High Degree Special Ruled Surfaces on Curves with General Moduli

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Abstract. Fix integers $g \geq 3$ and $d \geq 6g - 4$. Here we describe the irreducible components of the set of all triples $(C, E, V)$, where $C$ is a smooth genus $g$ curve with general moduli, $E$ is a rank 2 vector bundle on $C$ with degree $d$ and $V$ is a linear subspace of $H^0(C, E)$ such that $\dim(V) = d + 2 - 2g$, $V$ spans $E$ and the morphism $\mathbb{P}(E) \to \mathbb{P}^{d-2g+1}$ induced by $V$ is birational onto its image. For another proof (and more) see arXiv:math/0809.0373.

Mathematics Subject Classification: 14H60; 14J26; 14N05

Keywords: scroll; surface scroll; rank two vector bundle; ruled surface

1. Introduction

As in [2], Th. 1.2, for all integers $d, g$ such that $g \geq 0$ and $d \geq 2g + 2$ let $\mathcal{H}_{d,g}$ denote the unique irreducible component of the Hilbert scheme of surface scrolls of degree $d$ and sectional genus $g$ in $\mathbb{P}^{d-2g+1}$ whose general member represents a non-special linearly normal smooth scroll and which maps dominantly on the moduli space $\mathcal{M}_g$ of genus $g$ smooth curve (with obvious modifications when $g = 1$). $\dim(\mathcal{H}_{d,g}) = (d - 2g + 2)^2 + 7(g - 1)$. Moreover $\mathcal{H}_{d,g}$ is the unique component of the set of non-degenerate and non-special scrolls of degree $d$ and genus $g$ in $\mathbb{P}^{d-2g+1}$ ([2]). If $g \geq 2$, then a general element of $\mathcal{H}_{d,g}$ is associated to a general pair $(C, E)$ where $C$ is general in $\mathcal{M}_g$ and $E$ is a general rank 2 stable vector bundle on $C$ with degree $d$ (and the converse holds) [3], Th. 5.4). Let $H[d, g]$ denote the reduction of the Hilbert scheme of all non-degenerate surface scrolls in $\mathbb{P}^{d-2g+1}$ with degree $d$ and with sectional genus $g$. If $g \geq 2$ let $H'[d, g]$ denote the union of the irreducible components of $H[d, g]$ which dominate $\mathcal{M}_g$. If we drop the non-speciality assumption (or, equivalently, by Riemann-Roch the linearly normal condition),

\footnote{The author was partially supported by MIUR and GNSAGA of INdAM (Italy).}
then other components which dominate $\mathcal{M}_g$ exist, at least if $d \geq 2g + 11$ ([2], Example 5.12). For any smooth genus $g$ curve let $H[d, C]$ denote the moduli space of degree $d$ surface scrolls $S \subset \mathbb{P}^{d-2g+1}$ such that $C$ is the normalization of a general hyperplane section of $S$. There are two interesting algebraic sets which deserve to be studied:

(a) The set $S_{d,g}$ of all triples $(C, E, V)$, where $C \in \mathcal{M}_g$, $E$ is a rank 2 vector bundle on $C$, $V$ is a $(d + 2 - 2g)$-dimensional linear subspace of $H^0(C, E)$ spanning $V$ and such that the morphism $\mathbb{P}(E) \to \mathbb{P}^{d+1-2g}$ is birational onto its image, up to isomorphisms of triples.

(b) The set $H[d, g]$.

We write $\tilde{H}'[d, g]$ for the open subset of $S_{d,g}$ obtained restricting the curve $C$ to be with general moduli. Moreover, if we fix the curve $C \in \mathcal{M}_g$, we also get the set $\tilde{H}[d, C]$ associated to the subset of $S_{d,g}$ with $C$ as the base curve. We write $\tilde{H}_{d,g}$ for the subset of $S_{d,g}$ corresponding to non-special vector bundles.

We write $H[d, C]$ for the set of non-special degree $d$ rank 2 vector bundles over the fixed curve $C$. There is a surjection $\phi$ between each set with a $\sqrt{\,}$ onto the corresponding set without the $\sim$. The fiber of any of these surjections over a scroll $S$ corresponds to the subset of $\text{Aut}(\mathbb{P}^{d+1-2g})$ inducing an automorphism of $S$.

We believe that the algebraic sets $S_{d,g}$ deserve to be studied for their own sake. Anyway, their study is a preliminary step for the study of $H_{d,g}$.

In this note we describe the irreducible components of $\tilde{H}'[d, g]$ and of $\tilde{H}[d, C]$ when $d \geq 6g - 6$ and $C$ is general (see Theorem 1). This result (and much more) was also proved in [4].

For all integers $g, r, d$ let $\rho(g, r, d) := (r + 1)d - rg - r(r + 1)$ denote (the Brill-Noether number).

**Theorem 1.** Fix an integer $g \geq 2$, an integer $d \geq 6g - 4$ and general $C \in \mathcal{M}_g$. Let $\Theta$ be the set of all pairs $(m, \alpha)$, where $\left[(g + 3)/3\right] \leq m \leq 2g - 2$, $\alpha > 0$, $m + \alpha > g$ and $\rho(g, m - g + \alpha, m) \geq 0$. For all integers $r, m$ such that $r > 0$, $\left[(g + 3)/3\right] \leq m \leq g$, and $\rho(g, r, m) = 0$ let $\tau(g, r, m)$ denote the number of all $g'_m$ on $C$. Let $\Theta'$ be the set of all triples $(m, \alpha, t)$, where $(m, \alpha) \in \Theta$, $t = 1$ if $\rho(g, m - g + \alpha, m) > 0$, while $t$ is any positive integer $\leq \tau(g, r, m)$ if $\rho(g, m - g + \alpha, m) = 0$. Set $\theta_g := \sharp(\Theta)$ and $\theta'_g := \sharp(\Theta')$.

(i) $\tilde{H}'[d, g]$ has $\theta_g + 2$ irreducible components. $\tilde{H}_{d,g}$ is the only component of $\tilde{H}'[d, g]$ whose general member is associated to an indecomposable vector bundle. One irreducible component of $\tilde{H}'[d, g]$ is associated to cones, i.e. its general member is associated to a triple $(C', \mathcal{O}_{C'} \oplus L, V)$, with $C'$ general in $\mathcal{M}_g$, $L \in \text{Pic}^d(C')$ and $V$ a general $(d + 2 - 2g)$-dimensional linear subspace of $H^0(C', \mathcal{O}_{C'} \oplus L)$. The general member of the irreducible component with label $(m, \alpha) \in \Theta$ is of the form $(C', L \oplus M, V)$, with $C'$ general in $\mathcal{M}_g$, $L \in \text{Pic}^d(C')$, $M \in W^{m+\alpha-g}(C')$ and $V$ a general $(d + 2 - 2g)$-dimensional linear subspace of $H^0(C', L \oplus M)$. The
irreducible components of $\tilde{H}'[d,g]$ whose general member has smooth image in $\mathbb{P}^{d+1-2g}$ are $\tilde{\mathcal{H}}_{d,g}$ and the ones associated to a pair $(m, \alpha)$ with $m - g + \alpha \geq 3$.

(ii) $\tilde{H}[d,C]$ has $\theta'_g + 2$ irreducible components whose description is as in part (i) taking $C' := C$ and $\Theta'$ instead of $\Theta$.

We work over an algebraically closed field $\mathbb{K}$. The Brill-Noether theory of special divisors on a curve with general moduli is true in arbitrary characteristic ([6]). For part (i) of Theorem 1 we assume char($\mathbb{K}$) = 0, because we quote a special case of [5]. B. Osserman extended Eisenbud-Harris limit linear series to the positive characteristic case ([7]), allowing the interested reader to extend that part of [5] to the case char($\mathbb{K}$) $\geq 2g - 2$.

We thank the anonymous referee of a previous version and the authors of [4] for several very useful observations.

2. THE PROOF

Fix an integer $g \geq 2$. A general member of $\mathcal{H}_{d,g}$ is associated to a pair $(C, E)$ with $C$ general in $\mathcal{M}_g$ and $E$ a general degree $d$ rank 2 stable vector bundle on $C$ ([1], §2, [3], Th. 5.4). In particular the general $S \in \mathcal{H}_{d,g}$ is linearly normal. Let $E$ be a rank two vector bundle on the smooth genus $g$ curve $X$. Set $s(E) := \deg(E) - 2\deg(L)$, where $L$ is a maximal degree line subsheaf of $E$. A classical theorem of C. Segre and M. Nagata says that $s(E) \leq g$ for all $E$. Take any maximal degree line subsheaf $L$ of $E$. The maximality of the integer $\deg(L)$ gives that $L$ is saturated in $E$, i.e. $E/L \in \text{Pic}(X)$. Since $\deg(\text{Hom}(E/L, L)) = -s(E)$, $E \cong E/L \oplus L$ if $s(E) < 2 - 2g$. It is easy to check that any integer $2 - 2g \leq s(E) \leq g$ is realized by some indecomposable rank two vector bundle on $X$. The definition of stability (resp. semistability) gives that $E$ is stable (resp. semistable, resp. properly semistable) if and only if $s(E) > 0$ (resp. $s(E) \geq 0$, resp. $s(E) = 0$). Since $E$ is an extension of $E/L$ by $L$ and $s(E) = \deg(E/L) - \deg(E)$, then $s(E) \equiv \deg(E) \pmod{2}$. Fix any irreducible component $\Gamma$ of $\tilde{H}[d,g]$ or of $\tilde{H}[d,C]$ or of $\tilde{H}[d,C]$ or of $H[d,C]$ and take a general $S \in \Gamma$, say associated to the pair $(C, E)$. Set $s(\Gamma) := s(E)$. A semicontinuity theorem for the integer $s(E)$ gives that $s(\Gamma)$ is well-defined. We have $s(E) \leq g$ and $s(E) \equiv d \pmod{2}$. We have $s(\mathcal{H}_{d,g}) = g$ if $d \equiv g \pmod{2}$ and $s(\mathcal{H}_{d,g}) = g - 1$ if $d \equiv g - 1 \pmod{2}$.

Remark 1. Let $C$ be a smooth genus $g$ curve and $E$ a rank 2 vector bundle on $C$. Set $d := \deg(E)$ and $s := s(E)$. Notice that $d \equiv s \pmod{2}$. The integer $s$ is often called the degree of stability of $E$. Let $L$ be a maximal degree line subsheaf of $E$. Hence $E/L \in \text{Pic}(C)$, $\deg(L) = (d - s)/2$, $\deg(E/L) = (d + s)/2$ and $E$ is an extension of $E/L$ by $L$. Thus $h^1(C, E) = 0$ if $(d - |s|)/2 \geq 2g - 1$, i.e. if $d \geq 4g - 2 + |s|$. Since $s(F) \geq 2 - 2g$ for any indecomposable rank 2 vector bundle $F$ on $C$, we get $h^1(C, F) = 0$ for any indecomposable rank 2 vector bundle on $C$ such that $\deg(F) \geq 6g - 4$. If $g = 1$ it is sufficient
to assume $\deg(F) \geq 1$. Thus if $g \geq 2$, $d \geq 6g - 4$, $C \in \mathcal{M}_g$ and $\Gamma$ is any component of $H[d, C]$, then either $\Gamma$ is in the closure of the fiber over $C$ of the map $\mathcal{H}_{d,g} \rightarrow \mathcal{M}_g$ or its general member is associated to a decomposable vector bundle. A similar statement follows for the irreducible components of $H[d, g]$, $\bar{H}[d, C]$ and $\bar{H}[d, g]$.

**Remark 2.** Fix an irreducible component $\Gamma$ of $\bar{H}[d, g]$ or of $H[d, g]$ whose general element $S$ is associated to a triple $(C, E, V)$ with $E \cong L \oplus M$, $V \subseteq H^0(C, E)$, $L, M \in \text{Pic}(C)$, $h^1(C, L) = 0$ and $h^1(C, M) > 0$. Set $m := \deg(M)$ and $r := h^0(C, M) - 1$. The generality of $S$ implies that $L$ is a general element of $\text{Pic}^{d-m}(C)$. The semicontinuity theorem for the degree of stability shows that $s(E') \leq 2m - d$ for all pairs $(C', E')$ associated to some element of $\Gamma$. If $M \cong \mathcal{O}_C$, then $S$ is a cone. From now on we assume that $S$ is not a cone. Hence $m \geq \text{gon}(C)$. Now assume that $C$ has general moduli. Hence $m \geq [(g + 3)/2]$ and $\rho(g, r, m) \geq 0$. If $\rho(g, r, m) > 0$, the irreducibility of $G^r_d(C)$ and the fact that we may take as $L$ any general element of $\text{Pic}^{d-m}(C)$ gives that $\Gamma$ contains all $(C, M \oplus L)$ with $(M, L)$ general in $W^r_m(C) \times \text{Pic}^{d-m}(C)$. If $\rho(g, r, d) = 0$, then the same is true (moving $C \in \mathcal{M}_g$) by [5]. If $\rho(g, r, m) > 0$, then even the fiber $\Gamma_C$ at $S$ of the fiber of the rational map from $\Gamma$ into $\mathcal{M}_g$ contains all scrolls associated to vector bundles $L \oplus M$ with $(L, M)$ varying in a non-empty open subset of $W^r_m(C) \times \text{Pic}^{d-m}(C)$.

**Remark 3.** Fix integers $g, m, \alpha, d$, such that $g \geq 2$, $\alpha > 0$, $m + \alpha - g \geq 1$, $d - m \geq 2g$, and $\rho(g, m + \alpha - g, m) > 0$. Fix a general genus $g$ curve $C$. Let $A$ (resp. $B$) be the set of all decomposable vector bundles $E = L \oplus M$ with $\deg(L) = d - m$, $\deg(M) = m$, $M$ spanned and $h^1(C, M) = \alpha$ (resp. $\deg(L) = d - m - 1$, $\deg(M) = m + 1$, $M$ spanned and $h^1(C, M) = \alpha - 1$).

Since $C$ is general, Brill-Noether theory gives that the set $A'$ (resp. $B'$) of all line bundles $M$ appearing in the definition of $A$ (resp. $B$) is non-empty, that it has pure dimension $\rho(g, m + \alpha - g, m)$ (resp. $\rho(g, m + \alpha - g, m + 1) = \rho(g, m + \alpha - g, m) + m + \alpha - g$) and that it is irreducible if $\rho(g, m + \alpha - g, m) > 0$ (resp. $\rho(g, m + \alpha - g, m + 1) > 0$). Notice that $h^0(C, L \oplus M) = d + 2 - 2g + \alpha$. Let $G(d + 2 - 2g, d + 2 - 2g + \alpha)$ denote the Grassmannian of all $(d + 2 - 2g)$-dimensional linear subspaces of $K^{\oplus(d+2-2g+\alpha)}$. $G(d + 2 - 2g, d + 2 - 2g + \alpha)$ is irreducible and $\dim(G(d + 2 - 2g, d + 2 - 2g + \alpha)) = \alpha(d + 2 - 2g)$. Notice that $\dim(G(d + 2 - 2g, d + 2 - 2g + \alpha - 1)) = (\alpha - 1)(d + 2 - 2g)$. Let $A''$ (resp. $B''$) be the set of all pairs $(E, V)$ with $E \in A$ (resp. $E \in B$) and $V$ a $(d + 2 - 2g)$-dimensional linear subspace of $H^0(C, E)$. Hence $\dim(A'') = \dim(B'') + d + 2 - 2g - (m + \alpha - g) > \dim(B'')$. Hence $A''$ is not in the closure of $B''$ and the same is true for the set of all scrolls in $H[d, g]$ coming from $A''$ and $B''$. The same is true for all curves $C$ with general moduli, i.e. for the subsets $A_1$ and $B_1$ of $\bar{H}[d, g]$ obtained from $A''$ and $B''$ varying $C$ among a non-empty open subset of $\mathcal{M}_g$. Since $s(E) = 2m - d < 2m - d + 2 = s(F)$ for all $(E, F) \in A \oplus B$, $B_1$ is disjoint from the closure of $A_1$ in $H[d, g]$.
Proof of Theorem 1. We first consider part (ii). For all \((m, \alpha, x) \in \Theta'\) let \(\Gamma_{m,\alpha,x}\) denote the irreducible algebraic subset of \(\tilde{H}[d, C]\) parametrized by the pairs \((L \oplus M, V)\) with \(L \oplus M\) labelled by \((m, \alpha, x)\) and \(V\) a general \((d + 2 - 2g)\)-dimensional linear subspace of \(H^0(C, L \oplus M)\). Let \(\Gamma\) be an irreducible component of \(\tilde{H}[d, C]\) different from the fiber over \(C\) of the map \(\tilde{H}_{d,g} \to \mathcal{M}_g\). Let \((E, V)\) be a general element of \(\Gamma\). Since \(d \geq 6g - 4\), \(E \cong L \oplus M\), with, say \(\deg(M) < \deg(L)\). The induced scroll \(S\) is a cone if and only if \(M \cong \mathcal{O}_C\), i.e. if and only if \(m = 0\). Assume \(m > 0\). Since \(M\) is spanned, \(\Gamma\) is the closure of a unique \(\Gamma_{m,\alpha,x}\) with \(\alpha := h^1(C, M)\) (Remark 2). To prove part (ii), except the smoothness assertion, it is sufficient to prove that \(\Gamma\) contains no \(\Gamma'_{m',\alpha',x'}\) with \((m', \alpha, x') \neq (m, \alpha, x)\). Assume that this is not the case, and take \((m', \alpha, x') \neq (m, \alpha, x)\) such that \(\Gamma'_{m',\alpha',x'} \subset \Gamma\). Look at Remark 3. If \(\rho(g, m_1 + \alpha_1 - g, m_1) = 0\), then all \(\Gamma_{m_1,\alpha_1,t}\), \(1 \leq t \leq \tau(g, m_1 + \alpha_1 - g, m_1)\), have the same dimension. Hence \((m', \alpha') \neq (m, \alpha)\). The semicontinuity theorem for cohomology gives \(\alpha' \geq \alpha\). The semicontinuity theorem for the stability degree \(s(E)\) gives \(m' \leq m\). We have \(\dim(\Gamma_{m,\alpha,t}) = \dim(\Gamma'_{m',\alpha',x'}) = \rho(g, m + \alpha - g, \alpha) + \alpha(d + 2 - 2g) - \rho(g, m' + \alpha' - g, \alpha) + \alpha'(d + 2 - 2g)\). Since \(\dim(\Gamma_{m,\alpha,t}) > \dim(\Gamma'_{m',\alpha',x'})\) and \(m' \leq m\), we easily get \(\alpha' > \alpha\). Since \(d + 2 - 2g > g\), \(|\rho(g, m + \alpha - g, \alpha)| - \rho(g, m' + \alpha' - g, \alpha)| \leq g\), we get \(\dim(\Gamma_{m,\alpha,t}) < \dim(\Gamma'_{m',\alpha',x'})\), contradiction. Now we check that last assertion of (i) and (ii). \(L\) is very ample. Since \(M\) is general in \(G^{m+\alpha-g}(C)\), \(M\) is very ample if and only if \(m + \alpha - g \geq 3\). Since \(d + 2 - 2g \geq 5\), we get that if \(m + \alpha - g \geq 3\), then \(S \cong \mathbb{P}(L \oplus M)\) and hence \(S\) is smooth. Now assume \(m + \alpha - g \leq 2\) and take a general \(S \in \Gamma_{m,\alpha,t}\), say represented by a pair \((L \oplus M, V)\). \(M\) is spanned, but not very ample. The generality of \(S\) implies the generality of \(M\) in \(W^{m+\alpha-g}_m\) when \(\rho(g, g + \alpha - m, m) > 0\). Hence there are \(P, Q \subset C\), such that \(P \neq Q\) and \(h^0(C, M(\mathbb{P} - Q)) = h^0(C, M) - 1\). Hence \(h^0(C, (L \oplus M)(\mathbb{P} - Q)) = h^0(C, L \oplus M) - 3\). Let \(S_1 \subset \mathbb{P}^{d+1-g+\alpha}\) denote the image of \(\mathbb{P}(L \oplus M)\) obtained using \(H^0(C, L \oplus M)\). Since \(h^0(C, (L \oplus M)(\mathbb{P} - Q)) = h^0(C, L \oplus M) - 3\), the fibers \(D_P\) and \(D_Q\) of the ruling of \(\mathbb{P}(L \oplus M)\) over \(P\) and over \(Q\) are coplanar. Hence \(D_P \cap D_Q \neq \emptyset\). Hence \(S_1\) is not smooth. Hence a general projection of \(S_1\) in \(\mathbb{P}^{d+1-2g}\) is not smooth. Hence a general element of \(\Gamma_{m,\alpha,t}\) is not smooth, concluding the proof of part (ii). Part (i) follows from part (ii), the fact that \(G^x_y(C')\) and \(W^x_y(C')\) are irreducible and non-empty for all general \(C'\) and all \(x, y\) such that \(\rho(g, x, y) > 0\) and the irreducibility statement for \(W^x_y\) and \(G^x_y\) over a dense open subset of \(\mathcal{M}_g\) proved in [5] in the case \(\rho(g, x, y) = 0\).

References


Received: September, 2008