Homomorphism Theorems in GT-Algebras

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Abstract
We introduce the notion of normal GT-filters in GT-algebras, and we establish construct the quotient GT-algebras via normal GT-filter, and we have the fundamental theorem of homomorphisms for GT-algebras as a consequence.

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1 Introduction

The variety of Tarski algebras was introduced by J. C. Abbott in [2]. These algebras are an algebraic counterpart of the \( \{ \lor, \rightarrow \} \)-fragment of the propositional classical calculus. S. A. Celani ([5]) introduced Tarski algebras with

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a modal operator as a generalization of the concept of Boolean algebra with
a modal operator which he researched into these fragments of the algebraic
viewpoint. Properties of filters in Tarski algebras were treated by S. A. Celani
([5]) and the authors ([6]). Recently, J. Kim, Y. Kim and E. H. Roh ([6])
considered decompositions and expansions of filters in Tarski algebras, and
also they have shown that there is no non-trivial quadratic Tarski algebras on
a field $X$ with $|X| \geq 3$. However, we feel that the concept of Tarski algebra
is relatively too strong for filters. Kim et al. ([7]) established a new algebra,
called a GT-algebra, which is a generalization of Tarski algebra, and gave a
method to construct a GT-algebra from a quasi-ordered set. In this paper,
we introduce the notion of normal GT-filters in GT-algebras, and we estab-
ish construct the quotient GT-algebras via normal GT-filter, and we have the
fundamental theorem of homomorphisms for GT-algebras as a consequence.

2 Preliminary Notes

Let us review some definitions and results.

**Definition 2.1.** [7] By a generalized Tarski algebra (GT-algebra, for short)
we mean an algebra $(X; \rightarrow, 1)$ of type $(2, 0)$ satisfying the following conditions:

(T1) $(\forall a \in X)(1 \rightarrow a = a)$.

(T2) $(\forall a \in X)(a \rightarrow a = 1)$.

(T3) $(\forall a, b, c \in X)(a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c))$.

Given a GT-algebra $X$, if it satisfies the condition

(T4) $(\forall a, b \in X)((a \rightarrow b) \rightarrow b = (b \rightarrow a) \rightarrow a)$,

we call the algebra a Tarski algebra. In a Tarski algebra $X$ we can define an
order relation $\leq$ by setting $a \leq b$ if and only if $a \rightarrow b = 1$ for any $a, b \in X$.
Note that $(X; \leq)$ is a poset ([3]).

A reflexive and transitive relation $\mathcal{R}$ on a set $X$ is called a quasi-ordering
of $X$, and the couple $(X, \mathcal{R})$ is called a quasi-ordered set ([4]). Note that if $X$
is a GT-algebra, then the relation $\leq$ by setting $x \leq y$ if and only if $x \rightarrow y = 1$
for any $a, b \in X$ is a quasi-ordering of $X$; with respect to this quasi-ordering 1
is the greatest element of $X$ ([8]).

**Example 2.2.** [8] Let $X := \{a, b, c, 1\}$ be a set with the following Cayley
table:

<table>
<thead>
<tr>
<th>$\rightarrow$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>1</td>
<td>$c$</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>1</td>
<td>$c$</td>
<td>1</td>
</tr>
<tr>
<td>$c$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$1$</td>
<td>$a$</td>
<td>$b$</td>
<td>$c$</td>
<td>1</td>
</tr>
</tbody>
</table>
Then \((X; \to, 1)\) is a GT-algebra and
\[
\mathcal{R} := \{(a, a), (a, b), (a, 1), (b, a), (b, b), (b, 1), (c, a), (c, b), (c, c), (c, 1), (1, 1)\}
\]
is a quasi-ordering of \(X\), which is not an anti-symmetric relation of \(X\).

**Lemma 2.3.** [7] Let \(X\) be a GT-algebra. Then

\[(p1) \ (\forall a \in X)(a \leq 1).\]

\[(p2) \ (\forall a, b \in X)(a \leq b \to a).\]

\[(p3) \ (\forall a, b \in X)(a \to (a \to b) = a \to b).\]

\[(p4) \ (\forall a, b \in X)(a \leq (a \to b) \to b).\]

\[(p5) \ (\forall a, b, c \in X)(a \leq b \Rightarrow c \to a \leq c \to b).\]

**Definition 2.4.** [7] Let \(X\) be a GT-algebra. A nonempty subset \(F\) of \(X\) is called a GT-filter of \(X\) if it satisfies the following conditions:

\[(F1) \ (\forall a, b \in X)(b \in F \Rightarrow a \to b \in F).\]

\[(F2) \ (\forall a, b \in X)(a \to b \in F, a \in F \Rightarrow b \in F).\]

**Theorem 2.5.** [7] Let \(F\) be a nonempty subset of a GT-algebra \(X\). Then \(F\) is a GT-filter of \(X\) if and only if it satisfies \(1 \in F\) and \((F2)\).

Let \(\mathcal{R}\) be a relation on a GT-algebra \(X\). Then \(\mathcal{R}\) is said to be compatible if \((a \to e, b \to f) \in \mathcal{R}\) whenever \((a, b) \in \mathcal{R}\) and \((e, f) \in \mathcal{R}\) for all \(a, b, e, f \in X\). A compatible equivalence relation on \(X\) is said to be a congruence on \(X\).

Let \(X\) be a GT-algebra and \(K(\neq \emptyset) \subseteq X\). Denote by \(\Theta_K\) the relation on \(X\) given by

\[(a, b) \in \Theta_K \text{ iff } a \to b \in K \text{ and } b \to a \in K.\]

**Theorem 2.6.** [7] Let \(K\) be a GT-filter of a GT-algebra \(X\). Then the relation \(\Theta_K\) is an equivalence relation on \(X\) and \([1]_{\Theta_K} = K\).

In Theorem 2.6, \(\Theta_K\) may not be compatible in general, as the following example.
Example 2.7. [7] Let $X := \{a, b, c, d, 1\}$ be a set with the following Cayley table:

\[
\begin{array}{ccccccc}
\rightarrow & a & b & c & d & e & f & g & 1 \\
\hline
a & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
b & c & 1 & c & g & 1 & 1 & g & 1 \\
c & f & f & 1 & f & 1 & f & 1 & 1 \\
d & c & e & c & 1 & e & 1 & 1 & 1 \\
e & a & f & c & d & 1 & f & g & 1 \\
f & c & e & c & g & e & 1 & g & 1 \\
g & a & b & c & f & e & f & 1 & 1 \\
1 & a & b & c & d & e & f & g & 1 \\
\end{array}
\]

Then $(X; \rightarrow, 1)$ is a GT-algebra, and the subset $K := \{d, 1\}$ is a GT-filter of $X$. Moreover, we can find 

$\Theta_K = \{(a, a), (b, b), (c, c), (d, d), (e, e), (e, 1), (f, f), (g, g), (1, e), (1, 1)\}$.

It is routine to check that $\Theta_K$ is an equivalence relation on $X$, which is not compatible since $(e, 1) \in \Theta_K$ and $(b, b) \in \Theta_K$, but $(e \rightarrow b, 1 \rightarrow b) = (f, b) \notin \Theta_K$.

3 Main Results

Definition 3.1. A GT-filter $F$ of a GT-algebra $X$ is said to be normal if it satisfies:

(F3) $(\forall a, b, c \in X)(a \rightarrow b \in F \Rightarrow (b \rightarrow c) \rightarrow (a \rightarrow c) \in F)$.

Obviously, $X$ and $\{1\}$ are normal GT-filters of $X$.

Example 3.2. Let $X := \{a, b, c, 1\}$ be a set with the following Cayley table:

\[
\begin{array}{ccc}
\rightarrow & a & b & c & 1 \\
\hline
a & 1 & b & 1 & 1 \\
b & a & 1 & 1 & 1 \\
c & a & b & 1 & 1 \\
1 & a & b & c & 1 \\
\end{array}
\]

It is easy to check that $(X; \rightarrow, 1)$ is a GT-algebra, and $\{c, 1\}, \{a, c, 1\}, \{b, c, 1\}$ are normal GT-filters of $X$. But $F := \{a, 1\}$ is not a normal GT-filter of $X$ since $1 \rightarrow a \in F$ and $(a \rightarrow c) \rightarrow (1 \rightarrow c) = c \notin F$. 
Now we construct the quotient GT-algebras via normal GT-filters. Let $K$ be a normal GT-filter of a GT-algebra $(X; \rightarrow, 1)$. Then we obtain from Theorem 2.6 that $\Theta_K$ is an equivalence relation on $X$ and $[1]_{\Theta_K} = K$.

Let $a, b, c, d \in X$ such that $(a, b) \in \Theta_K$ and $(c, d) \in \Theta_K$. Then we have $a \rightarrow b \in K$ implies $(a \rightarrow c) \rightarrow (b \rightarrow c) \in K$, and $b \rightarrow a \in K$ implies $(b \rightarrow c) \rightarrow (a \rightarrow c) \in K$. Thus we get

$$(a \rightarrow c, b \rightarrow c) \in \Theta_K.$$ 

Since $c \rightarrow d \in K$ implies $(b \rightarrow c) \rightarrow (b \rightarrow d) \in K$, and $d \rightarrow c \in K$ implies $(b \rightarrow d) \rightarrow (b \rightarrow c) \in K$. Hence we have

$$(b \rightarrow c, b \rightarrow d) \in \Theta_K.$$ 

We conclude that $(a \rightarrow c, b \rightarrow d) \in \Theta_K$. Therefore $\Theta_K$ is a congruence relation on $X$.

Denote the equivalence class containing $a$ by $[a]_{\Theta_K}$, i.e.,

$$[a]_{\Theta_K} := \{ x \in X | (a, x) \in \Theta_K \}.$$ 

We note that $(a, b) \in \Theta_K$ if and only if $[a]_{\Theta_K} = [b]_{\Theta_K}$.

Denote $X/\Theta_K := \{ [a]_{\Theta_K} | a \in X \}$ and define

$$[a]_{\Theta_K} \rightarrow' [b]_{\Theta_K} := [a \rightarrow b]_{\Theta_K}.$$ 

The operation “$\rightarrow'$” is well-defined, since $\Theta_K$ is a congruence relation on $X$. We claim that $(X/\Theta_K; \rightarrow', [1]_{\Theta_K})$ is a GT-algebra. Clearly

$$[1]_{\Theta_K} \rightarrow' [a]_{\Theta_K} = [a]_{\Theta_K} \quad \text{and} \quad [a]_{\Theta_K} \rightarrow' [a]_{\Theta_K} = [1]_{\Theta_K}$$

for all $[a]_{\Theta_K} \in X/\Theta_K$. Let $[a]_{\Theta_K}, [b]_{\Theta_K}, [c]_{\Theta_K} \in X/\Theta_K$. Then we have

$$[a]_{\Theta_K} \rightarrow' ([b]_{\Theta_K} \rightarrow' [c]_{\Theta_K}) = [a \rightarrow (b \rightarrow c)]_{\Theta_K} = [(a \rightarrow b) \rightarrow (a \rightarrow c)]_{\Theta_K} = ([a]_{\Theta_K} \rightarrow' [b]_{\Theta_K}) \rightarrow' ([a]_{\Theta_K} \rightarrow' [c]_{\Theta_K}).$$

We summarize:

**Theorem 3.3.** Let $K$ be a normal GT-filter of a GT-algebra $(X; \rightarrow, 1)$. If we define

$$[a]_{\Theta_K} \rightarrow' [b]_{\Theta_K} := [a \rightarrow b]_{\Theta_K}$$

for all $a, b \in X$, then $(X/\Theta_K; \rightarrow', [1]_{\Theta_K})$ is a GT-algebra, which is called the quotient GT-algebra via $K$. 
Now, we state a fundamental theorem of a homomorphism.

**Definition 3.4.** Let $X, Y$ be GT-algebras. A mapping $f : X \to Y$ is called an homomorphism if

$$(\forall a, b \in X)(f(a \rightarrow b) = f(a) \rightarrow f(b)).$$

A homomorphism $f$ is called a monomorphism (resp., epimorphism) if it is injective (resp., surjective). A bijective homomorphism is called an isomorphism. Two GT-algebras $X$ and $Y$ are said to be isomorphic, written $X \cong Y$, if there exists an isomorphism $f : X \to Y$. For any homomorphism $f : X \to Y$, the set $\{a \in X|f(a) = 0\}$ is called the kernel of $f$, denoted by $\text{Ker}(f)$ and the set $\{f(a)|a \in X\}$ is called the image of $f$, denoted by $\text{Im}(f)$.

**Example 3.5.** Let $X := \{a, b, 1\}$ and $Y := \{x, 1\}$ be GT-algebras, whose Cayley tables are as follows.

$$
\begin{array}{c|ccc}
  & a & b & 1 \\
\hline
a & 1 & 1 & 1 \\
b & a & 1 & 1 \\
1 & a & b & 1 \\
\end{array}
\quad
\begin{array}{c|cc}
  & x & 1 \\
\hline
x & 1 & 1 \\
1 & x & 1 \\
\end{array}
$$

Define a mapping $f : X \to Y$ by $a \mapsto x, b \mapsto 1, 1 \mapsto 1$. It can be readily check that $f$ is a homomorphism from a GT-algebra $X$ to a GT-algebra $Y$.

Example 3.5 shows that the cardinal number of a homomorphic image of a finite GT-algebra may not be a factor of the cardinal number of the domain. Obviously, $|X| = 3$ and $|Y| = 2$, but 2 is not factor of 3.

**Lemma 3.6.** Let $f : X \to Y$ be a homomorphism from a GT-algebra $X$ to a GT-algebra $Y$. Then

(i) $f(1) = 1$.

(ii) $(\forall a, b \in X)(a \leq b \Rightarrow f(a) \leq f(b))$.

**Proof.** Straightforward.

The next proposition satisfies an ordinary algebraic homomorphism, whose verification is routine and omitted.

**Lemma 3.7.** Let $f : X \to Y$ be a homomorphism from a GT-algebra $X$ to a GT-algebra $Y$. Then

(i) $f$ is epimorphic if and only if $\text{Im}(f) = Y$

(ii) $f$ is monomorphic if and only if $\text{Ker}(f) = \{1\}$
(iii) \( f \) is isomorphic if and only if the inverse mapping \( f^{-1} : Y \to X \) is isomorphic.

**Theorem 3.8.** Let \( f : X \to Y \) be a homomorphism from a GT-algebra \( X \) onto a Tarski algebra \( Y \). Then \( \text{Ker}(f) \) is a normal GT-filter of \( X \).

**Proof.** Obviously, \( 1 \in \text{Ker}(f) \). Let \( a \to b \in \ker(f) \) and \( a \in \text{ker}(f) \). Then \( 1 = f(a \to b) = f(a) \to f(b) = 1 \to f(b) = f(b) \). Hence \( b \in \text{ker}(f) \). Let \( a \to b \in \text{ker}(f) \) Then for any \( c \in X \), we have
\[
f((b \to c) \to (a \to c)) = 1.
\]
Hence we obtain \( (b \to c) \to (a \to c) \in \text{Ker}(f) \). Therefore \( \text{ker}(f) \) is a normal GT-filter of \( X \).

In Example 3.5, \( X \) is a GT-algebra, which is not a Tarski algebra since \( (a \to b) \to b \neq (b \to a) \to a \), and \( Y \) is a Tarski algebra, and \( f \) is an epimorphism. Obviously, \( \text{Ker}(f) = \{b, 1\} \) is a normal GT-filter of \( X \).

**Theorem 3.9.** (Homomorphism Theorem) If \( f : X \to Y \) is a homomorphism from a GT-algebra \( X \) onto a Tarski algebra \( Y \), then the quotient GT-algebra \( X/\Theta_{\text{Ker}(f)} \) and \( Y \) are isomorphic, i.e., \( X/\Theta_{\text{Ker}(f)} \cong Y \).

**Proof.** Define a mapping
\[
\mu : X/\Theta_{\text{Ker}(f)} \to Y \text{ by } \mu([a]_{\Theta_{\text{Ker}(f)}}) = f(a).
\]
If \([a]_{\Theta_{\text{Ker}(f)}} = [b]_{\Theta_{\text{Ker}(f)}}\), then \( a \to b, b \to a \in \text{Ker}(f) \), and so we get
\[
f(a) \to f(b) = 1 = f(b) \to f(a)
\]
in \( Y \). Thus we have \( f(a) = f(b) \), i.e., \( \mu([a]_{\Theta_{\text{Ker}(f)}}) = \mu([b]_{\Theta_{\text{Ker}(f)}}) \). This means that \( \mu \) is well-defined. Let \([a]_{\Theta_{\text{Ker}(f)}} = [b]_{\Theta_{\text{Ker}(f)}} \in X/\Theta_{\text{Ker}(f)} \) with \([a]_{\Theta_{\text{Ker}(f)}} \neq [b]_{\Theta_{\text{Ker}(f)}}\). Then \((a, b) \notin \Theta_{\text{Ker}(f)}\), and hence
\[
either \ a \to b \notin \text{Ker}(f) \ or \ b \to a \notin \text{Ker}(f)\).
\]
Without loss of generality, we may assume \( a \to b \notin \text{Ker}(f) \). It follows that \( f(a) \to f(b) = f(a \to b) \neq 1 \), and hence \( f(a) \neq f(b) \). This means that \( \mu \) is one-one. For any \( b \in Y \), there is an \( a \in X \) such that \( b = f(a) \), since \( f \) is onto. Hence \( \mu([a]_{\Theta_{\text{Ker}(f)}}) = f(a) = b \), which means that \( \mu \) is onto. Since
\[
\mu([a]_{\Theta_{\text{Ker}(f)}} \to [b]_{\Theta_{\text{Ker}(f)}}) = \mu([a \to b]_{\Theta_{\text{Ker}(f)}})
= f(a \to b)
= f(a) \to f(b)
= \mu([a]_{\Theta_{\text{Ker}(f)}}) \to \mu([b]_{\Theta_{\text{Ker}(f)}}),
\]
\( \mu \) is a homomorphism. Thus we obtain \( X/\Theta_{\text{Ker}(f)} \cong Y \), completing the proof. \( \square \)
Theorem 3.10. Let $X$ and $Y$ be GT-algebras and $Z$ be a Tarski algebra, and let $h: X \rightarrow Y$ be an epimorphism and $g: X \rightarrow Z$ be a homomorphism. If $\text{Ker}(h) \subseteq \text{Ker}(g)$, then there is a unique homomorphism $f: Y \rightarrow Z$ satisfying $f \circ h = g$.

Proof. For any $b \in Y$, there exists an $a \in X$ such that $b = h(a)$. Given an element $a$, we put $c := g(a)$. Define a mapping

$$f: Y \rightarrow Z$$

such that $f(b) = c$. To prove that $f$ is well-defined and $f \circ h = g$. If $b = h(a_1) = h(a_2), a_1, a_2 \in X$, then $0 = h(a_1) \rightarrow h(a_2) = h(a_1) \rightarrow a_2$. Hence $a_1 \rightarrow a_2 \in \text{Ker}(h)$. Since $\text{Ker}(h) \subseteq \text{Ker}(g)$, we have $0 = g(a_1) \rightarrow a_2 = g(a_1) \rightarrow a_2$. Similarly, we get $g(a_2) \rightarrow a_1 = 0$. Thus $g(a_2) = g(a_1)$. This means that $f$ is well-defined. Clearly $g(a) = f(h(a))$ for any $a \in X$. To show that $f$ is a homomorphism. Let $b_1, b_2 \in Y$. For any $a_1, a_2 \in X$ such that $b_1 = h(a_1), b_2 = h(a_2)$, we have $f(b_1 \rightarrow b_2) = f(h(a_1) \rightarrow h(a_2)) = f(h(a_1) \rightarrow a_2)) = g(a_1) \rightarrow a_2 = g(a_1) \rightarrow g(a_2) = f(h(a_1)) \rightarrow f(h(a_2)) = f(b_1) \rightarrow f(b_2)$. Hence $f$ is a homomorphism. The uniqueness of $f$ follows directly from the fact that $h$ is an epimorphism. $\square$

Theorem 3.11. Let $X, Y$ and $Z$ be GT-algebras, and let $g: X \rightarrow Z$ be a homomorphism and $h: Y \rightarrow Z$ be a monomorphism with $\text{Im}(g) \subseteq \text{Im}(h)$, then there is a unique homomorphism $f: X \rightarrow Y$ satisfying $h \circ f = g$.

Proof. For each $a \in X$, $g(a) \subseteq \text{Im}(g) \subseteq \text{Im}(h)$. Since $h$ is a monomorphism, there exists a unique $b \in Y$ such that $h(b) = g(a)$. Define a map

$$f: X \rightarrow Y$$

such that $f(a) = b$. Then $h \circ f = g$. We show that $f$ is a homomorphism. If $a_1, a_2 \in X$, then $g(a_1 \rightarrow a_2) = h(f(a_1 \rightarrow a_2))$. On the other hand, since $g$ is a homomorphism, $g(a_1 \rightarrow a_2) = g(a_1) \rightarrow a_2 = h(f(a_1)) \rightarrow h(f(a_2)) = h(f(a_1) \rightarrow f(a_2))$. Hence $h(f(a_1 \rightarrow a_2)) = h(f(a_1) \rightarrow f(a_2))$. Since $h$ is a monomorphism, we have $f(a_1 \rightarrow a_2) = f(a_1) \rightarrow f(a_2)$. The uniqueness of $f$ follows from the fact that $h$ is a monomorphism. $\square$

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References


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