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On the Weiss Conjecture for Analytic Semigroups

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Abstract

Let $\mathbb{T}(t)$ be an analytic semigroup with generator A on some Banach space. Le Merdy in [12] showed the validity of the so-called Weiss conjecture : $C \in \mathcal{L}(D(A), Y)$ is an admissible observation operator if and only if $\{\|C(\lambda I - A)^{-1}\| \; Re\lambda \geq \alpha \text{ for some } \alpha > 0\}$ is a bounded set, is equivalent to the finite-time admissibility of the fractional power $(-A)^{1/2}$ for A. The proof was essentially based on the H^∞-functional calculus. Here, we give a new (and a much short) proof of this result and does not make any recourse to the H^∞- functional calculus. Next we show that the analyticity assumption of $\mathbb{T}(t)$ cannot be omitted. Finally we give a new proof for the Weiss Conjecture for analytic and normal semigroups on Hilbert spaces.

Keywords: infinite dimensional systems, semigroups, admissibility, sectorial operators, fractional power

1 Introduction

In this paper we consider the abstract differential system:

$$\left\{ \begin{array}{rcll} \dot{x}\,(t) & = & Ax(t), & t \geq 0, \\ x(0) & = & x_0, & \\ y(t) & = & Cx(t), & \end{array} \right. \tag{1}$$

where A is a generator of C_0-semigroup $\mathbb{T} := (\mathbb{T}(t))_{t \geq 0}$ on a Banach state space X, and C is a bounded operator from the domain of $A, D(A)$ with respect to

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the graph norm to the second output Banach space, that is, there exists a constant $M_0 > 0$ such that

$$\|Cx\|_Y \leq M_0(\|x\|_X + \|Ax\|_X), \quad x \in D(A).$$

Then for any $x \in D(A)$ and any $t > 0, \mathbb{T}(t)x \in D(A)$, and the resulting function $t \mapsto C\mathbb{T}(t)x$ is continuous from $(0, \infty)$ into Y. An important is question is whether the system (1) is well-posed. Of course since A generates a C_0-semigroup on X, we know that the state equation has he unique solution $x(t) = \mathbb{T}(t)x_0$. However, since C is not a bounded operator on X, it not clear whether the output equation is well-posed. Pointwise in time the output equation only makes sense if C is bounded on X. However, one could relax this to the question if the output trajectory is locally square integrable. Thereafter the following definition of an admissible observation operator has been introduced. According to [18], we say that C is finite-time admissible if, for some (and hence for all) $\tau \in (0, \infty)$ there exists M_τ such that

$$\int_0^\tau \|C\mathbb{T}(t)x\|_Y^2 dt \leq M_\tau \|x\|_X^2. \tag{2}$$

A variant of admissibility called infinite-time admissibility when the integral on $(0, \tau)$ in (2) is replaced by the whole time axis $(0, \infty)$ has also been extensively studied (see e.g., [3], [4], [21], [12], [2],[14], [22]). These notions of admissibility are invariant under scalings $e^{-\alpha \cdot}\mathbb{T}(\cdot)$. Hence if we want to investigate finite-time admissibility of observation operators, then we may assume that the semigroup is exponentially stable. We refer to ([4],[20][19]) and the references therein for historical background and application of admissibility. In [18] one can found that finite-time admissibility always implies that there exists $M > 0$ such that

$$\|C(sI - A)^{-1}\|^2 \leq M/Res, \quad (s \text{ in some right half-plane}). \tag{3}$$

It has been proved in [18] that the converse does not hold in the general Banach space context. However, in [18] it was been conjectured that if X and Y are Hilbert spaces, then C is finite-time admissible observation operator if and only (3) hold. Since then, this problem, which is known as the Weiss-conjecture, has received much attention. Zwart in [22] present some sufficient conditions for finite (or infinite) time admissibility. In particular the following sufficient condition is presented: If there exist $M > 0, \rho \in \mathbb{R}$, and $\alpha > \frac{1}{2}$ such that for all s with real part bigger than ρ we have $(\mathrm{Re}(s))^\alpha \|(sI - A)^{-1}B\| \leq M$, then B is admissible.

It was first shown in [13] that the isometric right shift semigroup $H^2(\mathbb{C}^+, X)$ satisfies the Weiss Conjecture for scalar observation operators (in the case $X = \mathbb{C}$). By using the Sz.- Nagy?Foias functional model for contraction semigroups on a Hilbert space and applying the same proof as in [13] for X a general separable Hilbert space, it was shown in [7] that the Weiss Conjecture holds for general contraction semigroups on separable Hilbert spaces with a scalar observation operator. The proof in [7] was later simplied in [16], Section 10.7 by using an isometric extension of the semigroup.

If $Y = \mathbb{C}$ and $\mathbb{T}(t)$ is a contraction semigroup, it was shown in [7] that the Weiss Conjecture holds, but the paper [8] showed that if $dim(Y) = \infty$ then the Weiss Conjecture can fail even for a semigroup of isometries. The papers [21] and [9] constructed a bounded, analytic semigroup $\mathbb{T}(t)$ for which the Weiss Conjecture fails with Y is of finite and infinite dimension respectively, but [12] gives other examples of bounded analytic semigroups $\mathbb{T}(t)$ not similar to contraction semigroups for which the Weiss Conjecture holds (for Y a general Banach space). Papers [6] and [19] contain the spacial case when the semigroup is normal and analytic. In the Banach space context, Le Merdy in his paper [12] showed that the Weiss conjecture holds for a bounded analytic semigroup if and only if the fractional power $(-A)^{1/2}$ is finite-time admissible for A. Essential use made of the H^∞-functional calculus. For a contractive analytic semigroup on Hilbert space X, it is shown in [12] that the Weiss conjecture holds. In particular, he extends the result by Hansen and Weiss [6] and by Weiss [19] concerning the case when the semigroup is bounded analytic and normal (and hence contractive). Concerning the infinite-time admissibility, in [12], it was proved that for a bounded analytic semigroup on a Banach space X, the Weiss conjecture holds if and only if $(-A)^{1/2}$ is infinite-time admissible for A. In [12], constantly use the analyticity of the semigroup $\mathbb{T}$. From [12], [21] and [9], we conclude that some bounded and analytic semigroups satisfy the Weiss Conjecture and some are not it. So, we may ask an intriguing question concerning the results obtained in [12] on the Weiss-Conjecture: Is $\mathbb{T}$ necessarily analytic?. In this note we give an affirmative answer.

The aim this paper is to give a much simpler and more straightforward proof of the result concerning the Weiss-conjecture for analytic semigroups stated in [12]. Moreover, it is proved that the analyticity assumption on the semigroup cannot be omitted. Our approach does not make any recourse to H^∞ functional calculus and its based on (elementary) the theory of fractional powers of linear operators (see Komatsu [11] for detail on this subject). In [12] Le Merdy gave sufficient conditions regarding A and the geometry of the underlying Banach space X under which the fractional power $(-A)^{1/2}$ is an admissible observation operator for A. In this paper it is shown that this may happen for analytic and normal semigroups on Hilbert space. This result can be viewed as a new-proof of the Weiss-conjecture for normal and analytic semigroups stated in [6]. As in Le Merdy [12], in the general Banach spaces framework, we introduce the following definition.

Definition 1.1 *Let A be the generator of a C_0-semigroup* $(\mathbb{T}_{t\geq 0})$ *on a Banach* *space X. We say that A satisfies the Weiss property if for any Banach space Y and $C \in \mathcal{L}(D(A), Y)$, the following statement are equivalent:*

(i) *C is finite-time admissible for A.*

(ii) *C satisfy the estimation (3).*

2 Some results on the Weis-conjecture

As announced in the introduction we shall use the fractional powers of sectorial operators to obtain our result on the Weiss-conjecture. Let us give some basic definition concerning fractional power of unbounded operator. A closed, densely defined operator A on a Banach space X is called sectorial if $(0, \infty) \subset \rho(A)$ and there is a constant K such that :

$$\|R(\lambda, A)\| \leq \frac{K}{1+\lambda} \text{ for all } \lambda > 0. \tag{4}$$

For a sectorial operator A, fractional power of $-A$ are well defined. If the resolvent satisfies (4), then there exist constant $\varepsilon > 0, \kappa > 0$ and $\psi \in (0, \pi)$ such that $\|R(\lambda, A)\| \leq \kappa(1+|\lambda|)^{-1}$ for all λ in the sector $\{\lambda \in \mathbb{C} : |arg\lambda| \leq \psi\} \cup \{\lambda \in \mathbb{C} : |\lambda| \leq \varepsilon\} \subset \rho(A)$.
For a sectorial operator A and a positive real $\beta > 0$ we define the operator $(-A)^{-\beta}$ by

$$(-A)^{-\beta}x = \frac{1}{2i\pi}\int_{\Gamma}(-z)^{-\beta}R(z, A)x dz \quad x \in X \tag{5}$$

where $(\mu)^{-\beta}$ is defined in terms of the principal branch of the logarithm.
Here $\Gamma = \Gamma(\psi, \varepsilon) = \Gamma^1(\psi, \varepsilon) \cup \Gamma^2(\psi, \varepsilon) \cup \Gamma^3(\psi, \varepsilon)$ denotes the upwards oriented path defined by

$$\Gamma^1(\psi, \varepsilon) = \{\lambda \in \mathbb{C} : |\lambda| \geq \varepsilon, arg\lambda = -\psi\},$$

$$\Gamma^2(\psi, \varepsilon) = \{\lambda \in \mathbb{C} : |\lambda| = \varepsilon, |arg\lambda| > \psi\},$$

$$\Gamma^3(\psi, \varepsilon) = \{\lambda \in \mathbb{C} : |\lambda| \geq \varepsilon, arg\lambda = \psi\}.$$

Note that we use the argument function with values in $(-\pi, \pi]$.
By Cauchy's theorem, the integral in (5) is equal to the integral over $\Gamma(\tilde{\psi}, \tilde{\varepsilon})$ for any $0 < \tilde{\psi} \leq \psi$ and $0 < \tilde{\varepsilon} \leq \varepsilon$. By virtue of (4) the integral

$$\frac{1}{2i\pi}\int_{\Gamma}(-z)^{-\beta}R(z, A)x dz$$

exists as a Bochner integral for all $x \in X$ and define a bounded operator $(-A)^{-\beta}$ which is injective for all $\beta > 0$. Thus, we can define $(-A)^{\beta}$ as the inverse of $(-A)^{-\beta}$. The operator $(-A)^{\beta}$ ($\beta \in \mathbb{R}$) are closed, injective, and satisfy the semigroup property

$$(-A)^{\alpha}(-A)^{\beta} = (-A)^{\alpha+\beta} \text{ if } \alpha \geq 0 \geq \beta \text{ or } \alpha, \beta \geq 0 \text{ or } \alpha, \beta \leq 0. \tag{6}$$

In particular, we have $(-A)^{-\beta}(-A)^{\beta}x = x$ for every $\beta \in \mathbb{R}$ and $x \in D((-A)^{\beta})$; in other words, $(-A)^{-\beta}$ is the inverse of the operator $(-A)^{\beta}$. We have inclusion

$$D((-A)^{\beta}) \subset D((-A)^{\alpha}) \text{ if } \beta \geq \alpha.$$

For $\beta > 0$, $(-A)^{\beta}$ is closed, densely defined operator and $D((-A)^{\beta}) = Rang((-A)^{-\beta})$. For more details , we refer to Komatsu [11].

Now let A be a sectorial operator on X and $x \in D(A)$. For all $\beta > 0$, we define the integral operator

$$\mathbb{J}x := \frac{1}{2i\pi} \int_{\Gamma} (-z)^{-\beta} CR(z, A)x dz \tag{7}$$

where $(\mu)^{-\beta}$ is defined in terms of the principal branch of the logarithm.

In order to give our result, we need the following Lemma.

Lemma 2.1 *Assume that A is a sectorial operator and $\beta \geq \frac{1}{2}$. If the observation operator C satisfies the estimate (3) then for $x \in D(A)$ the integral $\frac{1}{2i\pi}\int_{\Gamma}(-z)^{-\beta}CR(z,A)xdz$ exists as a Bochner integral, the formula (7) extends to a bounded linear operator $\mathbb{J} : X \to Y$ and for all $x \in D(A), \mathbb{J}x = C(-A)^{-\beta}x$.*

Proof. Fix $x \in D(A)$ and define $g(z) = (-z)^{-\beta}CR(z, A)x \quad z \in \Gamma$. The fact that $x \in D(A)$ and $z \in \Gamma$, we get

$$CR(z, A)x = \frac{Cx}{z} + \frac{CR(z, A)Ax}{z}. \tag{8}$$

Cauchy's Theorem applied to the half plane yields that

$$\int_{\Gamma} (-z)^{-(\beta+1)} dz = 0.$$

It follows by (8) that

$$\frac{1}{2i\pi} \int_{\Gamma} (-z)^{-\beta} CR(z, A)x dz = \frac{1}{2i\pi} \int_{\Gamma} (-z)^{-(\beta+1)} CR(z, A)Ax dz.$$

which exists as a Bochner integral due to the fact that the operator C satisfies the estimation (3).

Let $\Gamma_N = \Gamma(\psi,\varepsilon) = \Gamma^1_N(\psi,\varepsilon) \cup \Gamma^2(\psi,\varepsilon) \cup \Gamma^3_N(\psi,\varepsilon)$ denotes the upwards oriented path defined by

$$\Gamma^1_N(\psi,\varepsilon) = \{\lambda \in \mathbb{C} : N \geq |\lambda| \geq \varepsilon, arg\lambda = -\psi\},$$

$$\Gamma^3_N(\psi,\varepsilon) = \{\lambda \in \mathbb{C} : N \geq |\lambda| \geq \varepsilon, arg\lambda = \psi\}.$$

It follows from Cauchy's Theorem that

$$\frac{1}{2i\pi}\int_{\Gamma}(-z)^{-\beta}CR(z,A)xdz = \lim_{N\to\infty}\frac{1}{2i\pi}\int_{\Gamma_n}(-z)^{-\beta}CR(z,A)xdz. \qquad (9)$$

Let $\mathcal{C}_N = \mathcal{C}_N(\psi)$ and $\overline{\Gamma}^2 = \overline{\Gamma}^2(\psi,\varepsilon)$ be the upwards oriented curve defined by

$$\mathcal{C}_N = \{\lambda \in \mathbb{C} : |\lambda| = N, |arg\lambda| \leq \psi\},$$

$$\overline{\Gamma}^2(\psi,\varepsilon) = \{\lambda \in \mathbb{C} : |\lambda| = \varepsilon, |arg\lambda| \leq \psi\}.$$

Using again Cauchy's theorem, we get

$$\frac{1}{2i\pi}\int_{\Gamma^1_N\cup\Gamma^3_N}(-z)^{-\beta}CR(z,A)xdz$$

$$= \frac{1}{2i\pi}\int_{\mathcal{C}_N}(-z)^{-\beta}CR(z,A)xdz + \frac{1}{2i\pi}\int_{\overline{\Gamma}^2}(-z)^{-\beta}CR(z,A)xdz.$$

Now, we are going to estimate the integrals over $\Gamma^2, \overline{\Gamma}^2$ and $\mathcal{C}_N$ separately. We start with the integral over $\mathcal{C}_N$. Then we obtain

$$\frac{1}{2i\pi}\int_{\mathcal{C}_N}(-z)^{-\beta}CR(z,A)xdz = \frac{1}{2\pi}\int_{-\psi}^{\psi}(Ne^{i(\theta-\pi)})^{-\beta}CR(Ne^{i\theta},A)Ne^{i\theta}xd\theta.$$

The fact that C satisfies (3), we obtain

$$\begin{aligned}\|\frac{1}{2i\pi}\int_{\mathcal{C}_N}(-z)^{-\beta}CR(z,A)xdz\|_Y &\leq \frac{1}{2\pi}\int_{-\psi}^{\psi}\|(Ne^{i(\theta-\pi)})^{-\beta}CR(Ne^{i\theta},A)Ne^{i\theta}x\|d\theta \\ &\leq \frac{M}{2\pi N^{\beta-1/2}}\int_{-\psi}^{\psi}\frac{d\theta}{\cos(\theta)^{\beta}}\|x\|.\end{aligned}$$

The same estimate holds for the integral over $\overline{\Gamma}^2$. Finally for the integral over Γ^2, we have

$$\begin{aligned}\|\frac{1}{2i\pi}\int_{\Gamma^2}(-z)^{-\beta}CR(z,A)xdz\| &\leq \frac{1}{2\pi}\int_{\Gamma^2}|(-z)^{-(\beta+1)}|\|CR(z,A)\|\|x\||dz| \\ &\leq \frac{M_\varepsilon}{\sqrt{\varepsilon}}\|x\|,\end{aligned}$$

since Γ^2 is a compact set and $z \mapsto CR(z,A)$ is analytic on Γ^2. Putting everything together, we find that there is a constant K not depending on N such that

$$\|\frac{1}{2i\pi}\int_{\Gamma_N}(-z)^{-\beta}CR(z,A)xdz\|_Y \leq K\|x\|_X, \quad x \in X.$$

And from (9) we obtain

$$\|\mathbb{J}x\|_Y \leq K\|x\|_X, \text{ for all } x \in D(A).$$

Therefore, $\mathbb{J}$ extends to a bounded linear operator on X.
Next we will show, that for $x \in D(A)$ we have $\mathbb{J}x = C(-A)^{-\beta}x$. This is equivalent to show that operator C and the integral $\int_\Gamma(-z)^{-\beta}R(z,A)xdz$ commute. According to (6), A commute with the integral $\int_\Gamma(-z)^{-\beta}R(z,A)xdz$ for every $x \in D(A)$. Now let us show that $\int_{\Gamma_n}(-z)^{-\beta}R(z,A)xdz$ converges in $D(A)$. The closedness of A yields

$$A\int_{\Gamma_N}(-z)^{-\beta}R(z,A)xdz = \int_{\Gamma_N}(-z)^{-\beta}R(z,A)Axdz, \quad x \in D(A).$$

Thus the integral $\int_{\Gamma_N}(-z)^{-\beta}R(z,A)xdz$ converges in $D(A)$. As C is continuous on $D(A)$, we obtain

$$\begin{aligned} C(-A)^{-\beta}x &= \lim_{N\to\infty}\frac{C}{2i\pi}\int_{\Gamma_N}(-z)^{-\beta}R(z,A)xdz \\ &= \lim_{N\to\infty}\frac{1}{2i\pi}\int_{\Gamma_N}(-z)^{-\beta}CR(z,A)xdz \\ &= \mathbb{J}x. \end{aligned}$$

This ends the proof.

Remark 1 *In this remark, we will show that a sectorial operator A generating a C_0-semigroup on A Hilbert space X is such that the fractional power $(-A)^\alpha$ with $0 < \alpha < 1/2$ is always admissible for A.*
Let A as above, and $0 \leq \alpha < 1$. Then there is a constant $M_\alpha \geq 0$ such that

$$\|s^\alpha(-A)^{1-\alpha}(sI - A)^{-1}\| \leq M_\alpha, \ \ s \in \Gamma.$$

Indeed, this follows from [10] and its proof. Let $C_\alpha := (-A)^{1-\alpha}$, then we have $(Re(s))^\alpha\|C_\alpha(sI-A)^{-1}\|$ is bounded on some right half-plane. This implies that C_α with $1/2 < \alpha < 1$ is admissible according to Zwart'result [22]. Generally this not always the case for $\alpha = 1/2$ (see [21], [9], [12]). In order that this should be true, Le Merdy considers the case when A generates an analytic semigroups (which is a necessary condition see Proposition 2.3 below). He then proves in [12] , using H^∞-functional calculus that $(-A)^{1/2}$ is admissible for A if and only if A satisfies the Weiss property.

Now we can give our result. As mentioned above this result was first obtained by LeMerdy [12]. However our proof is much shorter and does not make any recourse to the H^∞- functional calculus.

Theorem 2.2 *Let A be a generator an exponentially stable analytic C_0-semigroup on a Banach space X. Then the following assertions are equivalent:*

(i) *$(-A)^{1/2}$ is admissible for A.*

(ii) *A satisfies the Weiss property.*

Proof. It is obvious that (ii) implies (i). We now turn to the proof of (i) implies (ii) and assume that $(-A)^{1/2}$ is admissible observation for A. Let us consider a continuous $C : D(A) \to Y$ satisfying estimate (3). The fact that A generates an exponentially stable C_0-semigroup, A is sectorial. Since $-A$ is invertible, then $(-A)^{1/2}$ is invertible as well. Hence given any $x \in D(A)$ and any $t > 0$ we may write

$$C\mathbb{T}(t)x = C(-A)^{-1/2}(-A)^{1/2}\mathbb{T}(t)x.$$

Thanks to Lemma (2.1), the operator $C(-A)^{-1/2}$ defined on $D(A)$ extends to a bounded linear operator on X. Hence

$$\|C\mathbb{T}(t)x\| \leq \|\mathbb{J}\|_{\mathcal{L}(X,Y)}\|(-A)^{1/2}\mathbb{T}(t)x\|.$$

Integrating this inequality on $(0, +\infty)$ yields

$$\int_0^\infty \|C\mathbb{T}(t)x\|^2 dt \quad \leq \quad \|\mathbb{J}\|^2_{\mathcal{L}(X,Y)}\int_0^\infty \|(-A)^{1/2}\mathbb{T}(t)x\|^2_Y dt.$$

It therefore follows from admissibility of $(-A)^{1/2}$ for A that C is admissible for A.

It is worth noting that contrary to [12], in our proof of the 'if' part no analyticity assumption it was needed. Regarding the outline of this proof, it seems to be a genuine generalization of Le Merdy's result. Unfortunately this is not the case and bellow, we will show that the analyticity assumption in Theorem 2.2 cannot be omitted. Consequently, if the conclusion of Theorem 2.2 holds true, then A must generates an analytic semigroup on X.

Proposition 2.3 *Let A be a generator an exponentially stable C_0-semigroup on a Banach space X. If $(-A)^{1/2}$ is admissible for A, then A generates an analytic semigroup on X.*

Proof. Let $(-A)^{1/2}$ be an admissible observation operator for A. First let us show that there exists $K > 0$ such that $\|(-A)^{1/2}\mathbb{T}(t)\| \leq \frac{K}{\sqrt{t}}, \;\; 0 < t \leq 1.$

Indeed let $x \in D(A)$, Hahn-Banach Theorem implies that there exists $\varphi_{t,x} \in X^*$ with $\|\varphi_{t,x}\| = 1$ such that :

$$\begin{array}{rcl} t\|(-A)^{1/2}\mathbb{T}(t)x\| & = & t|\langle(-A)^{1/2}\mathbb{T}(t)x, \varphi_{t,x}\rangle| \\ & = & t|\langle(-A)^{1/2}\mathbb{T}(t-s)\mathbb{T}(s)x, \varphi_{t,x}\rangle| \;\; (0 \leq s \leq t) \\ & = & t|\langle(-A)^{1/2}\mathbb{T}(t-s)x, \mathbb{T}^*(s)\varphi_{t,x}\rangle| \\ & \leq & t\|(-A)^{1/2}\mathbb{T}(t-s)x\|\|\mathbb{T}^*(s)\varphi_{t,x}\|. \end{array}$$

Hence, using Cauchy-Schwartz inequality we obtain

$$\begin{array}{rcl} t\|(-A)^{1/2}\mathbb{T}(t)x\| & \leq & \int_0^t \|(-A)^{1/2}\mathbb{T}(t-s)x\|\|\mathbb{T}^*(s)\varphi_{t,x}\|ds \\ & \leq & (\int_0^t \|(-A)^{1/2}\mathbb{T}(s)x\|^2 ds)^{1/2}(\int_0^t \|\mathbb{T}^*(s)\varphi_{t,x}\|^2 ds)^{1/2}. \end{array} \tag{10}$$

Since $(-A)^{1/2}$ is supposed to be admissible for A we have

$$\int_0^t \|(-A)^{1/2}\mathbb{T}(s)x\|^2 ds \leq M^2\|x\|^2, \tag{11}$$

for some constant $M > 0$ not depending on x. Since $\mathbb{T}(t)$ is exponentially bounded on X, $\mathbb{T}^*(t)$ is exponentially stable on X^*. Combining (11) and (10) we deduce that

$$\|\sqrt{t}(-A)^{1/2}\mathbb{T}(t)x\| \leq MM'\|x\| \;\; x \in D(A). \tag{12}$$

By density we deduce that (12) is true for any $x \in X$.
Now we show $\mathbb{T}(\cdot)$ that is analytic. Since $\mathbb{T}$ is bounded, it is sufficient (see [15]) to show that there exists $M > 0$ such that :

$$\|tA\mathbb{T}(t)\| \leq M, \;\; \text{for} \;\; 0 < t \leq 1.$$

Let x be an arbitrary element in $D(A^2)$, apply (12) we obtain

$$\begin{array}{rcl} \|tA\mathbb{T}(t)x\| & = & \|\sqrt{t}(-A)^{1/2}\mathbb{T}(t/2)\sqrt{t}(-A)^{1/2}\mathbb{T}(t/2)x\| \;\; (\mathbb{T}(t) \text{ commute with } (-A)^{1/2}) \\ & \leq & \|\sqrt{t}(-A)^{1/2}\mathbb{T}(t/2)\|^2\|x\| \\ & \leq & M\|x\|. \end{array}$$

Since A is closed and $D(A^2)$ is dense in X, we deduce from above that for all $x \in X$ we have $\mathbb{T}(t)x \in D(A)$ and

$$\|tA\mathbb{T}(t)x\| \leq M\|x\| \;\;\; 0 \leq t \leq 1.$$

Thus A generates an analytic semigroup on X.

For Hilbert spaces and for a large class of Banach spaces, Le Merdy in [12], gave sufficient conditions on A under which the fractional power $(-A)^{1/2}$ is an admissible observation operator for A. In Particular, it was showed that this may happen for (normal) contractive and analytic semigroups on Hilbert spaces.

In our last statement we give a new proof for the admissibility of $(-A)^{1/2}$ for a given operator A which generates a normal and analytic semigroup on Hilbert space. This result can be considered as a new proof of the Weiss conjecture for analytic and normal semigroups (see [19]). The ingredients of the proof are the spectral theorem, which asserts that each self-adjoint or, more generally, normal operator on Hilbert space is (isomorphic to) a multiplication operator on some L^2- space (see [1] or [17]) and that the admissibility is preserved under the taking of similarities.

Theorem 2.4 *Let X be a Hilbert space and A generates a normal and analytic semigroup on X. Then A satisfy the Weiss property.*

Proof. Since the Weiss property is preserved under the taking of similarities, the theorem above implies that for a normal analytic semigroup $\mathbb{T}(t)$ with generator A, $(-A)^{1/2}$ satisfies the Weiss property if the multiplication operator A_m on $L^2(\Omega,\mu)$ generator of the multiplication semigroup $\mathbb{T}_m$ is such that $(-A_m)^{1/2}$ satisfies the Weiss property. And by Theorem 2.2 it will suffice to prove that $(-A_m)^{1/2}$ is an admissible observation operator for A_m. Thus let $\mathbb{T}_m$ be the multiplication semigroup in $L^2(\Omega,\mu)$. Then we have $\mathbb{T}_m(t)f(x) = e^{t\cdot m(x)}f(x)$, with generator $A_m f = m\cdot f$, for all $f\in D(A_m)$ where $D(A_m) = \{f \in L^2(\Omega,\mu) : m\cdot f \in L^2(\Omega,\mu)\}$ and $Af(x) = m(x)f(x)$. Without loss of generality we can assume that $m(x)$ is real negative almost everywhere in Ω. Since the admissibility is invariant under scalings $e^{-\alpha\cdot}\mathbb{T}(\cdot)$, we may assume that the semigroup $\mathbb{T}_m(t)$ is exponentially stable. Let us consider $(-A_m)^{1/2}$ with $D((-A_m)^{1/2}) = \{f \in L^2(\Omega,\mu), (-m(x))^{1/2}f(x) \in L^2(\Omega,\mu)\}$ and $(-A_m)^{1/2}f(x) = (-m(x))^{1/2}f(x)$ for all $f \in D((-A_m)^{1/2})$ and prove that

$$\int_{-\infty}^{\infty} \|(-A_m)^{1/2}R(i\lambda,A_m)f\|^2_{L^2(\Omega,\mu)}d\lambda \leq M\|f\|^2_{L^2(\Omega,\mu)}$$

for some $M > 0$ and for all $f\in L^2(\Omega,\mu)$.
For $f \in L^2(\Omega,\mu)$, it is straightforward to see that

$$\|(-A_m)^{1/2}R(i\lambda,A_m)f\|^2_{L^2(\Omega,\mu)} \quad = \quad \int_\Omega \frac{-m(x)}{\lambda^2+m^2(x)}\|f(x)\|^2 d\mu(x).$$

Fubini's Theorem yields

$$\begin{aligned}\int_{-\infty}^{\infty}\|(-A_m)^{1/2}R(i\lambda,A_m)f\|^2_{L^2(\Omega,\mu)}d\lambda &= \int_{-\infty}^{\infty}\int_{\Omega}\frac{-m(x)}{\lambda^2+m^2(x)}\|f(x)\|^2d\mu(x)d\lambda\\ &= \int_{\Omega}\|f(x)\|^2\int_{-\infty}^{\infty}\frac{-m(x)}{\lambda^2+m^2(x)}d\lambda d\mu(x)\\ &\leq \pi\int_{\Omega}\|f(x)\|^2d\mu(x)\\ &= \pi\|f\|^2_{L^2(\Omega,\mu)}.\end{aligned}$$

Thus for all $f \in L^2(\Omega,\mu)$, $(-A_m)^{1/2}R(\cdot,A_m)f$ lies in $H^2(L^2(\Omega,\mu))$ and by the Paley-Wiener theorem (operator-valued form) (see e.g., [5]) we deduce that $(-A_m)^{1/2}\mathbb{T}_m(\cdot)$ lies in $L^2((0,\infty);L^2(\Omega,\mu))$. This ends the proof.

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