International Mathematical Forum, 3, 2008, no. 5, 205 - 208

On Nuclearity Maps

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Abstract

In this paper, we characterize λ-nuclear maps in term of pseudo-$\lambda\lambda^{\times}$-nuclear maps. Then we use our result to give a characterization of a $\lambda(A)$-nuclear map where $\lambda(A)$ is a smooth sequence space of finite or infinite type.

Mathematics Subject Classification: 46A45, 46A11

Keywords: Nuclear maps, Pseudo λ-nuclear maps, Nuclearity, Smooth sequence spaces

1 Basic Concepts

The **Köthe dual** $\lambda^{\times}$ of the sequence space λ is defined to be

$$\lambda^{\times} := \{\zeta : \sum_n |\eta_n \zeta_n| < +\infty \ \ \forall \eta \in \lambda\}.$$

If η, ζ are two sequences of scalars, we write $\zeta = O(\eta)$ if there exists $M > 0$ such that $|\zeta_n| \leq M\,|\eta_n|$ for all $n \in \mathbf{N}$.

A set A of sequences of non-negative real numbers is called a **Köthe set**, if it satisfies the following conditions:

1. For each pair of elements $a, b \in A$ there is $c \in A$ with $a_n = O(c_n)$ and $b_n = O(c_n)$.

2. For every integer $r \in N$ there exists $a \in A$ with $a_r > 0$.

The Köthe space $\lambda(A)$ of the Köthe set A is the space of all sequences $x = (x_n)$ such that

$$p_a(x) := \sum_n |x_n|\, a_n < +\infty$$

A Köthe set P will be called a **power set of infinite type** if it satisfies the following conditions:

1. For each $a \in P$, $0 < a_n \leq a_{n+1}$ for all n.
2. For each $a \in P$, there exists $b \in P$ such that $a_n^2 = O(b_n)$.

A Köthe space of the form $\lambda(P)$ where P is a power set of infinite type is called a G_∞**-space** or a **smooth sequence space of infinite type**[5].

A Köthe set Q will be called a **power set of finite type** if it satisfies the following conditions:

1. Each $q \in Q$ is a positive non-increasing sequence.
2. For each $q \in Q$ there exists $p \in Q$ with $\sqrt{q_n} = O(p_n)$.

A Köthe space of the form $\lambda(Q)$ where Q is a power set of finite type is called a G_1**-space** or a **smooth sequence space of finite type**[5].

Let $\alpha = (\alpha_n)$ be an unbounded non-decreasing sequence of positive real numbers. Then

$$P_1 = \{((1 - 1/k)^{\alpha_n}) \colon\, k \in \mathbf{N}\}$$

is countable Köthe set. The corresponding Köthe spaces $\Lambda_1(\alpha) = \lambda(P_1)$ is called the **power series space of finite type**[3].

Definition 1.1 *[1][2] A linear map T from a normed space E into a normed space F is called a* ***λ-nuclear map*** *if there exist a sequence (α_n) in λ and sequences (a_n) and (y_n) in E' and F respectively such that (a_n) is bounded and (y_n) has the property that for each $b \in F'$, $(\langle y_n, b\rangle) \in \lambda^\times$ and such that*

$$Tx = \sum\nolimits_n \alpha_n \langle x, a_n\rangle y_n,$$

for all $x \in E$.

Definition 1.2 *[1][2] A linear map T from a normed space E into a normed space F is called a* ***pseudo-λ-nuclear map*** *if there exist a sequence (α_n) in λ and bounded sequences (a_n) and (y_n) in E' and F respectively such that*

$$Tx = \sum\nolimits_n \alpha_n \langle x, a_n\rangle y_n,$$

for all $x \in E$.

2 Main results.

Prior to presenting our main results it is worth mentioning that for any sequence space λ, one can construct a Köthe space, $\lambda^\times$, which is generated by the Köthe set

$$C\lambda := \{(|x_n|) \colon\, (x_n) \in \lambda\}.$$

The following known results are crucial in proving our results.

Lemma 2.1 *[1] Let F be a Banach space, λ sequence space and $(\alpha_n) \in \lambda$. Let (y_n) be a sequence in F such that $(\langle y_n, b\rangle) \in \lambda^\times$ for each $b \in F'$. Then*

$$\sup_{b\in F', ||b||\leq 1} \sum_n |\alpha_n \langle y_n, b\rangle| < +\infty.$$

Lemma 2.2 [4] *If $\lambda(P)$ is a nuclear Köthe space, then $\lambda(P) = B(P)$ where $B(P) := \{x\colon\ xa \in \ell_\infty\ \ \forall a \in P\}$.*

The following definition is introduced in order to facilitate our subsequent arguments.

Definition 2.1 *A sequence space λ is called* ***strong dual*** *if $\lambda^\times$ is nuclear.*

Proposition 2.1 *For a strong dual space λ, a bounded linear map T from a Banach space E into a Banach space F is λ-nuclear iff it is a pseudo-$\lambda\lambda^\times$-nuclear map.*

Proof. ($\Rightarrow$) Assume that $T : E \to F$ is λ -nuclear. Then there exist a sequence (α_n) in λ and sequences (a_n) and (y_n) in E' and F respectively such that the sequence (a_n) is bounded and $(\langle y_n, b\rangle) \in \lambda^\times$ for all $b \in F'$ and such that

$$Tx = \sum_n \alpha_n \langle x, a_n\rangle y_n.$$

Given $\alpha \in \lambda$. Let $M = \sup_{b\in F', ||b||\leq 1} \sum_n |\alpha_n \langle y_n, b\rangle|$. Then by Lemma 2.1, M is finite. Since $|\alpha_n| ||y_n|| = \sup\{|\alpha_n \langle y_n, b\rangle| \colon\ b \in F',\ ||b|| \leq 1\} \leq M$, we have $(|\alpha_n| ||y_n||) \in \ell_\infty$. Since α is arbitrary, we have $(||y_n||) \in B(C\lambda)$. By Lemma 2.2, we have $(||y_n||) \in \lambda(C\lambda) = \lambda^\times$. Let $\beta_n = \alpha_n ||y_n||$ and $x_n = y_n/||y_n||$. Then

$$Tx = \sum_n \beta_n \langle x, a_n\rangle x_n.$$

Since the sequence (β_n) is in $\lambda\,\lambda^\times$, and since (a_n) and (x_n) are bounded sequences in E' and F respectively, T is a pseudo-$\lambda\,\lambda^\times$-nuclear map.

($\Leftarrow$) Assume that T is a pseudo-$\lambda\,\lambda^\times$-nuclear map. Then there exist sequences (α_n) in λ and (β_n) in $\lambda^\times$, and bounded sequences (a_n) and (y_n) in E' and F respectively such that

$$Tx = \sum_n \alpha_n \beta_n \langle x, a_n\rangle y_n.$$

Let $z_n = \beta_n y_n$. Then $Tx = \sum_n \alpha_n \langle x, a_n\rangle z_n$. Since $(\alpha_n) \in \lambda$, (a_n) is bounded sequence in E' and $(\langle z_n, b\rangle) \in \lambda^\times$ for all $b \in F'$, T is a λ-nuclear map. ■

Lemma 2.3 *If $\lambda(P)$ is a nuclear G_∞-space, then $\lambda(P) = \lambda(P)\,\lambda(P)^\times$.*

Proof. Since $(1, 1, \ldots) \in \lambda(P)^\times$, we have $\lambda(P) \subseteq \lambda(P)\,\lambda(P)^\times$. Also, since $xa \in \lambda(P)$ for all $x \in \lambda(P)$ and $a \in P$, we have $\lambda(P)\,\lambda(P)^\times \subseteq \lambda(P)$. ■

Lemma 2.4 *If* $\lambda(Q)$ *is a nuclear* G_1*-space, then* $\lambda(Q)^\times = \lambda(Q)\,\lambda(Q)^\times$.

Proof. Since $(1, 1, \dots) \in \lambda(Q)$, we have $\lambda(Q)^\times \subseteq \lambda(Q)\,\lambda(Q)^\times$. Also, since $xy \in \lambda(Q)$ for all $x, y \in \lambda(Q)$, we have $\lambda(Q)\,\lambda(Q)^\times \subseteq \lambda(Q)^\times$. ■

Lemma 2.5 *If* $\lambda(P)$ *is a nuclear smooth sequence space of infinite or finite type, then* $\lambda(P)$ *is strong dual.*

Proof. Follows from Grothendieck-Pietsch criterion for nuclearity and the fact that for a nuclear smooth sequence space $\lambda(P)$ of finite or infinite type $(n^2 x_n) \in \lambda(P)$ for $(x_n) \in \lambda(P)$. ■

For a nuclear smooth sequence space $\lambda(P)$ of infinite type, our next results give a characterization of $\lambda(P)$-nuclear maps in term of pseudo-$\lambda(P)$-nuclear maps.

Theorem 2.1 *Given a nuclear* G_∞*-space* $\lambda(P)$*, a linear map* T *from a Banach space* E *into a Banach space* F *is a* $\lambda(P)$*-nuclear map if and only if it is a pseudo-*$\lambda(P)$*-nuclear map.*

Proof. Follows from Lemmas 2.3, 2.5 and Proposition 2.1. ■.

For a nuclear smooth sequence space $\lambda(Q)$ of finite type, in the next result we characterize $\lambda(Q)$-nuclear maps in term of pseudo-$\lambda(Q)^\times$-nuclear maps.

Theorem 2.2 *Given a nuclear* G_1*-space* $\lambda(Q)$*, a linear map* T *from a Banach space* E *into a Banach space* F *is* $\lambda(Q)$*-nuclear if and only if it is a pseudo-*$\lambda(Q)^\times$*-nuclear.*

Proof. This follows from Lemmas 2.4,2.5 and Proposition 2.1. ■

Corollary 2.1 [3] *A linear map* T *from a Banach space* E *into a Banach space* F *is* $\Lambda_1(\alpha)$*-nuclear iff it is* $\Lambda_1(\alpha)^\times$*-nuclear*

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Received: September 3, 2007