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Bands of Weakly r-Archimedean Γ-Semigroups

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Abstract

Semigroups having a decomposition into a band of r-archimedean semigroups have been studied in many papers. In the present paper, we give some new results extending these semigroups. We have shown that a Γ-semigroup M is a band of weakly r-archimedean sub-Γ-semigroups of M if and only if it satisfies the condition for all $a, x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$, we have $(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a \gamma a\beta y) \in \eta_r \cap \eta_r^{-1}$. Moreover, we have also shown that the following statements are equivalent:

(a) M is a band of weakly r-archimedean left ideals of M.

(b) M is a band of r-archimedean left ideals of M.

(c) M satisfies the condition for all $a, x, y \in M$ and $\alpha, \beta, \gamma \in \Gamma$, $(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \eta_r \cap \eta_r^{-1}$.

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1 Introduction and Preliminaries

Semigroups which can be decomposed into a band of r (l or t)-archimedean semigroups have been studied in many papers. Bogdanović, Ćirić and Novikov investigated in [1] bands of l-archimedean semigroups, and Putcha investigated in [4] bands of t-archimedean semigroups.

In this paper, we have to introduce the concept of weakly r-archimedean Γ-semigroups and r-archimedean Γ-semigroups and give necessary and sufficient conditions in order that a Γ-semigroup M is a band of weakly r-archimedean sub-Γ-semigroups of M. Similar results hold if we replace the symbol "r" by "l".

To present the main results we first recall the definition of a Γ-semigroup which is important here.

Let M and Γ be any two nonempty sets. M is called a Γ-*semigroup* [6] if for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$, we have (i) $a\alpha b \in M$ and (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$. A nonempty subset K of M is called a *sub*-Γ-*semigroup* of M if $a\gamma b \in K$ for all $a, b \in K$ and $\gamma \in \Gamma$. A nonempty subset L of M is called a *left ideal* of M if $M\Gamma L \subseteq L$. Then every left ideal of a Γ-semigroup M is a sub-Γ-semigroup of M.

Examples of Γ-semigroups can be seen in [3, 5, 7, 9] and [10], respectively.

From now on we always assume that M stands for a Γ-semigroup. For nonempty subsets A and B of M and a nonempty subset Γ' of Γ, let $A\Gamma' B := \{a\gamma b \mid a \in A, b \in B \text{ and } \gamma \in \Gamma'\}$. If $A = \{a\}$, then we also write $\{a\}\Gamma' B$ as $a\Gamma' B$, and similarly if $B = \{b\}$ or $\Gamma' = \{\gamma\}$. If ρ is a relation on M, then the inverse of ρ, denote ρ^{-1}, is the set $\{(b, a) \in M \times M \mid a\rho b\}$. An equivalence relation ρ on M is called a *left* (*right*) *congruence* [8] if for any $a, b, c \in M$ and $\gamma \in \Gamma$, $(a, b) \in \rho$ implies $(c\gamma a, c\gamma b) \in \rho$ $((a\gamma c, b\gamma c) \in \rho)$. ρ is called a *congruence* on M if it is a both a left congruence and a right congruence on M. If ρ is a congruence on M, then $M/\rho := \{(x)_\rho \mid x \in M\}$ is a Γ-semigroup with $(x)_\rho\gamma(y)_\rho = (x\gamma y)_\rho$ for all $x, y \in M$ and $\gamma \in \Gamma$. A congruence ρ on M is called a *band congruence* if for all $a \in M$ and $\gamma \in \Gamma, (a\gamma a, a) \in \rho$.

Examples of congruences can be seen in [8] and [10], respectively.

The following definitions are introduced analogously to some definitions in [1].

M is called *r-archimedean* (*l-archimedean*) if for any $a, b \in M, b \in a\Gamma M$ $(b \in M\Gamma a)$ or there exists an integer $m \geq 2$ such that $b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in a\Gamma M$ $(b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in M\Gamma a)$ for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma$. A sub-$\Gamma$-semigroup (left ideal) T of M is called an *r-archimedean sub-Γ-semigroup* (*left ideal*) if T is r-archimedean. M is called a *band of r-archimedean sub-Γ-semigroups* (*left ideals*) *of* M if there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is an r-archimedean sub-Γ-semigroup (left ideal) of M for all $x \in M$. A sub-Γ-semigroup (left ideal) T of M is called a *weakly r-archimedean sub-Γ-semigroup* (*left ideal*) if for any $a, b \in T, b \in a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\gamma_1 b\gamma_2 b \ldots b\gamma_{m-1} b \in a\Gamma M$ for some $\gamma_1, \gamma_2, \ldots, \gamma_{m-1} \in \Gamma$. M is called a *band of weakly r-archimedean sub-Γ-semigroups (left ideals) of M* if there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is a weakly r-archimedean sub-Γ-semigroup (left ideal) of M for all $x \in M$.

It is easy to verify that every r-archimedean sub-Γ-semigroup (left ideal) of a Γ-semigroup M is a weakly r-archimedean sub-Γ-semigroup (left ideal) of M.

In the sequel, the following relations on M are used frequently:

$$\begin{aligned} \eta_r &:= \{(a,b) \mid b \in a \cup a\Gamma M \text{ or there exists an integer } m \geq 2 \text{ such that} \\ &\quad b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup a\Gamma M \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma\}, \\ \eta_l &:= \{(a,b) \mid b \in a \cup M\Gamma a \text{ or there exists an integer } m \geq 2 \text{ such that} \\ &\quad b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b \in a \cup M\Gamma a \text{ for some } \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma\}. \end{aligned}$$

The following two lemmas are also necessary for the main results.

Lemma 1.1 *If σ_1 and σ_2 are left congruences on M, then so are $(\sigma_1)^n$ and $(\sigma_1 \circ \sigma_2)^n$ for any $n \in \mathbb{N}$.*

Proof. Similar to the proof of Lemma 5.8 [2], we obtain it. □

Similar result holds if we replace the word "left" by "right". Then we get the following.

Corollary 1.2 *If σ_1 and σ_2 are congruences on M, then so are $(\sigma_1)^n$ and $(\sigma_1 \circ \sigma_2)^n$ for any $n \in \mathbb{N}$.*

Lemma 1.3 *If E is an equivalence relation on M, then*

$$\begin{aligned} E^\flat &:= \{(a,b) \mid (a,b), (x\alpha a, x\alpha b), (a\beta y, b\beta y), (x\alpha a\beta y, x\alpha b\beta y) \in E \text{ for all} \\ &\quad x, y \in M \text{ and } \alpha, \beta \in \Gamma\}. \end{aligned}$$

is the largest congruence on M contained in E.

Proof. Similar to the proof of Proposition 5.13 [2], we obtain it. □

2 Main Results

In this section, we give some characterizations of weakly r-archimedean Γ-semigroups that are given by the relation $\eta_r \cap \eta_r^{-1}$ on M and some characterizations of bands of weakly r-archimedean sub-Γ-semigroups.

Lemma 2.1 *A sub-*Γ*-semigroup* T *of* M *is weakly* r*-archimedean if and only if* $(a,b) \in \eta_r \cap \eta_r^{-1}$ *for all* $a,b \in T$.

Proof. Assume that a sub-Γ-semigroup T is weakly r-archimedean and let $a,b \in T$. Then $b \in a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1}b \in a\Gamma M$ for some $\alpha_1, \alpha_2, \dots, \alpha_{m-1} \in \Gamma$, and $a \in b\Gamma M$ or there exists an integer $n \geq 2$ such that $a\beta_1 a\beta_2 a \dots a\beta_{n-1}a \in b\Gamma M$ for some $\beta_1, \beta_2, \dots, \beta_{n-1} \in \Gamma$. Thus $b \in a \cup a\Gamma M$ or $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1}b \in a \cup a\Gamma M$, and $a \in b \cup b\Gamma M$ or $a\beta_1 a\beta_2 a \dots a\beta_{n-1}a \in b \cup b\Gamma M$. Hence $(a,b) \in \eta_r$ and $(b,a) \in \eta_r$, so $(a,b) \in \eta_r \cap \eta_r^{-1}$.

Conversely, assume that $(a,b) \in \eta_r \cap \eta_r^{-1}$ for all $a,b \in T$. Now let $a,b \in T$. Then $(a,b) \in \eta_r \cap \eta_r^{-1}$, so $(a,b) \in \eta_r$. Thus $b \in a \cup a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1}b \in a \cup a\Gamma M$ for some $\alpha_1, \alpha_2, \dots, \alpha_{m-1} \in \Gamma$. Hence, let $\gamma \in \Gamma$. If $b \in a \cup a\Gamma M$, then $b\gamma b \in a\Gamma M \cup a\Gamma M\Gamma M \subseteq a\Gamma M$. If $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1}b \in a \cup a\Gamma M$, then $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1}b\gamma b \in a\Gamma M \cup a\Gamma M\Gamma M \subseteq a\Gamma M$. Therefore T is weakly r-archimedean. □

As a consequence of this result, we obtain Theorem 2.2.

Theorem 2.2 *A* Γ*-semigroup* M *is a band of weakly* r*-archimedean sub-*Γ*-semigroups of* M *if and only if it satisfies the condition for all* $a,x,y \in M$ *and* $\alpha, \beta, \gamma \in \Gamma$,

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \eta_r \cap \eta_r^{-1}. \quad (\star)$$

Proof. Assume that M is a band of weakly r-archimedean sub-Γ-semigroups of M. Then there exists a band congruence ρ on M such that the ρ-class $(x)_\rho$ of M containing x is a weakly r-archimedean sub-Γ-semigroup of M for all $x \in M$. Now let $a,x,y \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since ρ is a band congruence on M, we have $(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \rho$. Then there exist $b_1, b_2, b_3, b_4 \in M$ such that $a, a\gamma a \in (b_1)_\rho, x\alpha a, x\alpha a\gamma a \in (b_2)_\rho, a\beta y, a\gamma a\beta y \in (b_3)_\rho, x\alpha a\beta y, x\alpha a\gamma a\beta y \in (b_4)_\rho$. Since $(b_1)_\rho, (b_2)_\rho, (b_3)_\rho$ and $(b_4)_\rho$ are weakly r-archimedean, it follows from Lemma 2.1 that

$$(a, a\gamma a),\ (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \eta_r \cap \eta_r^{-1}.$$

Conversely, assume that M satisfies the condition $(\star)$.

(i) Clearly, for any $a \in M, (a,a) \in \eta_r$.

(ii) Let $a,b,c \in M$ be such that $(a,b) \in \eta_r$ and $(b,c) \in \eta_r$. Then $b \in a \cup a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1}b \in a \cup a\Gamma M$ for some $\alpha_1, \alpha_2, \dots, \alpha_{m-1} \in \Gamma$, and $c \in b \cup b\Gamma M$ or there exists an integer $n \geq 2$ such that $c\beta_1 c\beta_2 c \dots c\beta_{n-1}c \in (b \cup b\Gamma M]$ for some $\beta_1, \beta_2, \dots, \beta_{n-1} \in \Gamma$. Thus $b = a$ or $b = a\alpha s_1$ for some $s_1 \in M$ and $\alpha \in \Gamma$ or $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1}b = a$

or $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b = a\alpha s_1$ for some $s_1 \in M$ and $\alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$, and $c = b$ or $c = b\beta s_2$ for some $s_2 \in M$ and $\beta \in \Gamma$ or $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b$ or $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b\beta s_2$ for some $s_2 \in M$ and $\beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$.

Now suppose that $b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b = a\alpha s_1$ and $c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b\beta s_2$. Put $p = c\beta_1 c\beta_2 c \ldots c\beta_{n-1} c = b\beta s_2$. By hypothesis, $(p, b\alpha_1 b\beta s_2) = (b\beta s_2, b\alpha_1 b\beta s_2) \in \eta_r \cap \eta_r^{-1}$. Then $(b\alpha_1 b\beta s_2, p) \in \eta_r$. Thus $p \in b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M$ or there exists an integer $m_1 \geq 2$ such that $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1} p \in b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M$ for some $\gamma_1, \gamma_2, \ldots, \gamma_{m_1-1} \in \Gamma$. Thus $p = b\alpha_1 b\beta s_2$ or $p = b\alpha_1 b\beta s_2 \delta_1 s_3$ for some $s_3 \in M$ and $\delta_1 \in \Gamma$ or $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1} p = b\alpha_1 b\beta s_2$ or $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1} p = b\alpha_1 b\beta s_2 \delta_1 s_3$ for some $s_3 \in M$ and $\delta_1 \in \Gamma$. Hence there exists an integer $k \geq n$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k-1} c \in b\alpha_1 b\beta s_2 \cup b\alpha_1 b\beta s_2 \Gamma M$ for some $\lambda_1, \lambda_2, \ldots, \lambda_{k-1} \in \Gamma$.

Case 1: $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1} p = b\alpha_1 b\beta s_2 \delta_1 s_3$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2 \delta_1 s_3) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2 \delta_1 s_3$, there exists an integer $k_1 \geq nm_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1} c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

Case 2: $p\gamma_1 p\gamma_2 p \ldots p\gamma_{m_1-1} p = b\alpha_1 b\beta s_2$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2$, there exists an integer $k_1 \geq nm_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1} c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

Case 3: $p = b\alpha_1 b\beta s_2 \delta_1 s_3$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2 \delta_1 s_3, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2 \delta_1 s_3) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2 \delta_1 s_3$, there exists an integer $k_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1} c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

Case 4: $p = b\alpha_1 b\beta s_2$. Similar to the case as above, since

$$(b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 r_1) = (b\alpha_1 b\beta s_2, b\alpha_1 b\alpha_2 b\alpha_1 b\beta s_2) \in \eta_r \cap \eta_r^{-1}$$

where $r_1 = b\beta s_2$, there exists an integer $k_1 \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_1-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_1-1} c \in b\alpha_1 b\alpha_2 b\alpha_1 r_1 \cup b\alpha_1 b\alpha_2 b\alpha_1 r_1 \Gamma M$.

If we continue in this way, there exist $r_{m-2} \in M$, an integer $k_{m-2} \geq n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{k_{m-2}-1} \in \Gamma$ such that $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_{m-2}-1} c \in b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2} \cup b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2} \Gamma M$. Therefore $c\lambda_1 c\lambda_2 c \ldots c\lambda_{k_{m-2}-1} c \in b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2} \cup b\alpha_1 b\alpha_2 b \ldots b\alpha_{m-1} b\alpha_1 r_{m-2} \Gamma M = a\alpha s_1 \alpha_1 r_{m-2} \cup a\alpha s_1 \alpha_1$

$r_{m-2}\Gamma M \subseteq a\Gamma M \subseteq a \cup a\Gamma M$. Hence $(a, c) \in \eta_r$. In another case, we can show that $(a, c) \in \eta_r$. By (i) and (ii), $\eta_r \cap \eta_r^{-1}$ is an equivalence relation on M.

(iii) Let

$$\begin{aligned} \rho \ := \ & \{(a,b) \mid (a,b), (x\alpha a, x\alpha b), (a\beta y, b\beta y), (x\alpha a\beta y, x\alpha b\beta y) \in \eta_r \cap \eta_r^{-1} \\ & \text{for all } x, y \in M \text{ and } \alpha, \beta \in \Gamma\}. \end{aligned}$$

Since $\eta_r \cap \eta_r^{-1}$ is an equivalence relation on M, it follows from Lemma 1.3 that ρ is the largest congruence on M contained in $\eta_r \cap \eta_r^{-1}$. By condition $(\star)$, ρ is a band congruence on M.

(iv) For any $x \in M$, let $a, b \in (x)_\rho$. Then $(a, b) \in \rho$, so $(a, b) \in \eta_r \cap \eta_r^{-1}$. Since ρ is a band congruence on M, $(x)_\rho$ is a sub-Γ-semigroup of M. It follows from Lemma 2.1 that $(x)_\rho$ is a weakly r-archimedean sub-Γ-semigroup of M. Therefore M is a band of weakly r-archimedean sub-Γ-semigroups of M.

Hence the proof is completed. □

We briefly recall here the definition of ordered Γ-semigroup. A partially ordered Γ-semigroup M is called an *ordered* Γ*-semigroup* [3] if for any $a, b, c \in M$ and $\gamma \in \Gamma, a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Theorem 2.3 *Let*

$$\begin{aligned} \rho \ := \ & \{(a,b) \mid (a,b), (x\alpha a, x\alpha b), (a\beta y, b\beta y), (x\alpha a\beta y, x\alpha b\beta y) \in \eta_r \cap \eta_r^{-1} \\ & \textit{for all } x, y \in M \textit{ and } \alpha, \beta \in \Gamma\}. \end{aligned}$$

be a congruence on M. Then M/ρ is an ordered Γ-semigroup.

Proof. Let $\preceq$ be a relation on M/ρ defined as following:

$$\preceq := \{((x)_\rho, (y)_\rho) \mid (x_1, y_1) \in \rho^m \text{ for some } x_1 \in (x)_\rho, y_1 \in (y)_\rho \text{ and } m \in \mathbb{Z}^+\}.$$

We shall show that $(M/\rho, \cdot, \preceq)$ is an ordered Γ-semigroup.

(i) For any $(x)_\rho \in M/\rho$. Since $(x, x) \in \rho$, $(x)_\rho \preceq (x)_\rho$. Thus $\preceq$ is reflexive.

(ii) Let $(x)_\rho, (y)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (x)_\rho$. Then there exist $x_1, x_2 \in (x)_\rho, y_1, y_2 \in (y)_\rho$ and $m, n \in \mathbb{Z}^+$ such that $(x_1, y_1) \in \rho^m$ and $(y_2, x_2) \in \rho^n$. Thus there exist $w_1, w_2, \ldots, w_{m-1}, w_1', w_2', \ldots, w_{n-1}' \in M$ such that

$$(x_1, w_1), (w_1, w_2), .., (w_{m-1}, y_1) \in \rho, \tag{1}$$

and

$$(y_2, w_1'), (w_1', w_2'), .., (w_{n-1}', x_2) \in \rho. \tag{2}$$

Since $(x_1, w_1) \in \rho$, we have $(x_1, w_1), (x\alpha x_1, x\alpha w_1), (x_1\beta y, w_1\beta y), (x\alpha x_1\beta y, x\alpha w_1\beta y) \in \eta_r \cap \eta_r^{-1}$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

Let $x, y \in M$ and $\alpha, \beta \in \Gamma$, and $p = x\alpha x_1\beta y$. Suppose that $(x\alpha x_1\beta y, x\alpha w_1\beta y) \in \eta_r \cap \eta_r^{-1}$. Then $(x\alpha w_1\beta y, x\alpha x_1\beta y) \in \eta_r$. Thus $x\alpha x_1\beta y \in x\alpha w_1\beta y \cup x\alpha w_1\beta y\Gamma M$ or there exists an integer $k_1 \geq 2$ such that $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2 (x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) \in x\alpha w_1\beta y \cup x\alpha w_1\beta y\Gamma M$ for some $\alpha_1, \alpha_2, \ldots, \alpha_{k_1-1} \in \Gamma$. Then $x\alpha x_1\beta y = x\alpha w_1\beta y$ or $x\alpha x_1\beta y = x\alpha w_1\beta y\delta_1 s_1$ for some $s_1 \in M$ and $\delta_1 \in \Gamma$ or $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) = x\alpha w_1\beta y$ or $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1 \beta y) = x\alpha w_1\beta y\delta_1 s_1$ for some $s_1 \in M$ and $\delta_1 \in \Gamma$.

Case 1: $x\alpha x_1\beta y = x\alpha w_1\beta y$. Then $p = x\alpha w_1\beta r_1$ where $r_1 = y$. Next, since $(w_1, w_2) \in \rho$, $(x\alpha w_1\beta r_1, x\alpha w_2\beta\, r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1\beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1) \beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 1.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 1.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 1.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 1.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 2: $x\alpha x_1\beta y = x\alpha w_1\beta y\delta_1 s_1$. Then $p = x\alpha w_1\beta r_1$ where $r_1 = y\delta_1 s_1$. Next, since $(w_1, w_2) \in \rho$, $(x\alpha w_1\beta r_1, x\alpha w_2\beta r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1 \beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1 \beta r_1) \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1) \beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 2.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 2.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 2.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 2.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\beta_1 p\beta_2 p\ldots p\beta_{k_2-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 3: $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) = x\alpha w_1\beta y$. Then $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p = x\alpha w_1\beta r_1$ where $r_1 = y$. Next, since $(w_1, w_2) \in \rho$, $(x\alpha w_1\beta r_1, x\alpha w_2\beta r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1\beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 3.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 3.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 3.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$. Then $(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 3.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 4: $(x\alpha x_1\beta y)\alpha_1(x\alpha x_1\beta y)\alpha_2(x\alpha x_1\beta y)\ldots(x\alpha x_1\beta y)\alpha_{k_1-1}(x\alpha x_1\beta y) = x\alpha w_1\beta y\delta_1 s_1$. Then $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p = x\alpha w_1\beta r_1$ where $r_1 = y\delta_1 s_1$. Next, since $(w_1, w_2) \in \rho, (x\alpha w_1\beta r_1, x\alpha w_2\beta r_1) \in \eta_r \cap \eta_r^{-1}$. Thus $(x\alpha w_2\beta r_1, x\alpha w_1\beta r_1) \in \eta_r$. Then $x\alpha w_1\beta r_1 \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ or there exists an integer $k_2 \geq 2$ such that $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) \in x\alpha w_2\beta r_1 \cup x\alpha w_2\beta r_1\Gamma M$ for some $\beta_1, \beta_2, \ldots, \beta_{k_2-1} \in \Gamma$. Then $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$ or $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$ or $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$ for some $s_2 \in M$ and $\delta_2 \in \Gamma$.

Case 4.1: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1$. Then $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 4.2: $x\alpha w_1\beta r_1 = x\alpha w_2\beta r_1\delta_2 s_2$. Then $p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

Case 4.3: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1$. Then $(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1$.

Case 4.4: $(x\alpha w_1\beta r_1)\beta_1(x\alpha w_1\beta r_1)\beta_2(x\alpha w_1\beta r_1)\ldots(x\alpha w_1\beta r_1)\beta_{k_2-1}(x\alpha w_1\beta r_1) = x\alpha w_2\beta r_1\delta_2 s_2$. Then $(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_1(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_2(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\ldots(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p)\beta_{k_2-1}(p\alpha_1 p\alpha_2 p\ldots p\alpha_{k_1-1}p) = x\alpha w_2\beta r_2$ where $r_2 = r_1\delta_2 s_2$.

If we continue in this way, we have $p = x\alpha y_1\beta r_m$ or there exists an integer $k \geq 2$ such that $p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p = x\alpha y_1\beta r_m$ for some $\lambda_1, \lambda_2, \ldots, \lambda_{k-1} \in \Gamma$. Since $(y_1, y_2) \in \rho, (x\alpha y_1\beta r_m, x\alpha y_2\beta r_m) \in \eta_r\cap\eta_r^{-1}$. Thus $(x\alpha y_2\beta r_m, x\alpha y_1\beta r_m) \in \eta_r$. Then $x\alpha y_1\beta r_m \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$ or there exists an integer $k' \geq 2$ such that $(x\alpha y_1\beta r_m)\gamma_1'(x\alpha y_1\beta r_m)\gamma_2'(x\alpha y_1\beta r_m)\ldots(x\alpha y_1\beta r_m)\gamma_{k'-1}'(x\alpha y_1\beta r_m) \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$ for some $\gamma_1', \gamma_2', \ldots, \gamma_{k'-1}' \in \Gamma$.

Put $q = x\alpha y_1\beta r_m$.

Case I: $p = q$ and $q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $p \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Case II: $p = q$ and $q\gamma_1' q\gamma_2' q\ldots q\gamma_{k'-1}' q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $p\gamma_1' p\gamma_2' p\ldots p\gamma_{k'-1}' p \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Case III: $p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p = q$ and $q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Case IV: $p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p = q$ and $q\gamma_1' q\gamma_2' q\ldots q\gamma_{k'-1}' q \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M$. Then $(p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p)\gamma_1'(p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p)\gamma_2'(p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p)\ldots(p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p)\gamma_{k'-1}'(p\gamma_1 p\gamma_2 p\ldots p\gamma_{k-1}p) \in x\alpha y_2\beta r_m \cup x\alpha y_2\beta r_m\Gamma M \subseteq x\alpha y_2\beta y \cup x\alpha y_2\beta y\Gamma M$. Hence $(x\alpha y_2\beta y, x\alpha x_1\beta y) \in \eta_r$.

Similar to the proof as above, by (2), we can prove that $(x\alpha x_1\beta y, x\alpha y_2\beta y) \in \eta_r$. Therefore $(x\alpha x_1\beta y, x\alpha y_2\beta y) \in \eta_r \cap \eta_r^{-1}$.

In another case, we have the following:

(1) If $(z_1, w_1) \in \eta_r \cap \eta_r^{-1}$, then $(x_1, y_2) \in \eta_r \cap \eta_r^{-1}$.

(2) If $(x\alpha z_1, x\alpha w_1) \in \eta_r \cap \eta_r^{-1}$, then $(x\alpha x_1, x\alpha y_2) \in \eta_r \cap \eta_r^{-1}$.

(3) If $(z_1\beta y, w_1\beta y) \in \eta_r \cap \eta_r^{-1}$, then $(x_1\beta y, y_2\beta y) \in \eta_r \cap \eta_r^{-1}$.

Therefore $(x_1, y_1) \in \rho$, so $(x)_\rho = (x_1)_\rho = (y_2)_\rho = (y)_\rho$. Hence $\preceq$ is anti-symmetric.

(*iii*) Let $(x)_\rho, (y)_\rho, (z)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho$ and $(y)_\rho \preceq (z)_\rho$. Then there exist $x_1 \in (x)_\rho, y_1, y_2 \in (y)_\rho, z_2 \in (z)_\rho$ and $m, n \in \mathbb{Z}^+$ such that

$(x_1, y_1) \in \rho^m$ and $(y_2, z_2) \in \rho^n$. Thus $x_1\rho^m y_1\rho y_2\rho^n z_2$, so $(x_1, z_2) \in \rho^{m+n+1}$. Hence $(x)_\rho \preceq (z)_\rho$. Therefore $\preceq$ is transitive.

(iv) Let $(x)_\rho, (y)_\rho \in M/\rho$ be such that $(x)_\rho \preceq (y)_\rho, (z)_\rho \in M/\rho$ and $\gamma \in \Gamma$. Then there exist $x_1 \in (x)_\rho, y_1 \in (y)_\rho$ and $m \in \mathbb{Z}^+$ such that $(x_1, y_1) \in \rho^m$. It follows from Corollary 1.2 that $(z\gamma x_1, z\gamma y_1), (x_1\gamma z, y_1\gamma z) \in \rho^m$. Since $z\gamma x_1 \in (z\gamma x)_\rho, x_1\gamma z \in (x\gamma z)_\rho, z\gamma y_1 \in (z\gamma y)_\rho$ and $y_1\gamma z \in (y\gamma z)_\rho$, we have $(z\gamma x)_\rho \preceq (z\gamma y)_\rho$ and $(x\gamma z)_\rho \preceq (y\gamma z)_\rho$. Thus $(z)_\rho\gamma(x)_\rho = (z\gamma x)_\rho \preceq (z\gamma y)_\rho = (z)_\rho\gamma(y)_\rho$ and $(x)_\rho\gamma(z)_\rho = (x\gamma z)_\rho \preceq (y\gamma z)_\rho = (y)_\rho\gamma(z)_\rho$. Hence $\preceq$ is compatible, so $(M/\rho, \cdot, \preceq)$ is an ordered Γ-semigroup.

This completes the proof. □

Immediately from Theorem 2.2 and 2.3, we have Corollary 2.4.

Corollary 2.4 *If a Γ-semigroup M is a band of weakly r-archimedean sub-Γ-semigroups of M, then there exists a congruence ρ on M such that M/ρ is an ordered Γ-semigroup.*

Lemma 2.5 *Let T be a left ideal of M. Then the following statements are equivalent:*

(a) *T is a weakly r-archimedean sub-Γ-semigroup of M.*

(b) *T is an r-archimedean sub-Γ-semigroup of M.*

(c) *For any $a, b \in T, (a, b) \in \eta_r \cap \eta_r^{-1}$.*

Proof. By Lemma 2.1, (b) implies (c) and (c) implies (a). Now, we shall prove that (a) implies (b). Suppose that T is a weakly r-archimedean sub-Γ-semigroup of M and let $a, b \in T$. It follows from Lemma 2.1 that $(a, b) \in \eta_r \cap \eta_r^{-1}$, so $(a, b) \in \eta_r$. Thus $b \in a \cup a\Gamma M$ or there exists an integer $m \geq 2$ such that $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b \in a \cup a\Gamma M$ for some $\alpha_1, \alpha_2, \dots, \alpha_{m-1} \in \Gamma$. For any $\gamma \in \Gamma$. If $b = a$ or $b = a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$, then $b\gamma b = a\gamma b \in a\Gamma T$ or $b\gamma b = a\alpha x\gamma b \in a\Gamma T$ since T is a left ideal of M. If $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b = a$ or $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b = a\alpha x$ for some $x \in M$ and $\alpha \in \Gamma$, then $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b\gamma b = a\gamma b \in a\Gamma T$ or $b\alpha_1 b\alpha_2 b \dots b\alpha_{m-1} b\gamma b = a\alpha x\gamma b \in a\Gamma T$ since T is a left ideal of M. Therefore T is an r-archimedean sub-Γ-semigroup of M. □

Combining Theorem 2.2 and Lemma 2.5, we obtain Corollary 2.6.

Corollary 2.6 *The following statements are equivalent:*

(a) *M is a band of weakly r-archimedean left ideals of M.*

(b) *M is a band of r-archimedean left ideals of M.*

(c) *M satisfies the condition for all* $a, x, y \in M$ *and* $\alpha, \beta, \gamma \in \Gamma$,

$$(a, a\gamma a), (x\alpha a, x\alpha a\gamma a), (a\beta y, a\gamma a\beta y), (x\alpha a\beta y, x\alpha a\gamma a\beta y) \in \eta_r \cap \eta_r^{-1}. \quad (\star)$$

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