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Representing and Recognizing 3-Manifolds Obtained from I-Bundles over the Klein Bottle[1]

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Abstract

As it is well-known, the boundary of the orientable I-bundle $K \tilde{\times} I$ over the Klein bottle K is a torus; thus - in analogy with torus bundle construction (see [18]) - any integer matrix A of order two with determinant -1 (resp. $+1$) uniquely defines an orientable (resp. non-orientable) 3-manifold $(K\tilde{\times}I)\cup(K\tilde{\times}I)/A$, which we denote by $KB(A)$. In the present paper an algorithmic procedure is described, which allows to construct, directly from any such matrix A, an edge-coloured graph representing the manifold $KB(A)$ associated to A. As a consequence, it is proved via regular genus (see [13]) that the Heegaard genus of any such manifold is less or equal to four; moreover, six elements of existing catalogues of orientable 3-manifolds represented by edge-coloured graphs (see [15] and [6]) are directly recognized as manifolds of type $KB(A)$.

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1. Introduction

3-manifolds obtained by pasting together two copies of the orientable I-bundle over the Klein bottle K frequently appear (together with torus bundles over the circle) in existing catalogues of "simple" 3-manifolds: see, for example, [16] and [3]. The present paper performs an approach to the study of such 3-manifolds

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via edge-coloured graphs as a combinatorial PL-manifolds representation tool (see [12], [2], [19], [10], [1], together with their references). In particular, an algorithmic procedure is described, which allows to construct, directly from any matrix $A \in GL(2;\mathbb{Z})$, a pseudosimplicial triangulation (and, hence, the edge-coloured graph $\Gamma_{KB}(A)$ visualizing it) of the manifold of type $KB(A) = (K \tilde{\times} I) \cup (K \tilde{\times} I)/A$ associated to A, i.e.

$$KB(A) = \mathbf{K}_1 \cup_{\tilde{\phi}_A} \mathbf{K}_2$$

where $\mathbf{K}_1$ and $\mathbf{K}_2$ are two copies of the orientable I-bundle over the Klein bottle K, $K \tilde{\times} [0,1]$, and $\tilde{\phi}_A : \partial\mathbf{K}_1 \to \partial\mathbf{K}_2$ is the homeomorphism of the bidimensional torus T into itself (note that both $\partial\mathbf{K}_1$ and $\partial\mathbf{K}_2$ are homeomorphic to T) induced - up to isotopy - by matrix A.

As a consequence, since edge-coloured graphs allow to define an n-dimensional combinatorial invariant for PL-manifolds - called *regular genus* (see [13] for its definition and, for example, [8] and [1] for subsequent related results) -, which coincides with Heegaard genus in the 3-dimensional setting, 3-manifolds of type $KB(A)$ are combinatorially proved to have Heegaard genus less or equal to four.

On the other hand, the described construction is applied in order to combinatorially recognize all manifolds of type $KB(A)$ belonging to Lins's catalogue [15] (resp. to Casali-Cristofori' catalogue [6]) of closed connected orientable 3-manifolds represented by edge-coloured graphs up to 28 (resp. 30) vertices, i.e. admitting a pseudosimplicial triangulation consisting of at most 28 (resp. 30) tetrahedra.

In fact, the quoted catalogues contain exactly six orientable 3-manifolds which are already known to be of type $KB(A)$, for suitable matrices $A \in GL(2;\mathbb{Z})$ with $\det(A) = -1$, through GM-complexity computation and homology group comparison or - in the 30 vertices case - through program *Three-manifold Recognizer*[2] (see [7]; Table 1 and Table 2). Here, the 4-coloured graph $\Gamma_{KB}(A)$ associated to each one of these matrices is effectively constructed and simplified by suitable combinatorial moves not affecting the homeomorphism class of the represented manifold, till to obtain an element of known crystallization catalogues, just encoding the manifold $KB(A)$.

We point out that the combinatorial nature of the representing tools, together with the algorithmic feature of the described construction, allow to imagine a suitable implementation of the whole process.[3]

[2] *Three-manifold Recognizer* is a program written by V. Tarkaev as an application of the results about recognition of 3-manifolds obtained by S.Matveev and his research group; it is available on the Web: http://www.topology.kb.csu.ru/∼recognizer/

[3] In analogy with Visual Basic program *TB(A)*, concerning torus bundles con-

Since - as already pointed out - manifolds obtained by pasting together two copies of the orientable I-bundle over the Klein bottle K frequently appear in existing catalogues of 3-manifolds, the author hopes the construction obtained in the present paper to be of use in order to perform interesting comparisons between different 3-manifold complexity notions.

2. Basic notions on 3-manifolds obtained from I-bundles over the Klein bottle

As it is well-known, the homeotopy group of bidimensional torus T, i.e. the mapping class group of punctured homeomorphisms $(T, x_0) \to (T, x_0)$ $(x_0 \in T)$, is isomorphic to the group of automorphisms of $\pi_1(T)$, i.e. to $GL(2;\mathbb{Z})$ (see [20]; Theorem 5.15.5).

This implies that any matrix $A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in GL(2;\mathbb{Z})$ induces, up to isotopy, a homeomorphism $\tilde{\phi}_A : T_1 \to T_2$, T_1 and T_2 being two copies of torus T; if (c_0, c_1) (resp. (c'_0, c'_1)) denotes a coordinate system $(meridian, longitude)$ of torus T_1 (resp. T_2), oriented so that the intersection number between c_0 and c_1 (resp. between c'_0 and c'_1) is $+1$, then $\tilde{\phi}_A$ maps c_0 (resp. c_1) into the curve $\tilde{\phi}_A(c_0) = a_{00}c'_0 + a_{01}c'_1$ (resp. $\tilde{\phi}_A(c_1) = a_{10}c'_0 + a_{11}c'_1$) and sends the (unique) intersection point $c_0 \cap c_1$ into the (unique) intersection point $c'_0 \cap c'_1$.

Let now $\mathbf{K}_1$ (resp. $\mathbf{K}_2$) be a copy of the orientable I-bundle over the Klein bottle K, $K \tilde{\times} [0,1]$, and let (c_0, c_1) (resp. (c'_0, c'_1)) be a fixed coordinate system on the bidimensional torus $\partial\mathbf{K}_1$ (resp. $\partial\mathbf{K}_2$), such that c_0 (resp. c'_0) projects onto a meridian - i.e. a nontrivial orientation-preserving circle - of the core Klein bottle and c_1 (resp. c'_1) double covers a longitude - i.e. an orientation-reversing circle - of the core Klein bottle.

It is easy to check that (in full analogy with torus bundle case: see [18]; sections 3.2 and 18.1) the homeomorphism $\tilde{\phi}_A : \partial\mathbf{K}_1 \to \partial\mathbf{K}_2$ uniquely determines a closed 3-manifold defined as

$$KB(A) = \frac{\mathbf{K}_1 \cup \mathbf{K}_2}{\sim_A},$$

where the equivalence relation $\sim_A$ on $\partial\mathbf{K}_1 \cup \partial\mathbf{K}_2$ is given by

$$x \sim_A \tilde{\phi}_A(x), \quad \forall x \in \partial\mathbf{K}_1.$$

struction, this program might be integrated within C++ program *DUKE III*, which is devoted to automatic analysis, manipulation and recognition of PL-manifolds via edge-coloured graphs; for both programs *TB(A)* and *DUKE III*, see http://cdm.unimo.it/home/matematica/casali.mariarita/DUKEIII.htm

Note that $A \in GL(2;\mathbb{Z})$ directly implies $\det(A) \in \{\pm 1\}$; more precisely, manifold $KB(A)$ is orientable (resp. non orientable) if and only if $\det(A) = -1$ (resp. $\det(A) = +1$).

The following technical lemma results to be useful, for our purposes, in order to restrict the attention to a subclass of matrices, inducing <u>all</u> manifolds of type $KB(A)$.

Lemma 2.1 *Let M be a manifold obtained by pasting together two copies of the orientable I-bundle over the Klein bottle K. Then, M is equivalent to $KB(A)$, where $A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in GL(2;\mathbb{Z})$ is such that:*

- $a_{i0} \cdot a_{i1} \geq 0 \quad \forall i \in \{0,1\}$;
- *either* $a_{0j} \cdot a_{1j} \geq 0 \ \forall j \in \{0,1\}$ *or* $a_{0j} \cdot a_{1j} \leq 0 \ \forall j \in \{0,1\}$.

Proof. First of all, let us assume $M = KB(\bar{A})$, where matrix $\bar{A}$ contains all non-null elements. It is very easy to check that $\det(\bar{A}) \in \{\pm 1\}$ excludes the possibility that a row (resp. a column) of $\bar{A}$ consists of concordant elements and the other row (resp. column) of discordant elements; hence, the second condition of the statement turns out to be always satisfied.
Moreover, the possible exchange of matrix $\bar{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with its inverse $\bar{A}^{-1} = (det\bar{A})^{-1} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (which is obviously associated to the inverse homeomorphism of boundary tori $\partial\mathbf{K}_1$ and $\partial\mathbf{K}_2$, and hence gives rise to the same manifold), allows to exclude the case of both rows and columns consisting of discordant elements.
Among the remaining cases, the only one not satisfying the two conditions of the statement is the case of a matrix whose rows (resp. columns) consist of discordant (resp. concordant) elements. In this case, the existence of an equivalent matrix satisfying the first condition of the statement directly follows from the possibility of exchanging the orientation of the chosen curves c_1 and c_1' within coordinate systems (c_0, c_1) and (c_0', c_1') for $\partial\mathbf{K}_1$ and $\partial\mathbf{K}_2$ (together with a change of orientation on $\partial\mathbf{K}_1$ and $\partial\mathbf{K}_2$, too), so that the associated matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is transformed into $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.
Finally, note that the last transformation allows to assume both conditions of the statement to hold, also in case of a matrix $\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \in GL(2;\mathbb{Z})$ containing null elements: in fact, if $a_{\bar{i}\bar{j}} = 0$, a possible exchange of orientation on the curves c_1 and c_1' (and on $\partial\mathbf{K}_1$ and $\partial\mathbf{K}_2$, too) yields $a_{i'\bar{j}} \cdot a_{\bar{i}j'} \geq 0$ and $a_{\bar{i}j'} \cdot a_{\bar{i}\bar{j}} = 0 (\geq 0)$, with $\{\bar{i}, i'\} = \{\bar{j}, j'\} = \{0,1\}$, while $a_{\bar{i}\bar{j}} \cdot a_{i'\bar{j}} = 0$ trivially

implies the condition concerning the columns to be satisfied.

□

Lemma 2.1 suggests (in analogy to [5]; Definition 2.2) the following definition:

Definition 2.2 A matrix $A \in GL(2;\mathbb{Z})$ will be said to be *in semi-normalized shape* if it satisfies both conditions of Lemma 2.1. The subset of $GL(2;\mathbb{Z})$ consisting of regular integer matrices of order two in semi-normalized shape will be denoted by the symbol $\overline{GL}(2;\mathbb{Z})$.

Finally, in the present paper (section 4), the following results about Heegaard genus of manifolds of type $KB(A)$ will be proved, as consequences of the algorithmic procedure for the construction of an edge-coloured graph representing $KB(A)$, for any matrix $A \in \overline{GL}(2;\mathbb{Z})$.

Proposition 2.3

a) $\mathcal{H}(KB(A)) \leq 4$*, for any* $A \in GL(2;\mathbb{Z})$.

b) If $A \in GL(2;\mathbb{Z})$ *contains a null element, then* $\mathcal{H}(KB(A)) \leq 3$.

3. Representation of PL-manifolds by means of edge-coloured graphs

Edge-coloured graphs are the objects of a representation theory dealing with the whole class of piecewise-linear (PL) manifolds, without assumptions about the dimension, the connectedness, the orientability or the boundary properties (see [12] or [1] for a general survey). In the present work, however, we restrict our attention to closed and connected manifolds of dimension $n = 3$; hence, we will briefly review only basic definitions and results of the theory concerning this particular case.

For general notions on PL category, we refer to [17].

Given a pseudocomplex (see [14]) K, triangulating a 3-manifold M, a *coloration* on K is a labelling of its vertices by $\Delta_3 = \{0, 1, 2, 3\}$, which is injective on each simplex of K.

The dual 1-skeleton of K is a (multi)graph $\Gamma = (V(\Gamma), E(\Gamma))$ embedded in $|K| = M$. Γ naturally inherits from the coloration of K an *edge-coloration*, i.e. a map $\gamma : E(\Gamma) \to \Delta_3$ defined in the following way: for each $e \in E(\Gamma)$, $\gamma(e) = c$ iff the vertices of the face dual to e are labelled in K by $\Delta_3 - \{c\}$. Note that an edge-coloration is characterized by being injective on each pair of adjacent edges of the graph.

The pair (Γ, γ) (and Γ itself, too, if no confusion arises) is called a *4-coloured graph representing* M or simply a *gem of* M (gem=graph encoded manifold), according to [15].

It is easy to see that, starting from Γ, we can always reconstruct $K(\Gamma) = K$ and hence the manifold M: see [12] and [1] for details.

The elements of the set $\Delta_3 = \{0, 1, 2, 3\}$ are said to be *colours* of Γ, while $e \in E(\Gamma)$ such that $\gamma(e) = i$ ($i \in \Delta_3$) is said to be an *i-coloured edge.* For every $i, j \in \Delta_3$ let $\Gamma_{i,j}$ (resp. $\Gamma_{\hat{i}}$) the subgraph obtained from (Γ, γ) by deleting all the edges of colour $c \in \Delta_3 - \{i, j\}$ (resp. by deleting all the edges of colour i). The connected components of $\Gamma_{i,j}$ (resp. $\Gamma_{\hat{i}}$) are said to be *$\{i, j\}$-coloured cycles* (resp. *$\hat{i}$-residues*) of Γ, and their number is denoted by $g_{i,j}$ (resp. $g_{\hat{i}}$). A 4-coloured graph (Γ, γ) representing a 3-manifold M is called a *crystallization* of M iff, for each $i \in \Delta_3$, the subgraph $\Gamma_{\hat{i}}$ is connected (i.e. iff $g_{\hat{i}} = 1\ \forall i \in \Delta_3$).

Several topological properties of M can be "read" as combinatorial properties of any crystallization (or more generally any gem) Γ of M: as an example, M is orientable iff Γ is bipartite.

It is very easy to check that every 3-manifold admits a gem representing it: in fact, the existence of a coloured triangulation (and, hence, an edge-coloured graph) representing M may be directly proved by considering the first baricentric subdivision of any simplicial triangulation of M, and by labelling every vertex by the dimension of the corresponding simplex. Pezzana Theorem and its subsequent improvements ([12] or [1]) ensure crystallizations to be an universal tool to represent manifolds, too.

Of course, many gems exist for any fixed 3-manifold M. In particular, if (Γ, γ) is a gem of M, any edge-coloured graph Γ' obtained from Γ by permutation of the vertex set and/or of the colour set (Γ and Γ' are usually said to be *colour-isomorphic graphs*) is obviously a gem of M, too. In [15] and [9], an alphanumerical *code* $c(\Gamma)$ is defined for any coloured graph Γ, so that $c(\Gamma) = c(\Gamma')$ iff Γ and Γ' are colour-isomorphic graphs [4].

On the other hand, *dipole moves* (see [11]) are elementary combinatorial moves on edge-coloured graphs so that any two gems of the same manifold M are transformed one into the other by a finite sequence of dipole moves.

If the 3-manifold M is assumed to be handle-free,[5] and Γ is any gem of M, then another combinatorial move - called *ρ-pair elimination* -, together

[4] Note that the recognizion of colour-isomorphic graphs may be easily implemented: for example, DUKE III program contains a suitable "code computation" function.

[5] A manifold M is said to be *handle-free* if its decomposition via connected sum contains no factor homeomorphic to an (orientable or non-orientable) $\mathbb{S}^2$-bundle over $\mathbb{S}^1$.

with dipole moves, allows to obtain a *rigid* crystallization Γ' of M, i.e. a crystalizzation where every pair of equally coloured edges belong to one common bicoloured cycle at most (see [15] and [4]), such that $\#V(\Gamma') \leq \#V(\Gamma)$.

The embedding of a coloured graph into a surface is said to be *regular* if the connected components split by the image of the graph onto the surface are open balls (called *regions* of the embedding) bounded by the image of bicoloured cycles; interesting results of crystallization theory (mainly related to the already quoted n-dimensional extension of Heegaard genus, called *regular genus*) relay on the existence of this type of embedding, for graphs representing manifolds of arbitrary dimension.

As far as the 3-dimensional case is concerned, it is well-known that, if (Γ, γ) is a bipartite (resp. non-bipartite) gem of M^3, then for every cyclic permutation $\epsilon = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$ of Δ_3, Γ admits a regular embedding into the closed orientable (resp. non-orientable) surface of genus $\rho_\epsilon(\Gamma) = g_{\epsilon_0,\epsilon_2} - g_{\hat{\epsilon}_1} - g_{\hat{\epsilon}_3} + 1$.

The regular genus $\rho(\Gamma)$ of (Γ, γ) is, by definition, the minimum $\rho_\epsilon(\Gamma)$, among all cyclic permutations ϵ of Δ_3. Finally, the *regular genus* of a 3-manifold M is defined as:

$$\mathcal{G}(M) = \min \{\rho(\Gamma) \ / (\Gamma, \gamma) \text{ is a gem of } \ M\}.$$

4. From matrices to 4-coloured graphs representing 3-manifolds of type $KB(A)$

The following paragraph will be entirely devoted to show how to construct edge coloured graphs representing the 3-manifolds obtained from I-bundles over the Klein bottle, directly from integer matrices inducing them.

Theorem 4.1 *Let $A \in \overline{GL}(2;\mathbb{Z})$. An algorithmic procedure exists, which allows to directly construct a 4-coloured graph $\Gamma_{KB}(A)$ representing the 3-manifold obtained by pasting together two copies of the orientable I-bundle over the Klein bottle, through the boundary homeomorphism associated to matrix A.*

Proof. The statement is directly proved by construction, via the following steps.

First step: We construct two cell-complexes $\bar{K}_1$ and $\bar{K}_2$ triangulating the torus T, so that a bijective cell-map $\bar{\Phi}_A : \bar{K}_1 \to \bar{K}_2$ exists, with $|\bar{\Phi}_A| = \tilde{\phi}_A$.

Let us denote $A_{ij} = max\{|a_{ij}|, 1\}$, $\forall i, j \in \{0, 1\}$.
In order to obtain $\bar{K}_2$, it is sufficient to consider on $\frac{I \times I}{\sim}$ (endowed with coordinate system (c'_0, c'_1)) the geometrical realization of curves $\alpha'_i = a_{i0}c'_0 + a_{i1}c'_1$ ($i \in \{0, 1\}$) consisting of the $A_{i0} + A_{i1} - 1$ edges, parallel to $\vec{v} \equiv (a_{i0}, a_{i1})$, having as end-points the $A_{i1} + 1$ vertices on $I \times \{0\} = I \times \{1\}$ of first coordinate $\frac{h}{A_{i1}}$, $h \in \{0, \ldots, A_{i1}\}$, and the $A_{i0} + 1$ vertices on $\{0\} \times I = \{1\} \times I$ of second coordinate $\frac{k}{A_{i0}}$, $k \in \{0, \ldots, A_{i0}\}$[6].
On the other hand, let $\bar{K}_1$ be the cellular subdivision of $\frac{I \times I}{\sim}$ (endowed with coordinate system (c_0, c_1)) constructed in the following way:

- Let us consider the $A_{00} + A_{01}$ vertices V_r, $r \in \{0, \ldots, A_{00} + A_{01} - 1\}$, on $I \times \{0\} = I \times \{1\}$ of first coordinate $\frac{r}{A_{00}+A_{01}-1}$ and the $A_{10} + A_{11}$ vertices W_s, $s \in \{0, \ldots, A_{10} + A_{11} - 1\}$, on $\{0\} \times I = \{1\} \times I$ of second coordinate $\frac{s}{A_{10}+A_{11}-1}$.[7]
- In case $A_{00} \geq A_{10}$ (and $A_{01} \geq A_{11}$, too[8]), then:
  - For every vertex of the set $\{W_0, \ldots, W_{A_{10}+A_{11}-1}\} \subset \{0\} \times I = \{1\} \times I$, let us consider an edge internal to $I \times I$ parallel to $\vec{w} \equiv (A_{10} + A_{11} - 1, -\mu(A)(A_{00} + A_{01} - 1))$, where $\mu(A) = +1$ (resp. $\mu(A) = -1$) if $a_{0j} \cdot a_{1j} \geq 0\ \forall j \in \{0, 1\}$ (resp. if $a_{0j} \cdot a_{1j} \leq 0\ \forall j \in \{0, 1\}$);
  - if $a_{i,j} \neq 0\ \forall i, j \in \mathbb{N}_2$, let us also consider the $A_{00} + A_{01} - A_{10} - A_{11}$ edges internal to $I \times I$, having both the end-points in the set $\{V_0, \ldots, V_{A_{00}+A_{01}-1}\} \subset I \times \{0\} = I \times \{1\}$, parallel to $\vec{w}' \equiv (A_{10} + A_{11}, -\mu(A)(A_{00} + A_{01} - 1))$.
- Otherwise (i.e., in case $A_{00} \leq A_{10}$ and $A_{01} \leq A_{11}$), then:
  - For every vertex of the set $\{V_0, \ldots, V_{A_{00}+A_{01}-1}\} \subset I \times \{0\} = I \times \{1\}$, let us consider an edge internal to $I \times I$ parallel to $\vec{w} \equiv (A_{10} + A_{11} - 1, -\mu(A)(A_{00} + A_{01} - 1))$, where $\mu(A) = +1$ (resp. $\mu(A) = -1$) if $a_{0j} \cdot a_{1j} \geq 0\ \forall j \in \{0, 1\}$ (resp. if $a_{0j} \cdot a_{1j} \leq 0\ \forall j \in \{0, 1\}$);
  - if $a_{i,j} \neq 0\ \forall i, j \in \mathbb{N}_2$, let us also consider the $A_{10} + A_{11} - A_{00} - A_{01}$ edges internal to $I \times I$, having both the end-points in the set $\{W_0, \ldots, W_{A_{10}+A_{11}-1}\} \subset \{0\} \times I = \{1\} \times I$, parallel to $\vec{w}' \equiv (A_{10} + A_{11} - 1, -\mu(A)(A_{00} + A_{01}))$.

[6]Note that, in the case $a_{ij} = 0$, the geometrical realization of α'_i simply coincides with the canonically identified edges $I \times \{0\} = I \times \{1\}$ (if $j = 1$) or $\{0\} \times I = \{1\} \times I$ (if $j = 0$).

[7]Obviously, via pairwise identification $\sim$ of opposite edges in $I \times I$, $V_0 = W_0$ and $V_{A_{00}+A_{01}-1} = W_{A_{10}+A_{11}-1}$. Moreover, note that the edge $I \times \{0\} = I \times \{1\}$ (resp. $\{0\} \times I = \{1\} \times I$) of $\bar{K}_1$ results to be subdivided into $A_{00} + A_{01} - 1$ (resp. $A_{10} + A_{11} - 1$) edges, as well as the geometrical realization of α'_0 (resp. α'_1) in $\bar{K}_2$.

[8]It is easy to check that condition $\det(A) \in \{\pm 1\}$ directly excludes both the case $A_{00} > A_{10},\ A_{01} < A_{11}$ and the case $A_{00} < A_{10},\ A_{01} > A_{11}$.

Note that both $\bar{K}_1$ and $\bar{K}_2$ consist of $\sum_{i,j\in\{0,1\}} |a_{ij}| - 1$ cells, among which $4-2n_0(A)$ are triangular cells and $\sum_{i,j\in\{0,1\}} |a_{ij}|-5+2n_0(A)$ are quadrangular ones, $n_0(A)$ being the number of null elements in A; moreover, the required bijective cell-map $\bar{\Phi}_A : \bar{K}_1 \to \bar{K}_2$, with $|\bar{\Phi}_A| = \tilde{\phi}_A$, is easily induced by $\tilde{\phi}_A(c_i) = \alpha'_i = a_{i0}c'_0 + a_{i1}c'_1$ (with correct orientations).

Second step: We construct two coloured triangulations K_1 and K_2 of the torus T, so that a bijective coloured simplicial map $\Phi_A : K_1 \to K_2$ exists , with $|\Phi_A| = \tilde{\phi}_A$.
K_1 (resp. K_2) is simply obtained from $\bar{K}_1$ (resp. $\bar{K}_2$) by performing a baricentric subdivision and by labelling every vertex of K_1 (resp. K_2) with the dimension of the corresponding cell of $\bar{K}_1$ (resp. $\bar{K}_2$). Hence, the bijective cell-map $\bar{\Phi}_A : \bar{K}_1 \to \bar{K}_2$ canonically induces a bijective coloured simplicial map $\Phi_A : K_1 \to K_2$, with the property $|\Phi_A| = \tilde{\phi}_A$.

Third step: We construct a coloured triangulation $\bar{H}_1$ (resp. $\bar{H}_2$) of the orientable I-bundle over the Klein bottle K, $K \tilde{\times} [0,1]$, so that $\partial\bar{H}_1 = K_1$ (resp. $\partial\bar{H}_2 = K_2$).
For, let P_1 (resp. P_2) be a copy of the *three-tetrahedron solid triangular prism* depicted in Figure 1(a), which gives rise - via suitable pairwise identification of six boundary triangles, as indicated in Figure 1(b) - to the so called *"square orientable" triangulation* of $K \tilde{\times} [0,1]$ (see [3]; Definition 3.2.5 and Figure 3.12), with a *boundary square* triangulating the torus, as indicated in Figure 1(c). Then, let $\bar{P}_1$ (resp. $\bar{P}_2$) be the cell complex consisting of exactly one 3-cell, whose boundary is obtained from ∂P_1 (resp. ∂P_2) by subdividing its boundary square according to the torus triangulation $\bar{K}_1$ (resp. $\bar{K}_2$); finally, let $\bar{H}_1$ (resp. $\bar{H}_2$) be the coloured simplicial triangulation of $K \tilde{\times} [0,1]$ obtained from $\bar{P}_1$ (resp. $\bar{P}_2$) by performing a baricentric subdivision, by labelling every vertex with the dimension of the corresponding cell of $\bar{P}_1$ (resp. $\bar{P}_2$) and by pairwise identifying all boundary faces, except those reproducing K_1 (resp. K_2), according to "square orientable triangulation" of $K \tilde{\times} [0,1]$.

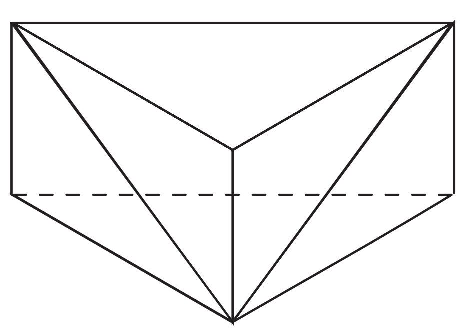
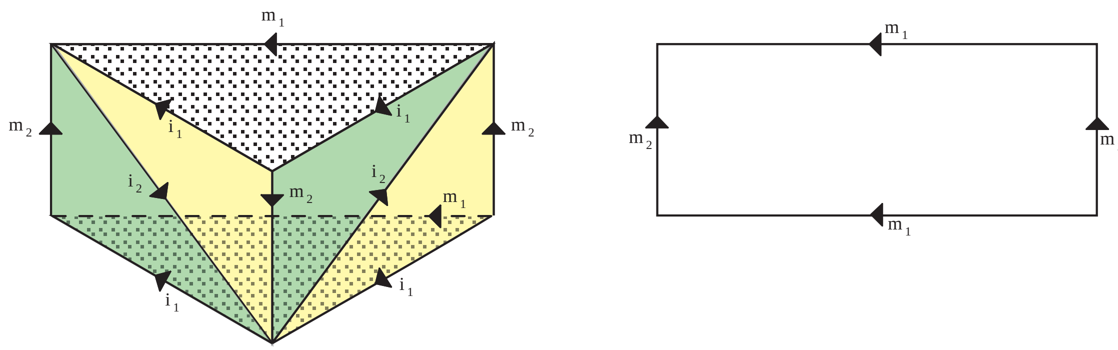


Figure 1(a)



Figures 1(b) and 1(c)

Fourth step: A coloured triangulation $\bar{K}_A$ of the closed 3-manifold $KB(A)$ is obtained from $\bar{H}_1$ and $\bar{H}_2$ by means of the identification of $\partial\bar{H}_1$ (reproducing K_1) with $\partial\bar{H}_2$ (reproducing K_2) according to $\bar{\Phi}_A$.

In order to complete the algorithmic construction, it is now sufficient to consider the edge coloured graph $\Gamma_{KB}(A)$ such that $\Gamma_{KB}(A) = \Gamma(\bar{K}_A)$ (as described in the previous paragraph).

$\square$

Example 1. If $A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \in GL(2;\mathbb{Z})$, then $KB(A)$ turns out to be equivalent to $KB(A')$, with $A' = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix} \in \overline{GL}(2;\mathbb{Z})$ (see Lemma 2.1). Then, the first step of the described algorithm yields the cell-complexes $\bar{K}_1$ and $\bar{K}_2$ triangulating the torus T depicted in Figure 2(a) (where the bijective cell-map $\bar{\Phi}_{A'} : \bar{K}_1 \to \bar{K}_2$ with $|\bar{\Phi}_{A'}| = \tilde{\phi}_{A'}$ is visualized by labelling the pairs of corresponding cells by equal symbols (for example, (x, x')). Further, Figure 2(b) illustrates the coloured triangulations K_1 and K_2 of the torus T obtained in the second step (where the equally labelled simplices are assumed to correspond each other in the bijective coloured simplicial map $\Phi_{A'} : K_1 \to K_2$, with $|\Phi_{A'}| = \tilde{\phi}_{A'}$). Finally, in Figure 2(c) the boundaries of two coloured simplicial subdivisions of the "square orientable triangulation" of $K \tilde{\times} [0,1]$ (yielding the triangulations $\bar{H}_1$ and $\bar{H}_2$, with an inner 3-colured vertex, as indicated in the third step) are depicted, and the equally labelled 2-simplices indicate the boundary identifications necessary to yield $K_{A'}$ from $\bar{H}_1 \cup \bar{H}_2$ (forth step). The resulting edge-coloured graph $\Gamma_{KB}(A') = \Gamma(K_{A'})$ is shown in Figure 2(d), where the 3-coloured edges are understood through the equal labelling of pairs of 3-adjacent vertices.

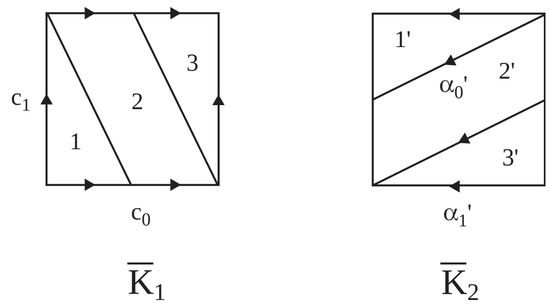


Figure 2(a)

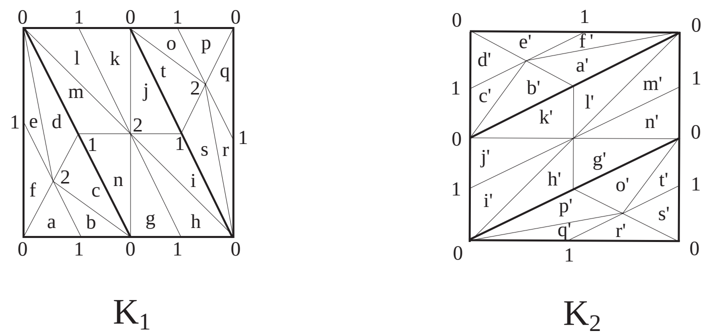
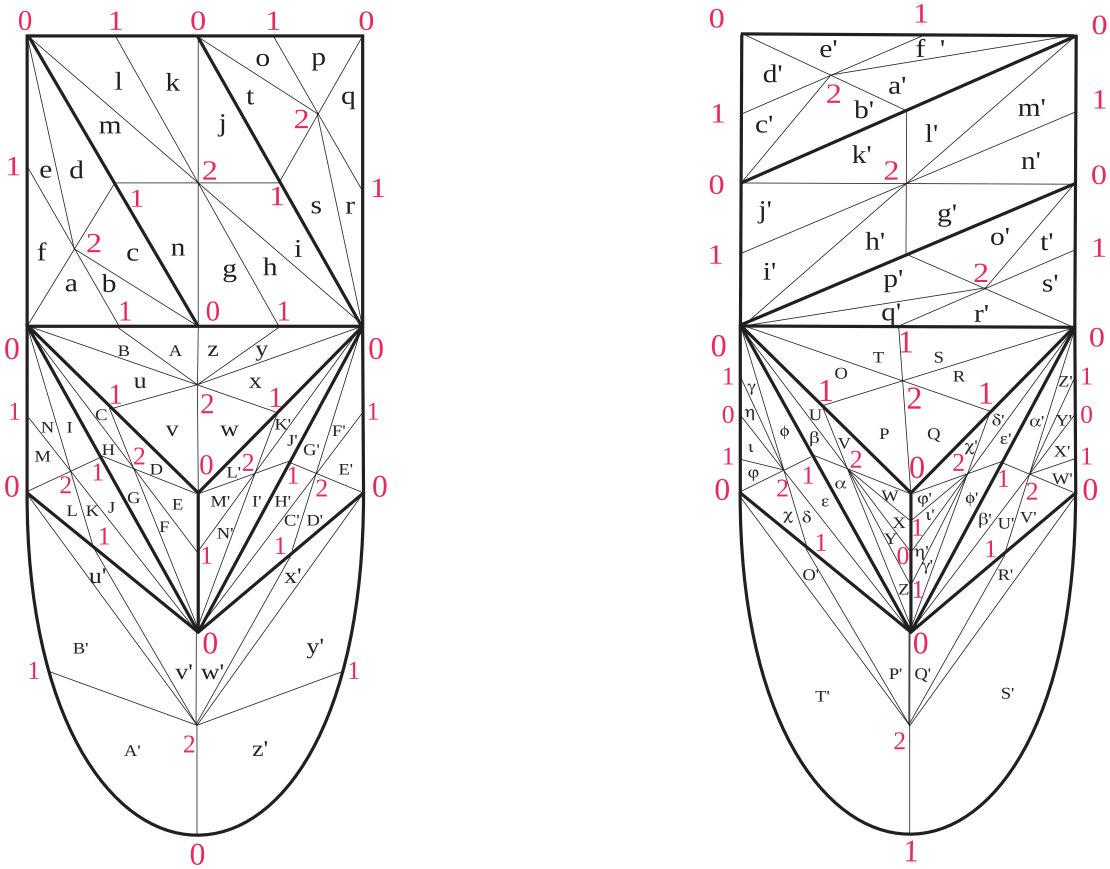


Figure 2(b)



$\overline{\mathrm{H}}_1$ $\overline{\mathrm{H}}_2$

Figure 2(c)

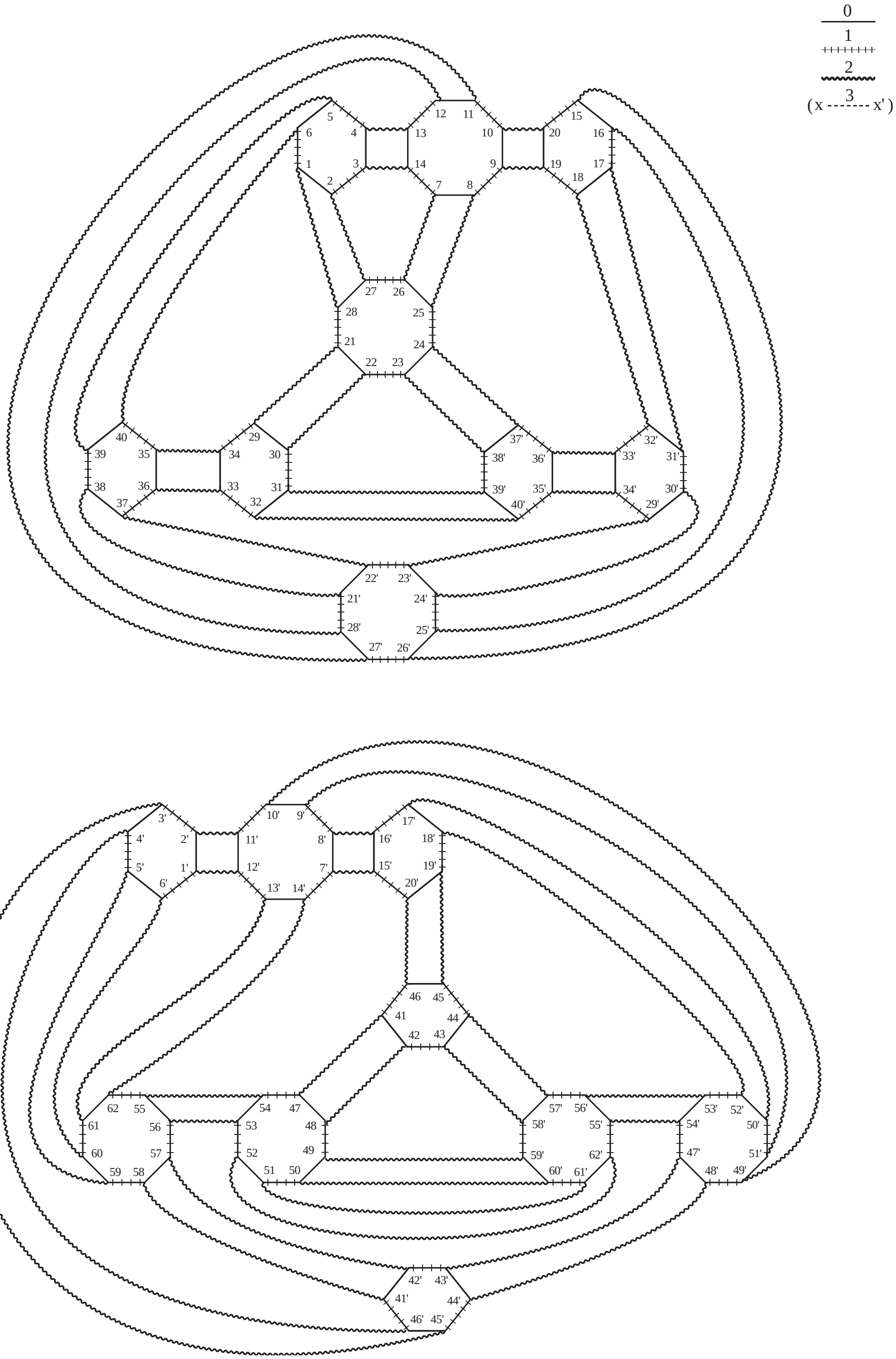


Figure 2(d)

Example 2. If $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \in \overline{GL}(2;\mathbb{Z})$, the first step of the described algorithm yields the cell-complexes $\bar{K}_1$ and $\bar{K}_2$ triangulating the torus T depicted in Figure 3(a) (where the bijective cell-map $\bar{\Phi}_{A'} : \bar{K}_1 \to \bar{K}_2$ with $|\bar{\Phi}_{A'}| = \tilde{\phi}_{A'}$ is visualized by equally labelling of the corresponding cells). Further, 3(b) illustrates the coloured triangulations K_1 and K_2 of the torus T obtained in the second step (where the equally labelled 2-simplices are assumed to correspond each other in the bijective coloured simplicial map $\Phi_{A'} : K_1 \to K_2$, with $|\Phi_{A'}| = \tilde{\phi}_{A'}$). Subsequent steps of the described algorithm follow as in Example 1.

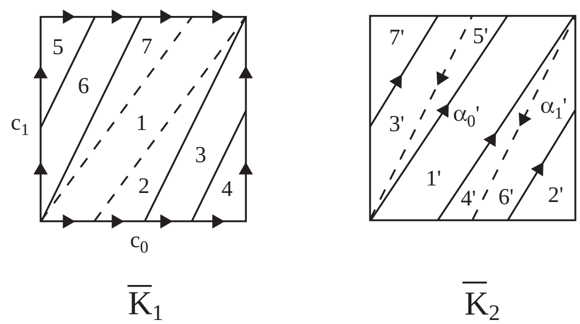


Figure 3(a)

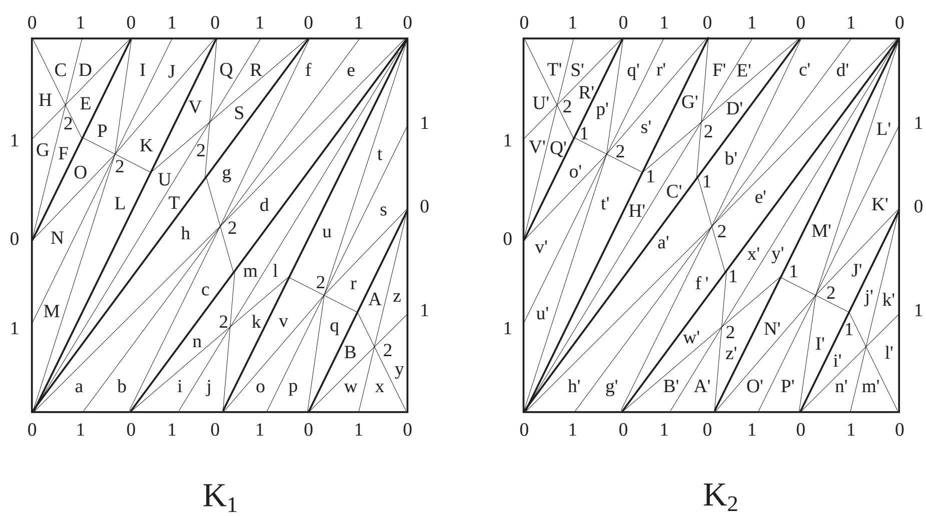


Figure 3(b)

Remark 1. Note that, in virtue of Lemma 2.1, any manifold obtained by pasting together two copies of the orientable I-bundle over the Klein bottle (i.e. any manifold of type $KB(A)$) turns out to admit a 4-coloured graph $\Gamma_{KB}(A)$ (obtained as an output of the algorithmic procedure of Theorem 4.1, for a suitable $A \in \overline{GL}(2;\mathbb{Z})$) representing it.

Remark 2. For every $A \in \overline{GL}(2;\mathbb{Z}))$, the 4-coloured graph $\Gamma_{KB}(A)$ representing $KB(A)$ enjoys the following combinatorial features (which may be easily checked by direct computation via the corresponding geometrical properties of the coloured triangulation $\bar{K}_A$):

$$2p = \#V(\Gamma_{KB}(A)) = 4(7\textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| - 2 + 5n_0(A) + A_{10} - A_{01});$$

$$g_{01} = 2(\textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| + 5);$$

$$g_{02} = g_{03} = g_{13} = 7\textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| - 2 + 5n_0(A) + A_{10} - A_{01};$$

$$g_{12} = 5\textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| - 8 + 5n_0(A) + A_{10} - A_{01};$$

$$g_{23} = 2(2\textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| + n_0(A));$$

$$g_{\hat{0}} = \textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| - 3 + n_0(A);$$

$$g_{\hat{1}} = 2\textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| + n_0(A);$$

$$g_{\hat{2}} = \textstyle\sum_{i,j\in\{0,1\}}|a_{ij}| + 5;$$

$$g_{\hat{3}} = 2.$$

Proposition 4.2 *For every $A \in \overline{GL}(2;\mathbb{Z})$, the following relations hold:*

a) $\rho(\Gamma_{KB}(A)) = \sum_{i,j\in\{0,1\}}|a_{i,j}| + 4;$

b) $\mathcal{G}(KB(A)) \leq 4.$

Further, if $A \in \overline{GL}(2;\mathbb{Z})$ contains a null element, then:

c) $\mathcal{G}(KB(A)) \leq 3.$

Proof. If $\bar{\epsilon} = (0,2,1,3)$, a direct computation yields:

$$\begin{aligned}\rho_{\bar{\epsilon}}(\Gamma_{KB}(A)) &= g_{01} - g_{\hat{2}} - g_{\hat{3}} + 1 = 2\sum_{i,j\in\{0,1\}}|a_{ij}| + 10 - (\sum_{i,j\in\{0,1\}}|a_{ij}| + 5) - 2 + 1 = \\ &= \sum_{i,j\in\{0,1\}}|a_{ij}| + 4\end{aligned}$$

On the other hand, it is easy to check that, for any permutation ϵ' of Δ_3, $\rho_{\epsilon'}(\Gamma_{KB}(A)) \geq \rho_{\bar{\epsilon}}(\Gamma(A))$ holds; thus, statement (a) follows.

In order to prove statement (b), it is necessary to note that, if σ, σ' are two cells of the cellular triangulation $\bar{K}_1$ of the bidimensional torus T, sharing a common boundary edge e, with $e \notin \partial(T)$, then the $\{0,1\}$-coloured cycle of $\Gamma_{KB}(A)$, dual to the $\{2,3\}$-labelled edge of $\bar{K}_A$ having the baricenter of σ (resp. σ') as an end-point, has exactly one common vertex with any $\{2,3\}$-coloured cycle of $\Gamma_{KB}(A)$, dual to one of the ($\{0,1\}$-labelled) edges of $\bar{K}_A$ subdividing e: this means that any such common vertex identifies a so called *generalized dipole*, which is a combinatorial structure that may be easily eliminated by a finite sequence of elementary moves on edge-coloured graphs, yielding a new graph, with one less $\{0,1\}$-coloured cycle, representing the same 3-manifold (see [11] for details). It is not difficult to check that, since $\bar{K}_1$ contains $\sum_{i,j\in\{0,1\}}|a_{ij}|-2$ edges not belonging to $\partial(T)$, the combinatorial structure of $\bar{K}_A$ allows to perform in $\Gamma_{KB}(A)$, for every $A\in\overline{GL}(2;\mathbb{Z})$, a finite sequence of $\sum_{i,j\in\{0,1\}}|a_{ij}|-2$ "independent" generalized dipole eliminations; moreover, $\Gamma_{KB}(A)$ contains at least two other "independent" generalized dipoles, involving colours $\{0,1\}$ and $\{2,3\}$, corresponding to inner 1-coloured vertices of the coloured triangulation $\bar{H}_1$ of $K\overset{\sim}{\times}[0,1]$. The whole sequence of generalized dipole eliminations gives rise to a new 4-coloured graph $\Gamma'_{KB}(A)$ representing $KB(A)$, so that

$$\rho_{\bar{\epsilon}}(\Gamma'_{KB}(A))=\rho_{\bar{\epsilon}}(\Gamma_{KB}(A))-[(\sum_{i,j\in\{0,1\}}|a_{ij}|-2)+2]=4.$$

This completes the proof of statement (b).

Finally, let us consider the case of a matrix $A\in\overline{GL}(2;\mathbb{Z})$ containing a null element. A direct check allows to verify that $\Gamma'_{KB}(A)$ contains at least another "independent" generalized dipole, involving colours $\{0,1\}$ and $\{2,3\}$, corresponding to the 1-coloured vertex of the coloured triangulation $\bar{H}_1$ of $K\overset{\sim}{\times}[0,1]$, which is the baricenter of the boundary edge $\bar{m}\in\{m_1,m_2\}$ (see Figure 1) which is not subdivided in $\bar{K}_1$.

Hence, the additional hypothesis that A contains a null element allows to obtain - through a further generalized dipole elimination - a new graph $\Gamma''_{KB}(A)$ representing $KB(A)$; statement (c) now directly follows:

$$\exists\bar{i},\bar{j}\in\{1,2\},\ a_{\bar{i}\bar{j}}=0\qquad\Longrightarrow\qquad\rho_{\bar{\epsilon}}(\Gamma''_{KB}(A))=\rho_{\bar{\epsilon}}(\Gamma'_{KB}(A))-1=3.$$

□

We are now able to easily prove the already quoted upper bound results about Heegaard genus of manifolds of type $KB(A)$.

Proof of Proposition 2.3.
Statement (a) directly follows from Proposition 4.2 (b), via Lemma 2.1 and Remark 1.
On the other hand, statement (b) is a direct consequence of Proposition 4.2 (c).

□

5. Recognition of manifolds of type $KB(A)$ among elements of existing crystallization catalogues

In [6], together with previous work [15], (resp. in [4]) a complete catalogue $\mathcal{C}^{(30)}$ (resp. $\tilde{\mathcal{C}}^{(26)}$) of orientable (resp. non-orientable) 3-manifolds admitting coloured triangulations up to 30 (resp. 26) tetrahedra is obtained, and its elements are deeply analyzed; as a consequence, the following results are proved:

Proposition 5.1

a) [15] *Exactly 69 non-homeomorphic prime orientable 3-manifolds exist, which admit a coloured triangulation consisting of at most 28 tetrahedra;*

b) [6] *Exactly 41 non-homeomorphic prime orientable 3-manifolds exist, which admit a coloured triangulation consisting of 30 tetrahedra and do not admit a coloured triangulation consisting of less than 30 tetrahedra;*

c) [4] *Exactly 7 non-homeomorphic prime non-orientable 3-manifolds exist, which admit a coloured triangulation consisting of at most 26 tetrahedra.* □

For details about the orientable represented manifolds, see [7].[9]

Now, we will apply the above described algorithmic construction of edge coloured graphs representing manifolds of type $KB(A)$, in order to combinatorially recognize all manifolds of type $KB(A)$ belonging to Lins's catalogue $\mathcal{C}^{(28)}$ [15] (resp. to Casali-Cristofori' catalogue $\mathcal{C}^{(30)}$ [6]) of closed connected orientable 3-manifolds represented by edge-coloured graphs up to 28 (resp. 30) vertices.

Proposition 5.2 *Exactly six manifolds of type $KB(A)$, for suitable matrices $A \in GL(2;\mathbb{Z})$ with $\det(A) = -1$, admit a pseudosimplicial triangulation consisting of ad most 30 tetrahedra.*
More precisely, if r_k^{2p} denotes the $k-th$ element with $2p$ vertices ($p \leq 15$) belonging to the crystallizations catalogues due to Lins [15] and to Casali-Cristofori [6], we have:

(a) *the orientable 3-manifold corresponding to r_5^{24} is the manifold $KB(A)$, with $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;*

[9] As far as the non-orientable case is concerned, Theorem I of [4] proves that the involved manifolds are: the four euclidean non-orientable 3-manifolds, the non-trivial $\mathbb{S}^2$ bundle over $\mathbb{S}^1$, the topological product between the real projective plane $\mathbb{RP}^2$ and $\mathbb{S}^1$, and the torus bundle over $\mathbb{S}^1$, with monodromy induced by matrix $\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

(b) *the orientable 3-manifold corresponding to* r^{26}_{5} *is the manifold* $KB(A)$, *with* $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$;

(c) *the orientable 3-manifold corresponding to* r^{28}_{6} *is the manifold* $KB(A)$, *with* $A = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$;

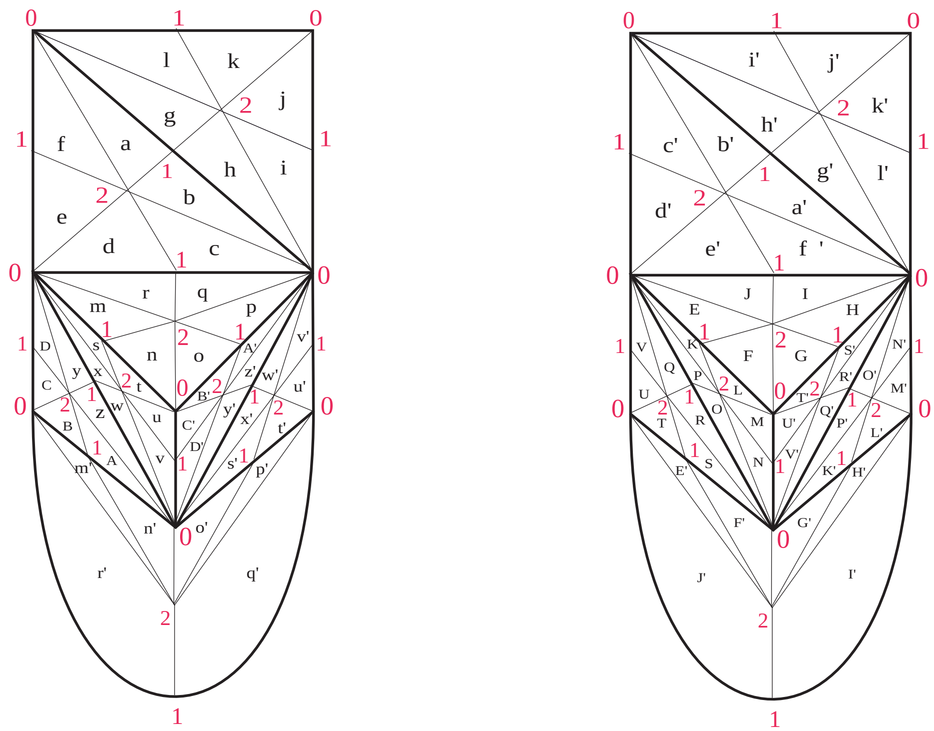
(d) *the orientable 3-manifold corresponding to* r^{28}_{19} *is the manifold* $KB(A)$, *with* $A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$;

(e) *the orientable 3-manifold corresponding to* r^{30}_{21476} *is the manifold* $KB(A)$, *with* $A = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$;

(f) *the orientable 3-manifold corresponding to* r^{30}_{45716} *is the manifold* $KB(A)$, *with* $A = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$.

Proof. Note that, within Lins's and Casali-Cristofori' catalogues ([15] and [6]), the topological identification of the six represented manifolds of type $KB(A)$, for suitable matrices $A \in GL(2;\mathbb{Z})$ with $\det(A) = -1$, has been already obtained through GM-complexity computation and homology group comparison or - in the 30 vertices cases - through Matveev's program *Three-manifold Recognizer* (see [7]; Table 1 and Table 2).
Hence, we have only to take into account the "first" element of the quoted catalogues which represents $KB(A)$, for each one of the six involved matrices.
(a) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $A \in \overline{GL}(2;\mathbb{Z})$, we can directly apply the algorithmic procedure of the previous section: the coloured pseudocomplex K_A is obtained from the two coloured simplicial triangulations $\bar{H}_1$ and $\bar{H}_2$ of $K \widetilde{\times} [0,1]$ (with an inner 3-coloured vertex) whose boundaries are depicted in Figure 4, by means of the pairwise identification of 2-simplices labelled x, x', for each label x.
It is not difficult to check that the associated order 96 4-coloured graph $\Gamma_{KB}(A)$ may be transformed by a finite sequence of eliminations of dipoles and ρ-pairs (for example, by making use of the corresponding functions of DUKE III program) into an order 32 rigid crystallization $\Gamma'_{KB}(A)$ having code

$$c(\Gamma'_{KB}(A)) = EABCDHFGJILKNMPO \quad LKMOFEDNHBJAPIGC \\ OINLKACMBHPDJGFE.$$



$\overline{\mathrm{H}}_1$ $\overline{\mathrm{H}}_2$

Figure 4

A direct computation allows to prove $\Gamma'_{KB}(A)$ to admit a so called *cluster*, which is a combinatorial structure that may be easily eliminated by a finite sequence of elementary moves on edge-coloured graphs, yielding a new graph $\Gamma''_{KB}(A)$, with two less vertices, representing the same 3-manifold (see Figure 5, or [15]; Proposition 24 for details). On the other hand, $\Gamma''_{KB}(A)$ may be further simplified via two 2-dipole eliminations (for example, by making use of the corresponding function of DUKE III program), so to obtain the order 26 4-coloured graph $\Gamma'''_{KB}(A)$ having code

$$c(\Gamma'''_{KB}(A)) = CABFDEIGHKJML \ \ ILDCJGFEMHBKA$$
$$DMGACIEKLBHFJ;$$

since this code identifies, up to permutation of vertices and colours, the element $r_3^{26} \in \mathcal{C}^{(28)}$, and since Lins's classification ensures the represented 3-manifold to be the same as $r_5^{24} \in \mathcal{C}^{(28)}$, part (a) of the statement follows.

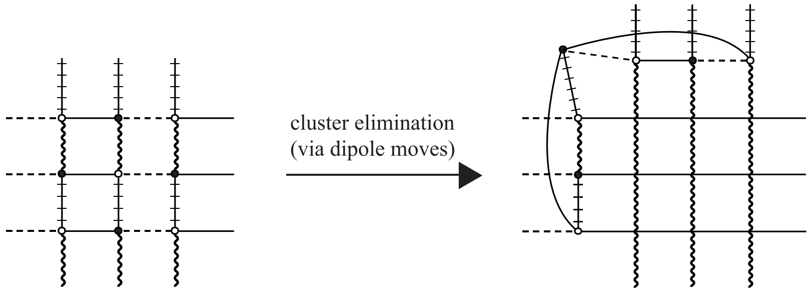
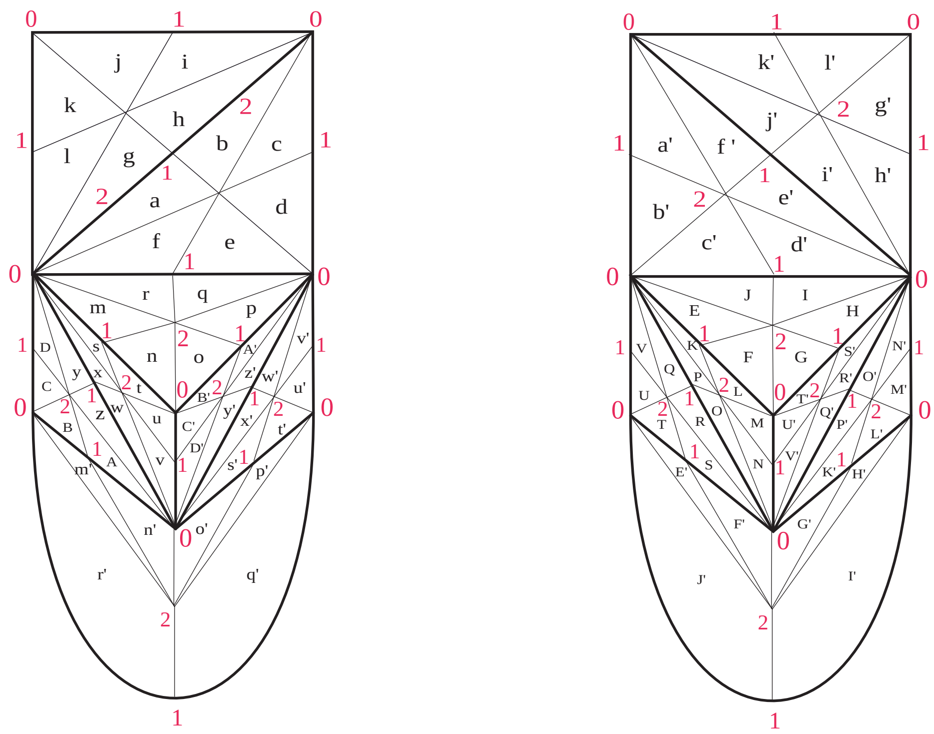


Figure 5

(b) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Since $A \in \overline{GL}(2;\mathbb{Z})$, we can directly apply the algorithmic procedure of the previous section: the coloured pseudocomplex K_A is obtained from the two coloured simplicial triangulations $\bar{H}_1$ and $\bar{H}_2$ of $K\tilde{\times}[0,1]$ (with an inner 3-coloured vertex) whose boundaries are depicted in Figure 6, by means of the pairwise identification of 2-simplices labelled x, x', for each label x.



$\overline{\mathrm{H}}_1$ $\overline{\mathrm{H}}_2$

Figure 6

It is not difficult to check that the associated order 96 4-coloured graph $\Gamma_{KB}(A)$ may be transformed by a finite sequence of eliminations of dipoles and ρ-pairs (for example, by making use of the corresponding functions of DUKE III program) into an order 32 rigid crystallization $\Gamma'_{KB}(A)$ having code

$$\begin{aligned} c(\Gamma'_{KB}(A)) = \ & EABCDHFGKIJNLMPO \quad NPLJFEDIHGCKOAMB \\ & LJIKHDOMCBFPEGNA. \end{aligned}$$

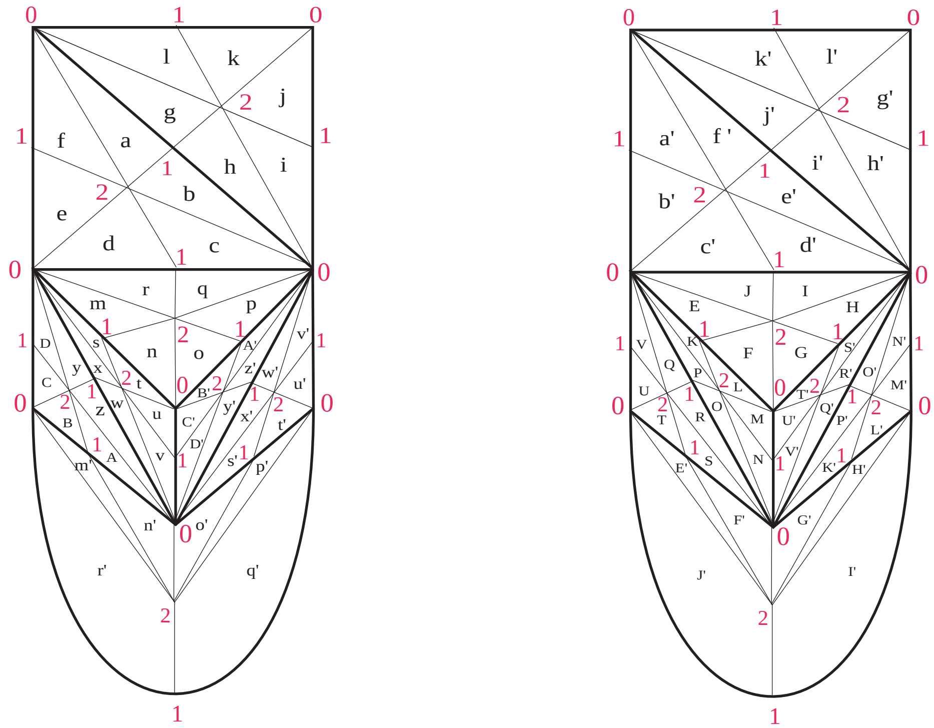
A direct computation allows to prove $\Gamma'_{KB}(A)$ to admit a *cluster*, which may be easily eliminated by a finite sequence of elementary moves on edge-coloured graphs, yielding a new graph $\Gamma''_{KB}(A)$, with two less vertices, representing the same 3-manifold (as already pointed out in part (a) of the present proof, see Figure 5 or [15]; Proposition 24 for details).
The code of $\Gamma''_{KB}(A)$ is

$$\begin{aligned} c(\Gamma''_{KB}(A)) = \ & DABCGEFJHIMKLON \ JONLDKHGFAIECMB \\ & HGKMAJNEDFCOIBL. \end{aligned}$$

Since this code identifies $r^{30}_{39343} \in \mathcal{C}^{(30)}$ (i.e. the element number 39343 of the crystallization catalogue $\mathcal{C}^{(30)}$), which belongs to the same "equivalence class" of $r^{26}_5 \in \mathcal{C}^{(28)}$, the represented manifolds surely belong to the same homeomorphism class: in fact, in [6], the complete classification of the manifolds encoded by catalogue $\mathcal{C}^{(30)}$ is performed via an automatic partition of crystallizations into equivalence classes which are proved to be in one-to-one correspondence with the homeomorphism classes of the represented manifolds. Hence, part (b) of the statement follows.

(c) Let $A = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$. Since $A \in \overline{GL}(2;\mathbb{Z})$, we can directly apply the algorithmic procedure of the previous section: the coloured pseudocomplex K_A is obtained from the two coloured simplicial triangulations $\bar{H}_1$ and $\bar{H}_2$ of $K \tilde{\times} [0,1]$ (with an inner 3-coloured vertex) whose boundaries are depicted in Figure 7, by means of the pairwise identification of 2-simplices labelled x, x', for each label x.
It is not difficult to check that the associated order 96 4-coloured graph $\Gamma_{KB}(A)$ may be transformed by a finite sequence of eliminations of dipoles and ρ-pairs (for example, by making use of the corresponding functions of DUKE III program) into an order 34 rigid crystallization $\Gamma'_{KB}(A)$ having code

$$\begin{aligned} c(\Gamma'_{KB}(A)) = \ & EABCDHFGKIJMLONQP \quad MPNGFEDIHCLKOJQBA \\ & NMLFKICQDHPGBAEOJ. \end{aligned}$$



$\overline{\mathrm{H}}_1$ $\overline{\mathrm{H}}_2$

Figure 7

The program *“Γ-class”* yielding the automatic partition of crystallizations into equivalence classes whose elements represent - up to homeomorphism - the same manifold (see [6] and [7] for details), allows to prove that $\Gamma'_{KB}(A)$ belongs to the same “equivalence class” of $r_6^{28} \in \mathcal{C}^{(28)}$; hence, part (c) of the statement follows.

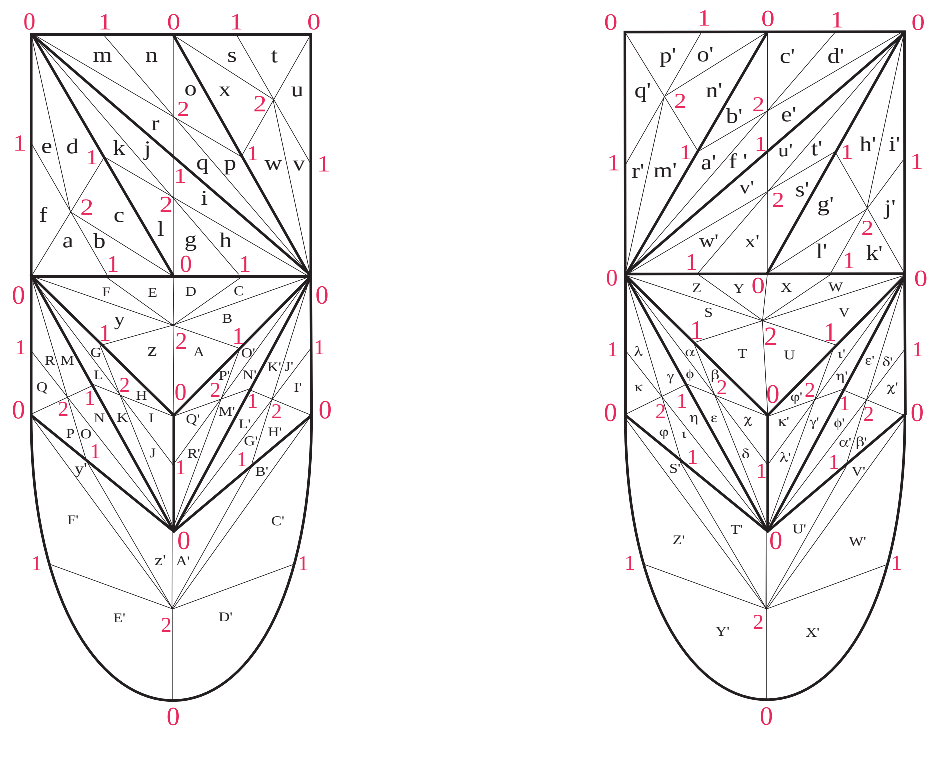
(d) Let $A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$. As already pointed out in Example 1, the algorithmic procedure of the previous section, applied to the equivalent matrix $A' = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix} \in \overline{GL}(2;\mathbb{Z})$, yields the order 124 4-coloured graph $\Gamma_{KB}(A') = \Gamma(K_{A'})$ depicted in Figure 2(d) (where the 3-coloured edges are understood through the equal labelling of the pairs of 3-adjacent vertices).

It is not difficult to check that $\Gamma_{KB}(A')$ may be transformed by a finite sequence of eliminations of dipoles and ρ-pairs (for example, by making use of the corresponding functions of DUKE III program) into an order 42 rigid crystallization $\Gamma'_{KB}(A')$ having code

$$\begin{aligned} c(\Gamma'_{KB}(A')) = \; & EABCDHFGJIMKLONQPSRUT \\ & MULSPEKIHGJNTCFORQDAB \\ & SLHMTBICOFNJPKGUAEQRD. \end{aligned}$$

The program "Γ-*class*" yielding the automatic partition of crystallizations into equivalence classes whose elements surely represent - up to homeomorphism - the same manifold (see [6] and [7] for details), allows to prove that $\Gamma'_{KB}(A')$ belongs to the same "equivalence class" of $r^{28}_{19} \in \mathcal{C}^{(28)}$; hence, part (d) of the statement follows.

(e) Let $A = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$. Since $A \notin \overline{GL}(2;\mathbb{Z})$, we apply the algorithmic procedure of the previous section to the equivalent matrix $A' = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \in \tilde{GL}(2;\mathbb{Z})$: the coloured pseudocomplex $K_{A'}$ is obtained from the two coloured simplicial triangulations $\bar{H}_1$ and $\bar{H}_2$ of $K \tilde{\times} [0,1]$ (with an inner 3-coloured vertex) whose boundaries are depicted in Figure 8, by means of the pairwise identification of 2-simplices labelled x, x', for each label x.



$\overline{\mathrm{H}}_1$ $\overline{\mathrm{H}}_2$

Figure 8

It is not difficult to check that the associated order 128 4-coloured graph $\Gamma_{KB}(A')$ may be transformed by a finite sequence of eliminations of dipoles and ρ-pairs (for example, by making use of the corresponding functions of DUKE III program) into an order 38 rigid crystallization $\Gamma'_{KB}(A')$ having code

$$\begin{aligned} c(\Gamma'_{KB}(A')) = {} & EABCDHFGKIJNLMQOPSR \\ & NMJQFEDOHCLKBAIRSPG \\ & RJIFOPNQLBSCKGMDHAE. \end{aligned}$$

The program "Γ-*class*" yielding the automatic partition of crystallizations into equivalence classes whose elements represent - up to homeomorphism - the same manifold (see [6] and [7] for details), allows to prove that $\Gamma'_{KB}(A)$ belongs to the same "equivalence class" of $r^{30}_{21476} \in \mathcal{C}^{(30)}$; hence, part (e) of the statement follows.

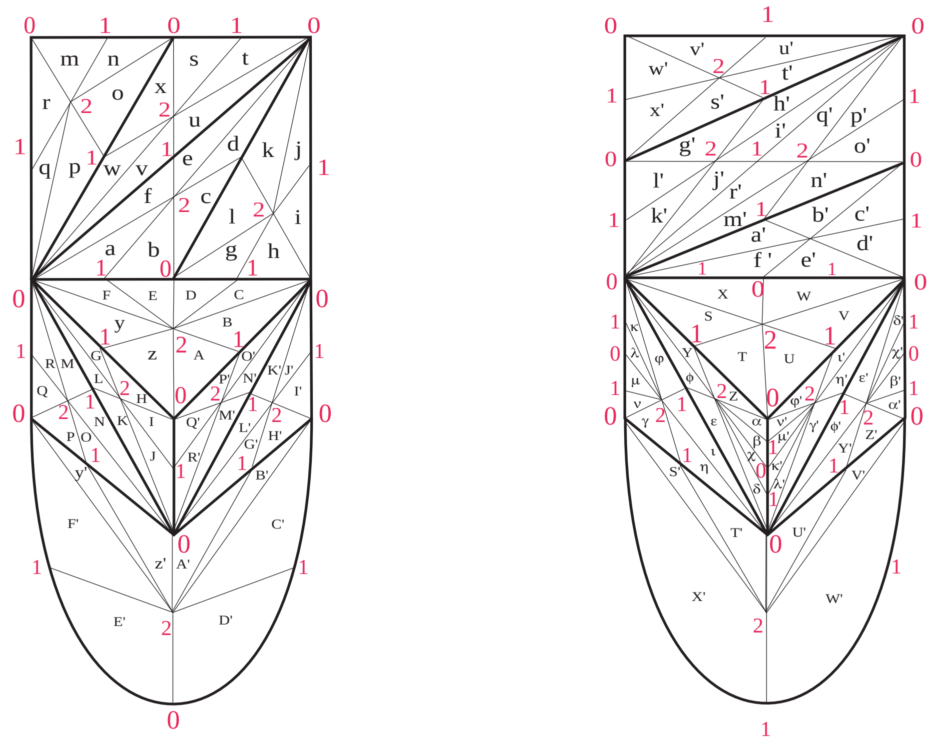
(f) Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix}$. Since $A \notin \overline{GL}(2;\mathbb{Z})$, we apply the algorithmic procedure of the previous section to the equivalent matrix $A' = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix} \in \tilde{GL}(2;\mathbb{Z})$ (see Lemma 2.1): the coloured pseudocomplex $K_{A'}$ is obtained from the two coloured simplicial triangulations $\bar{H}_1$ and $\bar{H}_2$ of $K \tilde{\times} [0,1]$ (with an inner 3-coloured vertex) whose boundaries are depicted in Figure 9, by means of the pairwise identification of 2-simplices labelled x, x', for each label x.

It is not difficult to check that the associated order 132 4-coloured graph $\Gamma_{KB}(A')$ may be transformed by a finite sequence of dipole eliminations (for example, by making use of the corresponding function of DUKE III program) into an order 34 rigid crystallization $\Gamma'_{KB}(A')$ having code

$$\begin{aligned} c(\Gamma'_{KB}(A')) = {} & DABCHEFGKIJMLONQP \quad KQJEDLOIHGPFNMCAB \\ & GPLKJIADQEHNBCMOF. \end{aligned}$$

The program "Γ-*class*" yielding the automatic partition of crystallizations into equivalence classes whose elements surely represent - up to homeomorphism - the same manifold (see [6] and [7] for details), allows to prove that $\Gamma'_{KB}(A)$ belongs to the same "equivalence class" of $r^{30}_{45716} \in \mathcal{C}^{(30)}$; hence, part (f) of the statement follows.

□



$\overline{\mathrm{H}}_1$ $\overline{\mathrm{H}}_2$

Figure 9

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