if \( 1 \in A \) then \( f(1) = 1 \), if \( a, b \in A \) and \( a + b \in A \) then \( f(a + b) = f(a) + f(b) \), if \( a, b \in A \) and \( a \cdot b \in A \) then \( f(a \cdot b) = f(a) \cdot f(b) \). We call an element \( r \in K \) arithmetically fixed if there is a finite set \( A \subseteq K \) (an arithmetic neighbourhood of \( r \) inside \( K \)) with \( r \in A \) such that each arithmetic map \( f : A \rightarrow K \) fixes \( r \), i.e. \( f(r) = r \). We prove: for infinitely many integers \( r \) for some arithmetic neighbourhood of \( r \) inside \( \mathbb{Z} \) this neighbourhood is a neighbourhood of \( r \) inside \( \mathbb{R} \) and is not a neighbourhood of \( r \) inside \( \mathbb{Z}[\sqrt{-1}] \); for infinitely many integers \( r \) for some arithmetic neighbourhood of \( r \) inside \( \mathbb{Z} \) this neighbourhood is not a neighbourhood of \( r \) inside \( \mathbb{Q} \); if \( K = \mathbb{Q}(\sqrt{5}) \) or \( K = \mathbb{Q}(\sqrt{33}) \), then for infinitely many rational numbers \( r \) for some arithmetic neighbourhood of \( r \) inside \( \mathbb{Q} \) this neighbourhood is not a neighbourhood of \( r \) inside \( K \); for each \( n \in (\mathbb{Z}\cap[3, \infty))\backslash\{2^2, 2^3, 2^4, \ldots\} \) there exists a finite set \( J(n) \subseteq \mathbb{Q} \) such that \( J(n) \) is a neighbourhood of \( n \) inside \( \mathbb{R} \) and \( J(n) \) is not a neighbourhood of \( n \) inside \( \mathbb{C} \).

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Let \( K \) be a ring and let \( A \) be a subset of \( K \). We say that a map \( f : A \rightarrow K \) is arithmetic if it satisfies the following conditions:

1. If \( 1 \in A \) then \( f(1) = 1 \),
2. If \( a, b \in A \) and \( a + b \in A \) then \( f(a + b) = f(a) + f(b) \),
3. If \( a, b \in A \) and \( a \cdot b \in A \) then \( f(a \cdot b) = f(a) \cdot f(b) \).

We call an element \( r \in K \) arithmetically fixed if there is a finite set \( A \subseteq K \) (an arithmetic neighbourhood of \( r \) inside \( K \)) with \( r \in A \) such that each arithmetic
map \( f : A \rightarrow K \) fixes \( r \), i.e. \( f(r) = r \). All previous articles on arithmetic neighbourhoods ([15], [5], [16]) dealt with a description of a situation where for an element in a field there exists an arithmetic neighbourhood. If \( K \) is a field, then any \( r \in K \) is arithmetically fixed if and only if \( \{ r \} \) is existentially first-order definable in the language of rings without parameters ([16]). Therefore, presentation of the arithmetic neighbourhood of the element \( r \) belonging to the field \( K \) is the simplest way of expression that \( \{ r \} \) is existentially definable in \( K \).

We want to find integers \( r \) with property (4), integers \( r \) with property (5), and rational numbers \( r \) with property (6).

(4) Each arithmetic neighbourhood of \( r \) inside \( \mathbb{Z} \) is also a neighbourhood of \( r \) inside each ring extending \( \mathbb{Z} \).
(5) Each arithmetic neighbourhood of \( r \) inside \( \mathbb{Z} \) is also a neighbourhood of \( r \) inside \( \mathbb{Q} \).
(6) Each arithmetic neighbourhood of \( r \) inside \( \mathbb{Q} \) is also a neighbourhood of \( r \) inside each ring extending \( \mathbb{Q} \).

Obviously, condition (4) implies condition (5).

By condition (1) for any ring \( K \) each arithmetic neighbourhood of 1 inside \( K \) is also a neighbourhood of 1 inside each ring extending \( K \). Since \( 0+0 = 0 \), by condition (2) for any ring \( K \) each arithmetic neighbourhood of 0 inside \( K \) is also a neighbourhood of 0 inside each ring extending \( K \).

We prove that for any ring \( K \) with \( 2 \neq 0 \) each arithmetic neighbourhood of 2 inside \( K \) is also a neighbourhood of 2 inside each ring \( L \) extending \( K \). Assume that \( A \) is an arithmetic neighbourhood of 2 inside \( K \) and \( f : A \rightarrow L \) is an arithmetic map. Then \( 1 \in A \), because in the opposite case the arithmetic map \( A \rightarrow \{0\} \) moves 2, which is impossible. Since \( f \) satisfies conditions (1) and (2), we get \( f(2) = f(1+1) = f(1) + f(1) = 1 + 1 = 2 \).

We prove that for any ring \( K \) with \( \frac{1}{2} \in K \) each arithmetic neighbourhood of \( \frac{1}{2} \) inside \( K \) is also a neighbourhood of \( \frac{1}{2} \) inside each ring \( L \) extending \( K \). Assume that \( A \) is an arithmetic neighbourhood of \( \frac{1}{2} \) inside \( K \) and \( f : A \rightarrow L \) is an arithmetic map. Then \( 1 \in A \), because in the opposite case the arithmetic map \( A \rightarrow \{0\} \) moves \( \frac{1}{2} \), which is impossible. Since \( f \) satisfies conditions (1) and (2), we get \( 1 = f(1) = f(\frac{1}{2} + \frac{1}{2}) = f(\frac{1}{2}) + f(\frac{1}{2}) \). Hence, \( f(\frac{1}{2}) = \frac{1}{2} \).

The above results imply that the numbers \( r = 1, r = 0, r = 2 \) satisfy conditions (4)–(6), and \( r = \frac{1}{2} \) satisfies condition (6).

Let \( K \) be an algebraically closed field and \( r \in K \) is arithmetically fixed. Then \( r \) belongs to the prime field in \( K \), see [16], cf. [15]. Let \( A = \{x_1, \ldots, x_n\} \) be a neighbourhood of \( r \), \( x_i \neq x_j \) if \( i \neq j \), and \( x_1 = r \). We choose all formulae \( x_i = 1 \ (1 \leq i \leq n) \), \( x_i + x_j = x_k \), \( x_i \cdot x_j = x_k \ (1 \leq i \leq j \leq n, 1 \leq k \leq n) \).
that are satisfied in A. Joining these formulae with conjunctions we get some formula \( \Phi \). Let \( V \) denote the set of variables in \( \Phi \). Since \( A \) is a neighbourhood of \( r \) inside \( K \), we have
\[
K \models \ldots \forall x_1 \ldots (\Phi \Rightarrow x_1 = r)
\]
\( x_s \in \{x_1\} \cup V \)

Of course, \( \{x_1\} \cup V = V \) ([15, the proof of Theorem 2] and [16, the proof of Theorem 1]) but this equality will not be used later.

**Proposition 1.** Let \( K \) be an algebraically closed field and \( r \in K \) is arithmetically fixed. Then each arithmetic neighbourhood of \( r \) inside \( K \) is also a neighbourhood of \( r \) inside each integral domain \( D \) extending \( K \).

**Proof.** We give a model-theoretic proof, an alternative proof follows from Hilbert’s Nullstellensatz. Let \( A \) be a neighbourhood of \( r \) inside \( K \). Let \( D_1 \) denote the algebraic closure of the fraction field of \( D \). It suffices to prove that \( A \) is a neighbourhood of \( r \) inside \( D_1 \). Since \( K \) is a subfield of \( D_1 \) and every embedding between algebraically closed fields is elementary ([6, pp. 103 and 57], we obtain
\[
D_1 \models \ldots \forall x_1 \ldots (\Phi \Rightarrow x_1 = r)
\]
\( x_s \in \{x_1\} \cup V \)

It implies that \( A \) is a neighbourhood of \( r \) inside \( D_1 \).

\( \square \)

Let \( \mathcal{T} \) denote the elementary theory of integral domains of characteristic 0.

**Proposition 2.** Let \( K \) be an algebraically closed field that extends \( \mathbb{Q} \) and \( r \in K \) is arithmetically fixed.

(I) If \( r = 0 \), then
\[
\mathcal{T} \models \ldots \forall x_1 \ldots (\Phi \Rightarrow x_1 = 0)
\]
\( x_s \in \{x_1\} \cup V \)

(II) If \( r = \frac{k}{w} \) for some \( k, w \in \{1, 2, 3, \ldots\} \), then
\[
\mathcal{T} \models \ldots \forall x_1 \ldots (\Phi \Rightarrow (1 + \ldots + 1) \cdot x_1 = 1 + \ldots + 1)
\]
\( x_s \in \{x_1\} \cup V \)

(III) If \( r = -\frac{k}{w} \) for some \( k, w \in \{1, 2, 3, \ldots\} \), then
\[
\mathcal{T} \models \ldots \forall x_1 \ldots (\Phi \Rightarrow (1 + \ldots + 1) \cdot x_1 + 1 + \ldots + 1 = 0)
\]
\( x_s \in \{x_1\} \cup V \)

**Proof.** We prove (II) and omit similar proofs of (I) and (III). It suffices to prove that the sentence
\[
\ldots \forall x_1 \ldots (\Phi \Rightarrow (1 + \ldots + 1) \cdot x_1 = 1 + \ldots + 1)
\]
\( x_s \in \{x_1\} \cup V \)

times \( w \)-times k-times
holds true in each integral domain of characteristic 0. Let $G$ be any integral domain of characteristic 0. Let $G_1$ denote the algebraic closure of the fraction field of $G$. There exists an algebraically closed field $M$ such that both $K$ and $G_1$ embed into $M$. Of course,

$$K \models \ldots \forall x_\alpha \ldots (\Phi \Rightarrow (1 + \ldots + 1) \cdot x_1 = 1 + \ldots + 1)$$

Since every two algebraically closed fields of the same characteristic are elementary equivalent [6, p. 57], we obtain

$$M \models \ldots \forall x_\alpha \ldots (\Phi \Rightarrow (1 + \ldots + 1) \cdot x_1 = 1 + \ldots + 1)$$

Since $G$ embeds into $M$, we obtain

$$G \models \ldots \forall x_\alpha \ldots (\Phi \Rightarrow (1 + \ldots + 1) \cdot x_1 = 1 + \ldots + 1)$$

\[\square\]

Let $n \in \mathbb{Z}, n \geq 3, S_n = \{1, 10, 20, 30\} \cup \{3, 3^2, 3^3, ..., 3^n\}, S = \bigcup_{n=3}^{\infty} S_n.$

**Theorem 1.** There is an arithmetic map $\gamma : S \to \mathbb{Z}[\sqrt{-1}]$ which moves all $r \in S \setminus \{1\}$. For each $r \in S_n \setminus \{1\}$ we have:

(7) $S_n$ is an arithmetic neighbourhood of $r$ inside $\mathbb{R}$, and so too inside $\mathbb{Q}$ and $\mathbb{Z}$.

(8) $S_n$ is not an arithmetic neighbourhood of $r$ inside $\mathbb{Z}[\sqrt{-1}]$.

**Proof.** We prove (7). Assume that $f : S_n \to \mathbb{R}$ is an arithmetic map. Then,

$f(1) = 1$,

$f(9) = f(3 \cdot 3) = f(3) \cdot f(3) = (f(3))^2,$

$f(27) = f(3 \cdot 9) = f(3) \cdot f(9) = f(3) \cdot (f(3))^2 = (f(3))^3,$

$f(30) = f(27 + 3) = f(27) + f(3) = (f(3))^3 + f(3),$

$f(10) = f(9 + 1) = f(9) + f(1) = (f(3))^2 + 1,$

$f(20) = f(10 + 10) = f(10) + f(10) = (f(3))^2 + 1 + (f(3))^2 + 1 = 2 \cdot (f(3))^2 + 2,$

$f(30) = f(20 + 10) = f(20) + f(10) = 2 \cdot (f(3))^2 + 2 + (f(3))^2 + 1 = 3 \cdot (f(3))^2 + 3.$

Therefore, $(f(3))^3 + f(3) = 3 \cdot (f(3))^2 + 3$. Hence $(f(3) - 3) \cdot ((f(3))^2 + 1) = 0$. Thus $f(3) = 3$, and by induction we obtain $f(3^k) = 3^k$ for each $k \in \{1, 2, 3, ..., n\}$. Consequently.

$f(10) = f(9 + 1) = f(9) + f(1) = 9 + 1 = 10,$

$f(20) = f(10 + 10) = f(10) + f(10) = 10 + 10 = 20,$

$f(30) = f(20 + 10) = f(20) + f(10) = 20 + 10 = 30.$

We have proved (7). We define $\gamma : S \to \mathbb{Z}[\sqrt{-1}]$ as
\{(1, 1), (10, 0), (20, 0), (30, 0)\} \cup \{(3, \sqrt{-1}), (3^2, (\sqrt{-1})^2), (3^3, (\sqrt{-1})^3), \ldots\}

The map \(\gamma\) is arithmetic and \(\gamma\) moves all \(r \in S \setminus \{1\}\), so condition (8) holds true.

We state a similar result without a proof. Let \(n \in \mathbb{Z}, n \geq 1\),
\[T_n = \{-2, 1, 5, 10, 20\} \cup \{4^1, \ldots, 4^n\}, \quad T = \bigcup_{n=1}^{\infty} T_n.\]
Let us define \(\tau : T \to \mathbb{Z}[\sqrt{-1}]\) as
\[\{(\pm 2, \sqrt{-1}), (1, 1), (5, 0), (10, 0), (20, 0)\} \cup \{(4, -1), (4^2, 1), (4^3, -1), (4^4, 1), \ldots\}\]
The map \(\tau\) is arithmetic and \(\tau\) moves all \(r \in T \setminus \{1\}\). For each \(r \in T_n \setminus \{-2, 1\}\) we have:
\(T_n\) is an arithmetic neighbourhood of \(r\) inside \(\mathbb{R}\), and so too inside \(\mathbb{Q}\) and \(\mathbb{Z}\), \(T_n\) is not an arithmetic neighbourhood of \(r\) inside \(\mathbb{Z}[\sqrt{-1}]\).

**Remark.** By Theorem 1 for infinitely many integers \(r\) fail both conditions (4) and (6). In Theorems 5, 6, and 7 we describe some other rational numbers \(r\) without property (6).

Let \(n \in \mathbb{Z}, n \geq 3\), \(B_n = \{1, 5, 25, 26\} \cup \{3, 3^2, 3^3, \ldots, 3^n\}, \quad B = \bigcup_{n=3}^{\infty} B_n.\)

**Theorem 2.** There is an arithmetic map \(\phi : B \to \mathbb{Q}\) which moves all \(r \in B \setminus \{1\}\). For each \(r \in B_n \setminus \{1, 5\}\) we have:
\(9\) \(B_n\) is an arithmetic neighbourhood of \(r\) inside \(\mathbb{Z}\),
\(10\) \(B_n\) is not an arithmetic neighbourhood of \(r\) inside \(\mathbb{Q}\).

**Proof.** We prove (9). Assume that \(f : B_n \to \mathbb{Z}\) is an arithmetic map. Then,
\(f(1) = 1,\)
\(f(9) = f(3 \cdot 3) = f(3) \cdot f(3) = (f(3))^2,\)
\(f(27) = f(3 \cdot 9) = f(3) \cdot f(9) = f(3) \cdot (f(3))^2 = (f(3))^3,\)
\(f(25) = f(5 \cdot 5) = f(5) \cdot f(5) = (f(5))^2,\)
\(f(26) = f(25 + 1) = f(25) + f(1) = (f(5))^2 + 1,\)
\(f(27) = f(26 + 1) = f(26) + f(1) = (f(5))^2 + 1 + 1 = (f(5))^2 + 2.\)

Therefore, \((f(3))^3 = f(27) = (f(5))^2 + 2\). The equation \(x^3 = y^2 + 2\) has \((3, \pm 5)\) as its only integer solutions, see [17, p. 398], [8, p. 124], [12, p. 104], [9, p. 66], [11, p. 57]. Thus, \(f(3) = 3\) and \(f(5) = \pm 5\). Hence, \(f(25) = f(5 \cdot 5) = f(5) \cdot f(5) = (\pm 5)^2 = 25, f(26) = f(25 + 1) = f(25) + f(1) = 25 + 1 = 26.\)

Since \(f(3) = 3,\) we get by induction \(f(3^k) = 3^k\) for each \(k \in \{1, 2, 3, \ldots, n\}\). We have proved (9). The equation \(x^3 = y^2 + 2\) has a rational solution \(\left(\frac{129}{100}, \frac{383}{1000}\right),\) see [2, p. 173], [13, p. 2], [9 p. 66], [11, p. 57]. We define \(\phi : B \to \mathbb{Q}\) as

\[
\left\{(1, 1), \left(\frac{5}{1000}, \frac{383}{1000}\right), \left(25, \left(\frac{383}{1000}\right)^2\right), \left(26, \left(\frac{383}{1000}\right)^2 + 1\right)\right\} \cup \left\{\left(3, \frac{129}{100}\right) \left(3^2, \left(\frac{129}{100}\right)^2\right), \left(3^3, \left(\frac{129}{100}\right)^3\right), \ldots\right\}
\]

\[
\left\{(1, 1), \left(\frac{5}{1000}, \frac{383}{1000}\right), \left(25, \left(\frac{383}{1000}\right)^2\right), \left(26, \left(\frac{383}{1000}\right)^2 + 1\right)\right\} \cup \left\{\left(3, \frac{129}{100}\right) \left(3^2, \left(\frac{129}{100}\right)^2\right), \left(3^3, \left(\frac{129}{100}\right)^3\right), \ldots\right\}
\]
The map $\phi$ is arithmetic and $\phi$ moves all $r \in B \setminus \{1\}$, so condition (10) holds true.

We present a simpler counterexample for $r = -1$. Let $G = \{-4, -1, 1, 3, 9, 12, 16\}, \eta : G \to \mathbb{Q}, \eta = \{(−4, \frac{1}{2}), (−1, 1), (1, 1), (3, \frac{1}{2}), (9, \frac{1}{4}), (12, \frac{3}{4}), (16, \frac{1}{2})\}$. The map $\eta$ is arithmetic and $\eta$ moves $−1$. We prove that $G$ is an arithmetic neighbourhood of $−1$ inside $\mathbb{Z}$. Assume that $f : G \to \mathbb{Z}$ is an arithmetic map. Since

$$f(-1) = f(-4 + 3) = f(-4) + f(3),$$

we get $f(-4) = f(-1) - f(3)$. Hence,

$$f(16) = f((-4) \cdot (-4)) = f(-4) \cdot f(-4) = (f(-1) - f(3))^2,$$

$$f(12) = f(-4 + 16) = f(-4) + f(16) = f(-1) - f(3) + (f(-1) - f(3))^2.$$

Since $1 = f((-1) \cdot (-1)) = f(-1) \cdot f(-1)$, we get $f(-1) = \pm 1$. Assume, on the contrary, that $f(-1) = 1$. Thus,

$$1 - f(3) + (1 - f(3))^2 = f(12) = f(3 + 3 \cdot 3) = f(3) + f(3 \cdot 3) = f(3) + (f(3))^2$$

Solving this equation for $f(3)$ we obtain $2 \cdot f(3) = 1$, a contradiction.

Let $Y = \{-4, -1, 1, 3, 9, 12, 14, 16, 20, 180, 196\}, \kappa : Y \to \mathbb{Q}(\sqrt{3}),$

$$\kappa = \{\left(-4, \frac{1}{2}\right), (-1, 1), (1, 1), \left(3, \frac{1}{2}\right), \left(9, \frac{1}{4}\right), \left(12, \frac{3}{4}\right),$$

$$\left(14, \frac{\sqrt{3}}{4}\right), \left(16, \frac{1}{4}\right), \left(20, -\frac{1}{4}\right), \left(180, -\frac{1}{16}\right), \left(196, \frac{3}{16}\right)\}$$

The map $\kappa$ is arithmetic and $\kappa$ moves $-1$. We prove that $Y$ is an arithmetic neighbourhood of $-1$ inside $\mathbb{Q}$. Let $f : Y \to \mathbb{Q}$ be an arithmetic map, and assume, on the contrary, that $f(-1) = 1$. As previously, we conclude that

$f(3) = \frac{1}{2}$ and $f(-4) = f(-1) - f(3) = 1 - \frac{1}{2} = \frac{1}{2}$. Hence,

$$f(9) = f(3 \cdot 3) = f(3) \cdot f(3) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

$$f(16) = f((-4) \cdot (-4)) = f(-4) \cdot f(-4) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$  

Since $f(16) = f(-4 + 20) = f(-4) + f(20)$, we get $f(20) = f(16) - f(-4) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$. Therefore,

$$f(180) = f(9 \cdot 20) = f(9) \cdot f(20) = \frac{1}{4} \cdot (-\frac{1}{4}) = -\frac{1}{16},$$

$$(f(4))^2 = f(14 \cdot 14) = f(180 + 16) = f(180) + f(16) = -\frac{1}{16} + \frac{1}{4} = \frac{3}{16},$$

a contradiction.

Let $M = \{-4, -1, 1, 3, 5, 9, 11, 42, 45, 121, 126\}, \chi : M \to \mathbb{Z}[\sqrt{-1}],$

$$\chi = \{(-4, 0), (-1, 1), (1, 1), (3, 1), (5, 1), (9, 1), (11, \sqrt{-1}), (42, 0), (45, 1), (121, -1), (126, 0)\}$$

The map $\chi$ is arithmetic and $\chi$ moves $-1$. We prove that $M$ is an arithmetic neighbourhood of $-1$ inside $\mathbb{R}$. Let $f : M \to \mathbb{R}$ be an arithmetic map, and assume, on the contrary, that $f(-1) = 1$. Since $1 = f(-4 + 3) = f(-4) + f(3)$, we get $f(-4) = 1 - f(3)$. Hence

$$1 = f(-4 + 5) = f(-4) + f(5) = 1 - f(3) + f(-4 + 3 \cdot 3) =$$

$$1 - f(3) + f(-4) + f(3 \cdot 3) = 1 - f(3) + 1 - f(3) + (f(3))^2$$
Solving this equation for $f(3)$ we obtain $f(3) = 1$. Therefore,

$$f(-4) = 1 - f(3) = 1 - 1 = 0,$$

$$f(5) = 0 + f(5) = f(-4) + f(5) = f(-4 + 5) = 1,$$

$$f(9) = f(3 \cdot 3) = f(3) \cdot f(3) = 1 \cdot 1 = 1,$$

$$f(45) = f(5 \cdot 9) = f(5) \cdot f(9) = 1 \cdot 1 = 1.$$

Since $f(45) = f(3 + 42) = f(3) + f(42) = 1 + f(42)$, we get $f(42) = f(45) - 1 = 1 - 1 = 0$. Thus, $f(126) = f(3 \cdot 42) = f(3) \cdot f(42) = 1 \cdot 0 = 0$. Since $0 = f(5 + 11 \cdot 11) = f(5) + f(11 \cdot 11) = 1 + (f(11))^2$, we get $(f(11))^2 = -1$, a contradiction.

Let $w$ denote the unique real root of the polynomial $x^3 - x^2 - x - 3$.

**Theorem 3.** There is an arithmetic map $\psi : \{-4\} \cup B \rightarrow \mathbb{Q}(w)$ which moves all $r \in \{-4\} \cup B \setminus \{1\}$. For each $r \in \{-4\} \cup B_n \setminus \{1\}$ we have:

(11) $\{-4\} \cup B_n$ is an arithmetic neighbourhood of $r$ inside $\mathbb{Q}$,

(12) $\{-4\} \cup B_n$ is not an arithmetic neighbourhood of $r$ inside $\mathbb{Q}(w)$.

**Proof.** We prove (11). Assume that $f : \{-4\} \cup B_n \rightarrow \mathbb{Q}$ is an arithmetic map. Since $1 = f(1) = f(-4 + 5) = f(-4) + f(5)$, we get

$$f(-4) = 1 - f(5) \tag{13}$$

Hence,

$$f(5) = f(-4 + (3 \cdot 3)) = f(-4) + f(3 \cdot 3) = f(-4) + f(3) \cdot f(3) = 1 - f(5) + (f(3))^2.$$ 

Therefore,

$$f(5) = \frac{1 + (f(3))^2}{2} \tag{14}$$

From equations (13) and (14), we obtain

$$f(-4) = 1 - f(5) = 1 - \frac{1 + (f(3))^2}{2} = \frac{1 - (f(3))^2}{2} \tag{15}$$

Proceeding exactly as in the proof of Theorem 2, we obtain $(f(3))^3 = (f(5))^2 + 2$. By this and equation (14), we get

$$(f(3))^3 = \left(\frac{1 + (f(3))^2}{2}\right)^2 + 2 \tag{16}$$

Equation (16) is equivalent to the equation

$$(f(3) - 3) \cdot ((f(3))^3 - (f(3))^2 - f(3) - 3) = 0$$

The equation $x^3 - x^2 - x - 3 = 0$ has no rational solutions, so we must have $f(3) = 3$. By induction we get $f(3^k) = 3^k$ for each $k \in \{1, 2, 3, ..., n\}$. Knowing that $f(3) = 3$, from equations (15) and (14) we obtain:

$$f(-4) = \frac{1 - (f(3))^2}{2} = \frac{1 - 3^2}{2} = -4$$
\[ f(5) = \frac{1 + (f(3))^2}{2} = \frac{1 + 3^2}{2} = 5 \]

Consequently,

\[ f(25) = f(5 \cdot 5) = f(5) \cdot f(5) = 5 \cdot 5 = 25 \]

\[ f(26) = f(25 + 1) = f(25) + f(1) = 25 + 1 = 26 \]

The proof of (11) is completed. We define \( \psi : \{ -4 \} \cup B \to \mathbb{Q}(w) \) as

\[
\left\{ \left(-4, \frac{1-w^2}{2}\right), (1,1), \left(5, \frac{1+w^2}{2}\right), \left(25, \left(\frac{1+w^2}{2}\right)^2\right), \left(26, \left(\frac{1+w^2}{2}\right)^2 + 1\right) \right\} \cup \{(3,w), (3^2, w^2), (3^3, w^3), \ldots\}
\]

The map \( \psi \) is arithmetic and \( \psi \) moves all \( r \in \{ -4 \} \cup B \backslash \{1\} \), so condition (12) holds true.

Let \( n \in \mathbb{Z}, n \geq 1, C_n = \{1, 3, 5, 13, 25, 65, 169, 194, 195\} \cup \{9, 9^2, 9^3, \ldots, 9^n\} \),

\( C = \bigcup_{n=1}^{\infty} C_n. \)

**Theorem 4.** There is an arithmetic map \( g : C \to \mathbb{Q} \) which moves all \( r \in C \backslash \{1\}. \)

(17) \( C_n \) is an arithmetic neighbourhood inside \( \mathbb{Z} \) for \( 9, 9^2, 9^3, \ldots, 9^n \),

(18) \( C_n \) is not an arithmetic neighbourhood inside \( \mathbb{Q} \) for \( 9, 9^2, 9^3, \ldots, 9^n \).

**Proof.** We prove (17). Assume that \( f : C_n \to \mathbb{Z} \) is an arithmetic map. Then,

\[
(f(5))^2 + (f(13))^2 + 1 = f(5^2) + f(13^2) + f(1) = f(5^2 + 13^2 + 1) = f((5 \cdot 13) \cdot 3) = f(5 \cdot 13) \cdot f(3) = f(5) \cdot f(13) \cdot f(3).
\]

If integers \( x, y, z \) satisfy \( x^2 + y^2 + 1 = xyz \) then \( z = \pm 3 \), see [8, p. 299], [9, pp. 58–59], [10, p. 31], [11, pp. 51–52], [1], cf. Theorem 4 in [7, p. 218]. Thus, \( f(3) = \pm 3 \). Hence \( f(9) = f(3 \cdot 3) = f(3) \cdot f(3) = (\pm 3)^2 = 9 \), and by induction we obtain \( f(9^k) = 9^k \) for each \( k \in \{1, 2, 3, \ldots, n\} \). The proof of (17) is completed. We define \( g : C \to \mathbb{Q} \) as

\[
\left\{ (1,1), \left(3, \frac{9}{4}\right), (5,2), (13,2), (25,4), (25,4), (65,4), (169,4), (194,8), (195,9) \right\} \cup \nabla \left\{ \left(9, \frac{81}{16}\right), \left(9^2, \left(\frac{81}{16}\right)^2\right), \left(9^3, \left(\frac{81}{16}\right)^3\right), \ldots, \left(9^n, \left(\frac{81}{16}\right)^n\right) \right\}
\]

The map \( g \) is arithmetic and \( g \) moves all \( r \in C \backslash \{1\} \), so condition (18) holds true. \( \square \)
We know (see Theorem 2 or Theorem 4) that infinitely many integers \( r \) do not satisfy condition (5). Now, we sketch a more elementary (but longer) proof of this fact. Let \( n \in \mathbb{Z}, n \geq 3, \)

\[
H_n = \{1, 2, 4, 16, 60, 64, 3600, 3604, 3620, 3622, 3623, 7^3 \cdot 13^2, 7^3 \cdot 13^2 + 1\} \cup \{13, 13^2, 7, 7^2, 7^3, \ldots, 7^n\}
\]

\( H_n \) is an arithmetic neighbourhood inside \( \mathbb{Z} \) for each \( r \in H_n \setminus \{13\} \). \( H_n \) is not an arithmetic neighbourhood inside \( \mathbb{Q} \) for 13, 13^2, 7, 7^2, 7^3, \ldots, 7^n. \) The proofs follow from the following observations:

\[
7^3 \cdot 13^2 + 1 = 16 \cdot 3623
\]

\[\forall x, y \in \mathbb{Z} \ (x^3 \cdot y^2 = 7^3 \cdot 13^2 \Rightarrow (x = 7 \land y = \pm 13))\]

\[
(7^3)^3 \cdot (8 \cdot 13)^3 = 7^3 \cdot 13^2
\]

**Theorem 5.** Let

\[
D = \{-36, \frac{1}{2}, 1, 2, \frac{5}{2}, 5, 12, 25, 50, 100, 12^2, 200, 400, 425, 430, 432, 36^2, 12^3\}.
\]

(19) \( D \) is an arithmetic neighbourhood inside \( \mathbb{Q} \) for 12, 12^2, 36^2, 12^3.

(20) \( D \) is not an arithmetic neighbourhood inside \( \mathbb{Q}(\sqrt[3]{5}) \) for 12, 12^2, 36^2, 12^3.

**Proof.** We prove (19). Assume that \( f : D \to \mathbb{Q} \) is an arithmetic map. Then, \( f(1) = 1 \) and \( f(2) = f(1 + 1) = f(1) + f(1) = 1 + 1 = 2 \). Since \( 1 = f(1) = f\left(\frac{1}{2} + \frac{1}{2}\right) = f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) \), we get \( f\left(\frac{1}{2}\right) = \frac{1}{2} \). Knowing \( f\left(\frac{1}{2}\right) \) and \( f(2) \), we calculate

\[
f\left(\frac{5}{2}\right) = f\left(2 + \frac{1}{2}\right) = f(2) + f\left(\frac{1}{2}\right) = 2 + \frac{1}{2} = \frac{5}{2},
f(5) = f\left(\frac{5}{2} + \frac{5}{2}\right) = f\left(\frac{5}{2}\right) + f\left(\frac{5}{2}\right) = \frac{5}{2} + \frac{5}{2} = 5,
f(25) = f(5 \cdot 5) = f(5) \cdot f(5) = 5 \cdot 5 = 25,
f(50) = f(25 + 25) = f(25) + f(25) = 25 + 25 = 50,
f(100) = f(50 + 50) = f(50) + f(50) = 50 + 50 = 100,
f(200) = f(100 + 100) = f(100) + f(100) = 100 + 100 = 200,
f(400) = f(200 + 200) = f(200) + f(200) = 200 + 200 = 400,
f(425) = f(400 + 25) = f(400) + f(25) = 400 + 25 = 425,
f(430) = f(425 + 5) = f(425) + f(5) = 425 + 5 = 430,
f(432) = f(430 + 2) = f(430) + f(2) = 430 + 2 = 432.
\]

Therefore, \( (f(12))^3 = (f(12) \cdot f(12)) \cdot f(12) = f(12) \cdot f(12) \cdot f(12) = f((12 \cdot 12) \cdot 12) = f((-36)^2 + 432) = f((-36)^2) + f(432) = (f(-36))^2 + 432. \) The equation \( x^3 = y^2 + 432 \) has \( (12, \pm 36) \) as its only rational solutions, see [3], [12, p. 107], [2, p. 174], [4, p. 296], [8, p. 247], [14, p. 54]. Thus, \( f(12) = 12 \) and \( f(-36) = \pm 36. \) Hence, \( f(12^2) = f(12) \cdot f(12) = 12^2, f(12^3) = f(12 \cdot 12^2) = f(12) \cdot f(12^2) = 12 \cdot 12^2 = 12^3, f(36^2) = f((-36) \cdot (-36)) = f(-36) \cdot f(-36) = (\pm 36)^2 = 36^2. \) The proof of (19) is completed. We find that
\[ 8^3 = (4 \cdot \sqrt{3})^2 + 432 \] and we define \( h : D \to \mathbb{Q}(\sqrt{5}) \) as

\[
\begin{align*}
\left\{ (-36, 4 \cdot \sqrt{5}, (12, 8), (12^2, 8^2), (36^2, 80), (12^3, 8^3)) \right\} & \cup \\
\text{id}\left( \left\{ \frac{1}{2}, 1, 2, \frac{5}{2}, 5, 25, 50, 100, 200, 400, 425, 430, 432 \right\} \right)
\end{align*}
\]

We summarize the check that \( h \) is arithmetic. Obviously, \( h(1) = 1 \). To check the condition

\[ \forall x, y, z \in D \ (x + y = z \implies h(x) + h(y) = h(z)) \]

it is enough to consider all the triples \((x, y, z) \in D \times D \times D\) for which \( x+y = z \), \( x \leq y \), and \( h \) is not the identity on \( \{x, y, z\} \). There is only one such triple: \((432, 36^2, 12^3)\).

To check the condition

\[ \forall x, y, z \in D \ (x \cdot y = z \implies h(x) \cdot h(y) = h(z)) \]

it is enough to consider all the triples \((x, y, z) \in D \times D \times D\) for which \( x \cdot y = z \), \( x \leq y \), \( x \neq 1 \), \( y \neq 1 \), and \( h \) is not the identity on \( \{x, y, z\} \). These triples are as follows:

\[ (-36, -36, 36^2), (12, 12, 12^2), (12, 12^2, 12^3) \]

The sentence (20) is true because \( h \) is arithmetic and \( h \) moves \( 12, 12^2, 36^2, 12^3 \).

\[ \square \]

**Corollary.** Let us define by induction the finite sets \( D_n \subseteq \mathbb{Q} \ (n = 0, 1, 2, \ldots) \).

Let \( D_0 = D, d_n \) denote the greatest number in \( D_n, D_{n+1} = D_n \cup \{d_n^2\} \). For each \( n \in \{0, 1, 2, \ldots\} \) we have:

\[ D_n \text{ is an arithmetic neighbourhood of } d_n \text{ inside } \mathbb{Q}, \]

\[ D_n \text{ is not an arithmetic neighbourhood of } d_n \text{ inside } \mathbb{Q}(\sqrt{5}). \]

Let \( u = \frac{1 + \sqrt{33}}{8}, n \in \mathbb{Z}, n \geq 3, E_n = \{\frac{1}{2}, 1, \frac{3}{2}, \frac{9}{4}\} \cup \{9, -2, (-2)^2, (-2)^3, \ldots, (-2)^n\}, E = \bigcup_{n=3}^{\infty} E_n. \)

**Theorem 6.** There is an arithmetic map \( \sigma : E \to \mathbb{Q}(\sqrt{33}) \) which moves 9 and all the numbers \((-2)^k\), where \( k \in \{1, 2, 3, \ldots\} \). For each \( r \in E_n \setminus \{\frac{1}{2}, 1, \frac{3}{2}, \frac{9}{4}\} \) we have:

(21) \( E_n \) is an arithmetic neighbourhood of \( r \) inside \( \mathbb{Q} \),

(22) \( E_n \) is not an arithmetic neighbourhood of \( r \) inside \( \mathbb{Q}(\sqrt{33}) \).

**Proof.** We prove (21). Assume that \( f : E_n \rightarrow \mathbb{Q} \) is an arithmetic map.

Since \( 1 = f(\frac{1}{2} + \frac{1}{2}) = f(\frac{1}{2}) + f(\frac{1}{2}) \), we get \( f(\frac{1}{2}) = \frac{1}{2} \). Hence,

\[ f\left( \frac{3}{2} \right) = f\left( 1 + \frac{1}{2} \right) = f(1) + f\left( \frac{1}{2} \right) = 1 + \frac{1}{2} = \frac{3}{2}. \]

Thus,

\[ f\left( \frac{9}{4} \right) = f\left( \frac{3}{2} \cdot \frac{3}{2} \right) = f\left( \frac{3}{2} \right) \cdot f\left( \frac{3}{2} \right) = \frac{3}{2} \cdot \frac{3}{2} = \frac{9}{4}. \]

Therefore,
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\( f(9) = f(\frac{9}{4} \cdot 4) = f(\frac{9}{4}) \cdot f((-2) \cdot (-2)) = \frac{9}{4} \cdot (f(-2))^2 \). It implies that

\[ 1 = f(-2 \cdot 4 + 9) = f(-2 \cdot 4) + f(9) = f(-2) \cdot f(4) + \frac{9}{4} \cdot (f(-2))^2 = \]

\[ f(-2) \cdot f(-2) + \frac{9}{4} \cdot (f(-2))^2 = (f(-2))^3 + \frac{9}{4} \cdot (f(-2))^2 \]

Solving this equation for \( f(-2) \) we obtain \( f(-2) = -2 \), the only rational root.

Another roots are \( \frac{-1-\sqrt{33}}{8} \) and \( \frac{-1+\sqrt{33}}{8} \). Knowing \( f(-2) \), we calculate

\[ f(9) = \frac{9}{4} \cdot (f(-2))^2 = \frac{9}{4} \cdot (-2)^2 = 9 \]

Applying induction, we obtain \( f((-2)^k) = (-2)^k \) for each \( k \in \{1, 2, 3, ..., n\} \).

We have proved (21). We define \( \sigma : E \to \mathbb{Q} (\sqrt{33}) \) as

\[ \text{id} \left( \left\{ \frac{1}{2}, 1, \frac{3}{2}, \frac{9}{4} \right\} \right) \cup \left\{ \left(9, \frac{9}{4} \cdot u^2\right), \left(-2, u\right), \left((-2)^2, u^2\right), \left((-2)^3, u^3\right), ... \right\} \]

The map \( \sigma \) is arithmetic and \( \sigma \) moves all \( r \in E_n \setminus \{\frac{1}{2}, 1, \frac{3}{2}, \frac{9}{4}\} \), so condition (22) holds true.

\[ \square \]

Theorem 7 which follows is more general than the previous ones. Let \( n \) be an integer, and assume that \( n \geq 3 \) and \( n \not\in \{2^2, 2^3, 2^4, ...\} \). We find the smallest integer \( \rho(n) \) such that \( n^3 \leq 2^{\rho(n)} \). From the definition of \( \rho(n) \) we obtain \( 2^{\rho(n)-1} < n^3 \). It gives

\[ 2^{\rho(n)} = 2 \cdot 2^{\rho(n)-1} < 2 \cdot n^3 < n \cdot n^3 = n^4 \]

Since \( n^3 \leq 2^{\rho(n)} < n^4 \), \( 2^{\rho(n)} \) has four digits in the number system with base \( n \). Let

\[ 2^{\rho(n)} = m_3 \cdot n^3 + m_2 \cdot n^2 + m_1 \cdot n + m_0 \]

where \( m_3 \in \{1, 2, ..., n-1\} \) and \( m_2, m_1, m_0 \in \{0, 1, 2, ..., n-1\} \). Let

\[ J(n) = \left\{ -1, 0, 1, -\frac{1}{2}, -\frac{1}{2^2}, -\frac{1}{2^3}, ..., -\frac{1}{2^{\rho(n)-1}}, n, n^2 \right\} \cup \right\{ k \cdot n^3 : k \in \{1, 2, ..., m_3\} \right\} \cup \right\{ m_3 \cdot n^3 + k \cdot n^2 : k \in \{1, 2, ..., m_2\} \right\} \cup \right\{ m_3 \cdot n^3 + m_2 \cdot n^2 + k \cdot n : k \in \{1, 2, ..., m_1\} \right\} \cup \right\{ m_3 \cdot n^3 + m_2 \cdot n^2 + m_1 \cdot n + k : k \in \{1, 2, ..., m_0\} \right\} \]

Of course, \( \{2^{\rho(n)}, 2^{\rho(n)}-1, 2^{\rho(n)}-2, ..., 2^{\rho(n)}-m_0\} \subseteq J(n) \).

Theorem 7. \( J(n) \) is an arithmetic neighbourhood of \( n \) inside \( \mathbb{R} \), and so too inside \( \mathbb{Q} \). \( J(n) \) is not an arithmetic neighbourhood of \( n \) inside \( \mathbb{C} \).
Proof. Assume that \( f : \mathcal{J}(n) \to \mathbb{R} \) is an arithmetic map. Since \( 0 + 0 = 0 \), \( f(0) = 0 \). Since \( f(0) = 0 \) and \( -1 + 1 = 0 \), \( f(-1) = -1 \). Since
\[
-1 = f(-1) = f \left( \frac{-1}{2} \right) + f \left( \frac{-1}{2} \right) = f \left( \frac{-1}{2} \right) + f \left( \frac{-1}{2} \right)
\]
we get \( f \left( \frac{-1}{2} \right) = -\frac{1}{2} \). Hence, from \( \frac{-1}{2} = \frac{-1}{2} + \frac{-1}{2} \) we obtain \( f \left( \frac{-1}{2} \right) = -\frac{1}{2} \). Applying induction we obtain \( f \left( \frac{-1}{2^{\rho(n)}} \right) = -\frac{1}{2^{\rho(n)}} \). We have
\[
-1 = f(-1) = f \left( 2^{\rho(n)} \right) \cdot \left( -\frac{1}{2^{\rho(n)}} \right) + f \left( \frac{-1}{2^{\rho(n)}} \right) \cdot \left( -\frac{1}{2^{\rho(n)}} \right) = f(2^{\rho(n)}) \cdot \left( -\frac{1}{2^{\rho(n)}} \right)
\]
Hence \( f(2^{\rho(n)}) = 2^{\rho(n)} \). Applying induction we get
\[
2^{\rho(n)} = f(2^{\rho(n)}) = f(m_3 \cdot n^3 + m_2 \cdot n^2 + m_1 \cdot n + m_0)
\]
\[
= m_3 \cdot (f(n))^3 + m_2 \cdot (f(n))^2 + m_1 \cdot f(n) + m_0
\]
We want to prove that \( f(n) = n \). It suffices to show that the function
\[
\mathbb{R} \ni x \xrightarrow{\zeta} m_3 \cdot x^3 + m_2 \cdot x^2 + m_1 \cdot x + m_0 \in \mathbb{R}
\]
takes the value \( 2^{\rho(n)} \) only for \( x = n \). Since \( \zeta \) is strictly increasing in the interval \([0, \infty)\), for each \( x \in [0, n) \) we have \( \zeta(x) < \zeta(n) = 2^{\rho(n)} \), and for each \( x \in (n, \infty) \) we have \( \zeta(x) > \zeta(n) = 2^{\rho(n)} \). We show that \( \zeta \) does not reach the value \( 2^{\rho(n)} \) for \( x \in (-\infty, 0] \). For each \( x \in (-\infty, 0) \) we have
\[
\zeta(x) = m_3 \cdot x^3 + m_2 \cdot x^2 + m_1 \cdot x + m_0 \leq x^3 + (n - 1) \cdot x^2 + n - 1 \quad (23)
\]
By (23), if \( x \in (-\infty, -n + 1] \) then
\[
\zeta(x) \leq x^3 + (n - 1) \cdot x^2 + n - 1 = (x + n - 1) \cdot x^2 + n - 1 \leq n - 1 < n^3 \leq 2^{\rho(n)}
\]
Thus, \( \zeta(x) \neq 2^{\rho(n)} \). By (23), if \( x \in [-n + 1, 0] \) then
\[
\zeta(x) \leq x^3 + (n - 1) \cdot x^2 + n - 1 \leq (n - 1) \cdot x^2 + n - 1 \leq (n - 1)^3 + n - 1 < n^3 \leq 2^{\rho(n)}
\]
Thus, \( \zeta(x) \neq 2^{\rho(n)} \). We have proved that \( f(n) = n \). It proves that \( \mathcal{J}(n) \) is an arithmetic neighbourhood of \( n \) inside \( \mathbb{R} \). We prove that \( \mathcal{J}(n) \) is not an arithmetic neighbourhood of \( n \) inside \( \mathbb{C} \). The number \( n \) is a single root of the polynomial
\[
m_3 \cdot x^3 + m_2 \cdot x^2 + m_1 \cdot x + m_0 - 2^{\rho(n)}
\]
because the derivative of this polynomial takes the non-zero value
\[
3 \cdot m_3 \cdot n^2 + 2 \cdot m_2 \cdot n + m_1 \geq 3 \cdot n^2 \geq 27
\]
at \( x = n \). Hence the polynomial
\[
m_3 \cdot x^3 + m_2 \cdot x^2 + m_1 \cdot x + m_0 - 2^{\rho(n)}
\]
has two conjugated roots \( z_1, z_2 \in \mathbb{C} \setminus \mathbb{R} \). Let \( z = z_1 \) or \( z = z_2 \). We define \( \theta : \mathcal{J}(n) \to \mathbb{C} \) as
\[
\text{id} \left( \left\{ -1, 0, 1, -\frac{1}{2}, -\frac{1}{2^2}, ..., -\frac{1}{2^{\rho(n)}} \right\} \right) \cup \{(n, z), (n^2, z^2)\} \cup \{(k \cdot n^3, k \cdot z^3) : k \in \{1, 2, ..., m_3\}\} \cup \{(m_3 \cdot n^3 + k \cdot n^2, m_3 \cdot z^3 + k \cdot z^2) : k \in \{1, 2, ..., m_2\}\} \cup \{(m_3 \cdot n^3 + m_2 \cdot n^2 + k \cdot n, m_3 \cdot z^3 + m_2 \cdot z^2 + k \cdot z) : k \in \{1, 2, ..., m_1\}\} \cup \{(m_3 \cdot n^3 + m_2 \cdot n^2 + m_1 \cdot n + k, m_3 \cdot z^3 + m_2 \cdot z^2 + m_1 \cdot z + k) : k \in \{1, 2, ..., m_0\}\}
\]

Of course, \( \theta(x) = x \) for each \( x \in \{2^{\rho(n)}, 2^{\rho(n)} - 1, 2^{\rho(n)} - 2, ..., 2^{\rho(n)} - m_0\} \). Since \( \theta(n) = z \neq n \), \( \theta \) moves \( n \). We summarize the check that \( \theta \) is arithmetic.

Obviously, \( \theta(1) = 1 \). To check the condition
\[
\forall x, y, z \in \mathcal{J}(n) \ (x + y = z \Rightarrow \theta(x) + \theta(y) = \theta(z))
\]
it is enough to consider all the triples \((x, y, z) \in \mathcal{J}(n) \times \mathcal{J}(n) \times \mathcal{J}(n)\) for which \( x + y = z \), \( x \leq y \), \( x \neq 0 \), \( y \neq 0 \), and \( \theta \) is not the identity on \( \{x, y, z\} \). These triples are as follows:
\[
(k \cdot n^3, l \cdot n^3, (k + l) \cdot n^3), \text{ where } k, l, k + l \in \{1, 2, ..., m_3\} \text{ and } k \leq l,
\]
\[
(n^2, m_3 \cdot n^3 + k \cdot n^2, m_3 \cdot n^3 + (k + 1) \cdot n^2), \text{ where } k, k + 1 \in \{0, 1, 2, ..., m_2\},
\]
\[
(n, m_3 \cdot n^3 + m_2 \cdot n^2 + k \cdot n, m_3 \cdot n^3 + m_2 \cdot n^2 + (k + 1) \cdot n), \text{ where } k, k + 1 \in \{0, 1, 2, ..., m_1\}.
\]

To check the condition
\[
\forall x, y, z \in \mathcal{J}(n) \ (x \cdot y = z \Rightarrow \theta(x) \cdot \theta(y) = \theta(z))
\]
it is enough to consider all the triples \((x, y, z) \in \mathcal{J}(n) \times \mathcal{J}(n) \times \mathcal{J}(n)\) for which \( x \cdot y = z \), \( x \leq y \), \( x \neq 1 \), \( y \neq 1 \), \( x \neq 0 \), \( y \neq 0 \), and \( \theta \) is not the identity on \( \{x, y, z\} \). These triples are as follows:
\[
(n, n, n^2), (n, n^2, n^3).
\]

\( \square \)

References

[1] E. S. Barnes, On the Diophantine equation \( x^2 + y^2 + c = xyz \), J. London Math. Soc. 28 (1953), 242–244.


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