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The Convergence Analysis of the Auxiliary Problem Method for Monotone Affine Variational Inequalities

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Abstract

In this paper, we provide the convergence analysis of the auxiliary problem method for solving the monotone affine variational inequalities (M.C. Ferris and O.L. Mangasarian, *Annals of Operations Research*, 47, 1993, 293-305). The convergence rate was also given under suitable conditions.

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1 Introduction

Consider the monotone affine variational inequality (MAVI) problem of finding x^* in X such that

$$(x - x^*)^\top (Mx^* + q) \geq 0, \quad \forall\, x \in X = \{x \mid Ax \geq b, x \geq 0\} \tag{1.1}$$

where $M \in R^{n\times n}$ is positive semi-definite and $A \in R^{m\times n}$.

The MAVI is a special case of the classical variational inequalities (VI). It is well known that the VI plays an important role in economics, operation research and nonlinear analysis, etc, and has been received much attention of researchers[1]. In recent years, many methods have been proposed to solve the VI, among various of efficient methods for solving VI, projection method is the simplest one([3]-[8]). To decrease the projection times at each iteration, several double-projection methods for solving VI appeared by introducing one hyperplane to the projection region [5, 6, 7]. Recently, Solodov and Svaiter [4],

He [3], Wang[8] applied a new class of projection-contraction (PC) methods to monotone VI. Different from the algorithm above, Ferris and Mangasarian[2] proposed an auxiliary problem method for solving the problem (1.1), i.e., computing $y^{k+1} = y(y^k)$ by solving the quadratic programming defined by

$$\begin{array}{ll} \min & (y - y^k)^\top (My + q) + \frac{1}{2}\gamma \|y - y^k\|^2 \\ \text{s.t.} & y \in X \end{array} \tag{1.2}$$

where $\gamma > 0$ is a constant. Certainly, a strictly convex quadratic programming need to be solved at each iteration. However, the convergence analysis was not provided. In this paper, we show that the method is globally convergent. Furthermore, we show that this method is $R-$linear convergent under milder assumptions. Some numerical experiments are also reported, and indicate that this method has nice stability and high numerical efficiency.

Some notations used in this paper are in order. The norm $\|\cdot\|$ denote the Euclidean 2-norm, the transpose of a matrix M be denoted by $M^\top$. Without making confusion, we denote a nonnegative vector $x \in R^n$ by $x \geq 0$.

2 Algorithm and Convergence

In this section, we first present the auxiliary problem method for solving the monotone variational inequalities proposed by Ferris and Mangasarian in [2].

Algorithm 2.1

Step1. Take $\varepsilon > 0$, $\gamma > \frac{1}{2\tau} > 0$, and take initial point $y^0 \in X$. Set $k \triangleq 0$;

Step2. Compute $y^{k+1} = y(y^k)$ by solving the quadratic programming defined by (1.2).

Step3. If $\|y^{k+1} - y^k\| \leq \varepsilon$ stop, otherwise, go to Step 2 with $k \triangleq k + 1$.

Before give an analysis to this method, we first give some needed assumptions.

Assumption 2.1 Suppose that the matrix M is positive semi-definite in X, and satisfies the property

$$x^\top Mx = 0 \Rightarrow Mx = 0, \quad \forall x \in X. \tag{2.1}$$

Obviously, (2.1) holds if the matrix M is symmetric positive semi-definite in X. Based on Assumption 2.1, we have the following conclusion.

Lemma 2.1 *Suppose that Assumption 2.1 holds, then there exists a positive scalar τ such that for all $x \in X$,*

$$x^\top Mx \geq \tau \|Mx\|^2. \tag{2.2}$$

Proof. If Lemma 2.1 does not hold, then there exists x_0 in X such that for any $\epsilon > 0$

$$x_0^\top M x_0 < \varepsilon \|M x_0\|^2. \tag{2.3}$$

Let $\epsilon \to 0$, we get $x_0^\top M x_0 \leq 0$, by M is positive semi-definite matrix in X, we have $x_0^\top M x_0 = 0$. Combining this with assumption 2.1, we obtain $M x_0 = 0$ which contradicts (2.3), and the desired result follows. □

Lemma 2.2 *Suppose that Assumption 2.1 holds, and y^* is a solution of AMVI, then there exists a positive constant τ such that $\langle My + q,\ y - y^* \rangle \geq \tau \|M(y - y^*)\|^2, \quad \forall y \in X$.*

Proof. Under Assumption 2.1 and (2.2), there exists a positive constant τ such that

$$\langle (My + q) - (My^* + q),\ y - y^* \rangle = (y - y^*)^\top M (y - y^*) \geq \tau \|M(y - y^*)\|^2.$$

Since y^* is a solution of (1.1), so for any $y \in X$, we have $\langle My^* + q,\ y - y^* \rangle \geq 0$, and the desired result follows. □

In this following, we quote some known results from [2] on the error bound for (1.1) which is crucial to convergence of Algorithm.

Lemma 2.3 *Let M be a positive semidefinite matrix, and $\gamma > 0$ is a constant, then for any $x \in R^n$, there exists a solution x^* of (1.1) such that*

$$\|x - x^*\| \leq \mu(M\ A\ q\ b)(t(x) + t(x)^{\frac{1}{2}}) + \|\omega(x) - x\|,$$

where

$$\omega(x) \in \arg\min_{\omega \in X} \{(\omega - x)^\top (M\omega + b) + \frac{\gamma}{2} \|\omega - x\|^2\},$$

$$t(x) = \left\| \begin{pmatrix} [M + \gamma I]^\top (\omega(x) - x) \\ -\omega(x)^\top [M + \gamma I]^\top (\omega(x) - x) \end{pmatrix}_+ \right\|.$$

Remark $\mu(M\ A\ q\ b) > 0$ is a constant (dependent on $(M\ A\ q\ b)$).

Lemma 2.4 *Suppose that M is positive semi-definite in X, $x \in R^n$ is a solution of AMVI if and only if $y(x) = x$ is the unique solution of (1.2).*

Proof. Since the matrix $(M + \frac{\gamma}{2} I)$ is positive definite, and $x^* \in X^*$, so for any $y \in X$,

$$(y - x^*)^\top (My + q) + \frac{\gamma}{2} \|y - x^*)\|^2 = (y - x^*)^\top (M + \frac{\gamma}{2} I)(y - x^*) + (y - x^*)^\top [Mx^* + q) \geq 0.$$

Since $y(x^*) = x^*$ is feasible for (1.2) and the corresponding objective function value is zero, we conclude that $y(x^*)$ is the unique solution of (1.2).

Conversely, for $x \in R^n$, if $y(x) = x$ is the unique solution of (1.2), then $t(x) = 0$. By Lemma 2.3, $x \in X^*$. □

Obviously, if $y^{k+1} = y^k$, by Lemma 2.4, then y^k is a solution of (1.1). In the following theoretical analysis, we assume that Algorithm 3.1 generates an infinite sequence.

Theorem 2.1 *Suppose Assumption 2.1 holds, then the sequence $\{y^k\}$ generated by Algorithm 3.1 globally converges to a solution of MAVI.*

Proof. Since $\gamma > 0$, and M is positive semi-definite in X, so (1.2) has an unique solution, i.e., y^{k+1} is uniquely determined. Moreover, (1.2) can be equivalently reformulated as the following variational inequalities

$$\langle 2(M + \frac{1}{2}\gamma I)(y^{k+1} - y^k), y - y^{k+1}\rangle + \langle My^k + q, y - y^{k+1}\rangle \geq 0, \quad \forall y \in X. \quad (2.4)$$

Consider the function $\Delta(y)$ defined by $\Delta(y) = \Phi(y) + \Psi(y)$, where y^* is a solution of (1.1) and $\Phi(y) = (y - y^*)^\top(M + \frac{1}{2}\gamma I)(y - y^*)$, $\Psi(y) = \langle My^* + q, y - y^*\rangle$. Since M is positive semi-definite in X, one has

$$\Delta(y) \geq \Phi(y) \geq (\gamma/2)\|y - y^*\|^2 \geq 0. \quad (2.5)$$

For the sequence $\{\Delta(y^k)\}$, set $\Theta(k, k+1) = \Delta(y^k) - \Delta(y^{k+1})$, then a direct computation yields that

$$\begin{aligned}
\Theta(k, k+1) &= (y^k - y^*)^\top(M + \tfrac{1}{2}\gamma I)(y^k - y^*) + \langle My^* + q, y^k - y^*\rangle \\
&\quad -(y^{k+1} - y^*)^\top(M + \tfrac{1}{2}\gamma I)(y^{k+1} - y^*) - \langle My^* + q, y^{k+1} - y^*\rangle \\
&= (y^k)^\top(M + \tfrac{1}{2}\gamma I)y^k - (y^*)^\top(M + \tfrac{1}{2}\gamma I)y^* - 2\langle (M + \tfrac{1}{2}\gamma I)y^*, y^k - y^*\rangle \\
&\quad -(y^{k+1})^\top(M + \tfrac{1}{2}\gamma I)y^{k+1} + (y^*)^\top(M + \tfrac{1}{2}\gamma I)y^* \\
&\quad +2\langle (M + \tfrac{1}{2}\gamma I)y^*, y^{k+1} - y^*\rangle + \langle My^* + q, y^k - y^{k+1}\rangle \\
&= (y^k)^\top(M + \tfrac{1}{2}\gamma I)y^k - (y^{k+1})^\top(M + \tfrac{1}{2}\gamma I)y^{k+1} \\
&\quad +2\langle (M + \tfrac{1}{2}\gamma I)y^*, y^{k+1} - y^k\rangle + \langle My^* + q, y^k - y^{k+1}\rangle \\
&= (y^k)^\top(M + \tfrac{1}{2}\gamma I)y^k - (y^{k+1})^\top(M + \tfrac{1}{2}\gamma I)y^{k+1} \\
&\quad -2\langle (M + \tfrac{1}{2}\gamma I)y^{k+1}, y^k - y^{k+1}\rangle \\
&\quad +2\langle (M + \tfrac{1}{2}\gamma I)(y^{k+1} - y^*), y^k - y^{k+1}\rangle + \langle My^* + q, y^k - y^{k+1}\rangle \\
&= (y^k - y^{k+1})^\top(M + \tfrac{1}{2}\gamma I)(y^k - y^{k+1}) \\
&\quad +2\langle (M + \tfrac{1}{2}\gamma I)(y^{k+1} - y^k), y^* - y^{k+1}\rangle + \langle My^* + q, y^k - y^{k+1}\rangle.
\end{aligned}$$

Set $y = y^k$ in Lemma 2.2, then $\langle My^k + q, y^k - y^*\rangle \geq \tau\|M(y^k - y^*)\|^2$. Hence, if we let $y = y^*$ in (2.4), then

$$\begin{aligned}
&2\langle (M + \tfrac{1}{2}\gamma I)(y^{k+1} - y^k), y^* - y^{k+1}\rangle + \langle My^* + q, y^k - y^{k+1}\rangle \\
&\geq -\langle My^k + q, y^* - y^{k+1}\rangle + \langle My^* + q, y^k - y^{k+1}\rangle \\
&= \langle My^k + q, y^k - y^*\rangle - \langle My^k + q, y^k - y^{k+1}\rangle + \langle My^* + q, y^k - y^{k+1}\rangle \\
&\geq \tau\|M(y^k - y^*)\|^2 - \langle M(y^k - y^*), y^k - y^{k+1}\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Theta(k, k+1) &\geq \tfrac{1}{2}\gamma\|y^k - y^{k+1}\|^2 + \tau\|M(y^k - y^*)\|^2 - \langle M(y^k - y^*), y^k - y^{k+1}\rangle \\
&\geq \tfrac{1}{2}\gamma\|y^k - y^{k+1}\|^2 - \tfrac{1}{4\tau}\|y^k - y^{k+1}\|^2
\end{aligned}$$

where the last inequality is based on Cauchy-Schwarz inequality.

Since $\gamma > \frac{1}{2\tau}$, we have $\Theta(k, k+1) > 0$, and by (2.5), the nonnegative sequence $\{\Delta(y^k)\}$ is strictly decreasing, so it converges. hence $\Theta(k, k+1) \to 0$ as $k \to \infty$, and

$$\lim_{k\to\infty} \|y^k - y^{k+1}\| = 0. \tag{2.6}$$

Moreover, $\{\Delta(y^k)\}$ is bounded since it is convergent, and so is $\{y^k\}$ according to (2.5), i.e., there exists a constant $\eta_0 > 0$ such that $\|y^k\| \leq \eta_0$. Combining representation of $t(x)$ in Lemma 2.3, there exists a constant $\eta_1 > 0$ such that

$$t(y^k) \leq \|M + \gamma I\|(1 + \|y^{k+1}\|)\|y^{k+1} - y^k\| \leq \eta_1\|y^{k+1} - y^k\|. \tag{2.7}$$

Combining Lemma 2.3 and (2.6)-(2.7) again, we know that there exists a constant $\eta_2 > 0$ and a vector $\bar{y}(y^k) \in X^*$ such that

$$dist(y^k, X^*) = \|y^k - \bar{y}(y^k)\| \leq \eta_2\|y^{k+1} - y^k\|^{\frac{1}{2}} \to 0 \quad (k \to \infty).$$

Since $\{y^k\}$ is bounded, let $\{y^{k_i}\}$ be a subsequence of $\{y^k\}$ and converges to $\bar{y}$. By (2.4), for any $y \in X$, we have

$$\begin{aligned}
\langle My^{k_i} + q, y - y^{k_i+1}\rangle &\geq \langle -2(M + \tfrac{1}{2}\gamma I)(y^{k_i+1} - y^{k_i}), y - y^{k_i+1}\rangle \\
&\geq -\|2(M + \tfrac{1}{2}\gamma I\|\|y^{k_i+1} - y^{k_i}\|\|y - y^{k_i+1}\|,
\end{aligned}$$

and$\|y^{k_i+1} - \bar{y}\| \leq \|y^{k_i+1} - y^{k_i}\| + \|y^{k_i} - \bar{y}\|$. By (2.6), we know that $\bar{y}$ is a solution of (1.1). Substituting $\bar{y}$ by y^* in the function $\Delta(y)$, and denoting the corresponding function by $\bar{\Delta}(y)$, we have

$$(\gamma/2)\|y - \bar{y}\|^2 \leq \bar{\Delta}(y) \leq \|M + \frac{\gamma}{2}I\|\|y - \bar{y}\|^2 + \|M\bar{y} + q\|\|y - \bar{y}\|. \tag{2.8}$$

Since $\{\bar{\Delta}(y^k)\}$ converges, substituting y in (2.8) by y^{k_i} leads to that $\bar{\Delta}(y^{k_i}) \to 0 (i \to \infty)$. Thus, $\bar{\Delta}(y^k) \to 0$ as $k \to \infty$. By using (2.5) again, for any positive integers p, q such that $p \geq q$ and $\varepsilon_1 > 0$, we have $\|y^p - y^q\| \leq (\|y^p - \bar{y}\| + \|y^q - \bar{y}\|) < 2\varepsilon_1$. This means that the sequence $\{y^k\}$ is a Cauchy sequence, and hence it globally converges to a solution of AMVI. □

Theorem 2.2 *Suppose Assumption 2.1 holds. If* $\beta =: \|[2(M+\frac{1}{2}\gamma I)]^{-1}[M+\gamma I]\| < 1$*, then the sequence* $\{y^k\}$ *generated by Algorithm 3.1 converges to a solution of (1.1)* $R-$*linearly.*

Proof. For (1.2), by KKT condition, there exist $u_1^k \in R_+^s,\ u_2^k \in R_+^n$ such that

$$2(M+\frac{1}{2}\gamma I)(y^{k+1}-y^k)+My^k+M^\top q = A^\top u_1^k + u_2^k,$$

i.e.,

$$y^{k+1} = [2(M+\frac{1}{2}\gamma I)]^{-1}[M+\gamma I]y^k + c^k \tag{2.9}$$

where $c^k = [2(M+\frac{1}{2}\gamma I)]^{-1}[A^\top u_1^k + u_2^k - M^\top q]$. Since $\{y^k\}$ is bounded according to the proof of Theorem 3.1, from (2.9), we know that there exist constants $\eta > 0,\ \alpha > 0$ such that $\|c^k\| \le \eta$, and

$$\|[2(M+\frac{1}{2}\gamma I)]^{-1}[M+\gamma I]\bar{y} - \bar{y} + c^k\| \le \alpha. \tag{2.10}$$

By (2.9) and (2.10), we obtain

$$\begin{aligned}
\|y^{k+1}-\bar{y}\| &= \|[2(M+\tfrac{1}{2}\gamma I)]^{-1}[M+\rho\gamma I](y^k-\bar{y})\| \\
&\quad + \|[2(M+\tfrac{1}{2}\gamma I)]^{-1}[M+\gamma I]\bar{y} - \bar{y} + c^k\| \\
&\le \|[2(M+\tfrac{1}{2}\gamma I)]^{-1}[M+\gamma I]\|\|(y^k-\bar{y})\| + \alpha \\
&= \beta\|(y^k-\bar{y})\| + \alpha \le \beta^2\|(y^{k-1}-\bar{y})\| + \beta\alpha \\
&\cdots\cdots\cdots\cdots \\
&\le \beta^{k+1}\|(y^0-\bar{y})\| + \beta^k\alpha = \beta^{k+1}[\|(y^0-\bar{y}) + \tfrac{\alpha}{\beta}\|].
\end{aligned}$$

Since $\beta < 1$, we know that the sequence $\{y^k\}$ converges to a solution of (1.1) $R-$linearly. □

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