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Asymptotic Behavior of Solutions for BCF Model Describing Crystal Surface Growth

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Abstract. This paper continues a study on the initial-boundary value problem for a nonlinear parabolic equation of forth order which was presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [8] for describing the process of growth of a crystal surface under molecular beam epitaxy(MBE). In the previous paper [5], we have constructed a dynamical system determined from the model equation. This paper is then devoted to investigating asymptotic behavior of trajectories of the dynamical system by constructing exponential attractors and a Lyapunov function.

Mathematics Subject Classification: 35K30, 37L30, 37L45

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1. Introduction

We continue a study on the initial-boundary value problem for a nonlinear parabolic equation of fourth order

$$(1.1)\qquad \begin{cases} \frac{\partial u}{\partial t} = -a\Delta^2 u - \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) & \text{in}\quad \Omega\times(0,\infty), \\ \frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0 & \text{on}\quad \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x) & \text{in}\quad \Omega \end{cases}$$

in a two-dimensional bounded domain $\Omega \subset \mathbb{R}^2$. This model was presented by Johnson-Orme-Hunt-Graff-Sudijono-Sauder-Orr [8] for describing the process of growing of a crystal surface on the basis of the BCF theory due to Burton-Cabrera-Frank [2] (cf. also [7, 12, 13, 14, 18]). Here, $u = u(x,t)$ denotes a

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displacement of height of surface from the standard level ($u = 0$) at a position $x \in \Omega$. We assume that u and Δu satisfy the homogeneous Neumann boundary conditions on $\partial\Omega$.

The term $-a\Delta^2 u$ in the equation of (1.1) denotes a surface diffusion of adatoms which is caused by the difference of the chemical potential. It is known that the chemical potential is proportional to the curvature of the surface. Indeed, the adatoms on the surface have tendency to migrate from the positions of a large curvature to those of a small curvature. The macroscopic representation of the surface diffusion by the term $-a\Delta^2 u$ is due to the classical work of Mullins [10], where $a > 0$ is a surface diffusion constant. In the meantime, $-\mu\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ denotes the effect of surface roughening. Such roughening is caused by the Schwoebel barriers [4, 15] (cf. also [18]) which prevent the adatoms from hopping from the upper terraces to lower ones. As a consequence, non-equilibrium uphill current is induced. The macroscopic representation of the roughening by the term $-\mu\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right)$ is formulated in the paper [8] mentioned above, where $\mu > 0$ is a coefficient of surface roughening. Some numerical simulations for one or two-dimensional model of (1.1) were performed by the papers [7, 12, 13, 14].

We will handle (1.1) in the underlying space $L^2(\Omega)$. In the preceding paper [5], we have already constructed a global solution in $L^2(\Omega)$ for each initial function $u_0 \in H^1_m(\Omega)$, $H^1_m(\Omega)$ being a closed subspace of $H^1(\Omega)$ consisting of functions $u \in H^1(\Omega)$ with null mean, i.e., $|\Omega|^{-1}\int_\Omega u\,dx = 0$. And, by showing continuity of the global solutions with respect to the initial functions, we have constructed a dynamical system $(S(t), H^1_m(\Omega), L^2(\Omega))$ determined from (1.1) with the phase space $H^1_m(\Omega)$ in the universal space $L^2(\Omega)$. In this paper, we will proceed to investigate the structure of $(S(t), H^1_m(\Omega), L^2(\Omega))$. First, we shall construct exponential attractors. The notion of exponential attractor was presented by Eden et al. [3] as a new attractor set which is a positively invariant set of $S(t)$ with finite fractal dimension and attracts every trajectory at an exponential rate. The authors of the paper [3] presented also the squeezing property of semigroup $S(t)$ by which one can easily construct exponential attractors. We shall then show that our semigroup determined from (1.1) actually enjoy the squeezing property. Second, we shall present a Lyapunov function the value of which decreases monotonically along every trajectory of $(S(t), H^1_m(\Omega), L^2(\Omega))$. Finally, using this fact, we shall prove that the ω-limit set $\omega(u_0)$ of any initial value $u_0 \in H^1_m(\Omega)$ consists of equilibria of $S(t)$.

Throughout the paper, Ω is a bounded domain of $\mathcal{C}^4$ class in $\mathbb{R}^2$. According to [6], the Poisson problem $-\Delta u = f$ in Ω under the homogeneous Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ enjoys the shift property that $f \in H^2(\Omega)$ implies $u \in H^4(\Omega)$.

2. Dynamical system

This section is devoted to reviewing the results obtained in the previous paper [5] concerning construction of a dynamical system determined from (1.1).

We first introduce an abstract formulation of (1.1) as the Cauchy problem for an abstract evolution equation

$$\begin{cases} \frac{du}{dt} + Au = F(u), & 0 < t < \infty, \\ u(0) = u_0 \end{cases} \tag{2.1}$$

in $L^2(\Omega)$, see [11]. For the abstract evolution equations, we refer the reader to [9, 16]. Here, A is a positive definite self-adjoint operator of $L^2(\Omega)$ which is given by $A = (-\sqrt{a}\Delta + 1)^2$ with domain

$$\mathcal{D}(A) = H^4_{N^2}(\Omega) = \{u \in H^4(\Omega);\ \tfrac{\partial u}{\partial n} = \tfrac{\partial}{\partial n}\Delta u = 0 \text{ on } \partial\Omega\}.$$

For $0 \le \theta \le 1$, A^θ are fractional powers of A ($A^0 = 1$ and $A^1 = A$). The domains of $\mathcal{D}(A^\theta)$ can be characterized as [5, Proposition 3.1] (cf. also [19]).

The nonlinear operator $F : \mathcal{D}(A^{\frac{7}{8}}) \to L^2(\Omega)$ is given by

$$F(u) = -\mu\nabla \cdot \left(\frac{\nabla u}{1 + |\nabla u|^2}\right) - 2\sqrt{a}\Delta u + u, \qquad u \in \mathcal{D}(A^{\frac{7}{8}}) \subset H^{\frac{7}{2}}(\Omega).$$

According to [5, Proposition 3.2], F satisfies a Lipschitz condition of the form

$$\begin{aligned} \|F(u) - F(v)\| \le C[&\|A^{\frac{1}{2}}(u - v)\| \\ &+ (\|A^{\frac{7}{8}}u\| + \|A^{\frac{7}{8}}v\|)\|A^{\frac{1}{4}}(u - v)\|], \qquad u, v \in \mathcal{D}(A^{\frac{7}{8}}). \end{aligned} \tag{2.2}$$

The space of initial functions is set by

$$K = H^1_m(\Omega) = \left\{u_0 \in H^1(\Omega);\ \frac{1}{|\Omega|}\int_\Omega u_0\,dx = 0\right\}.$$

Obviously, K is a closed subspace of $H^1(\Omega)$.

We in fact showed by [5, Theorem 4.1] that, for each $u_0 \in K$, there exists a unique global solution to (2.1) in the function space:

$$u \in \mathcal{C}([0,\infty); H^1(\Omega)) \cap \mathcal{C}^1((0,\infty); L^2(\Omega)) \cap \mathcal{C}((0,\infty); H^4_{N^2}(\Omega)). \tag{2.3}$$

The condition $\int_\Omega u_0\,dx = 0$ implies $\int_\Omega u(t)\,dx = 0$ for every $0 < t < \infty$. Denote the global solution $u(\cdot)$ by $u(\cdot; u_0)$. For each time $t \ge 0$, a mapping $S(t) : K \to K$ is defined by $S(t)u_0 = u(t; u_0)$. By the uniqueness of solutions, $S(t)$ becomes a nonlinear semigroup acting on K, i.e., $S(t)S(s) = S(t + s)$.

Several favorable properties of $S(t)$ have already been proved in [5]. First, $S(t)$ enjoys a dispersing estimate

$$\|S(t)u_0)\|_{H^1} \le C[e^{-\rho t}\|u_0\|_{H^1} + 1], \qquad 0 \le t < \infty,\ u_0 \in K \tag{2.4}$$

with some exponent $\rho > 0$, see [5, (5.1)]. Second, $S(t)$ enjoys a smoothing property such that

$$\|S(t)u_0\|_{H^4} \le (1 + t^{-\frac{3}{4}})p(\|u_0\|_{H^1}), \qquad 0 < t < \infty,\ u_0 \in K \tag{2.5}$$

with some continuous increasing function $p(\cdot)$, see [5, (4.11)]. Finally, $S(t)$ satisfies the Lipschitz condition with respect to initial functions in the topology of $L^2(\Omega)$. More precisely, for any bounded subset B of K and for any bounded interval $[0, T]$, it holds that

$$\|S(t)u_0 - S(t)v_0\|_{L^2} \le C_{B,T}\|u_0 - v_0\|_{L^2}, \qquad u_0,\ v_0 \in B;\ 0 \le t \le T, \tag{2.6}$$

where $C_{B,T} > 0$ is some constant dependent on B and T, see [5, (4.13)].

Using these properties of $S(t)$, we were able to construct an absorbing and invariant set of $S(t)$ which is a compact subset of $L^2(\Omega)$ and is a bounded subset of $H^4(\Omega)$. That is, as shown in [5, Theorem 5.2], one can construct a subset $\mathcal{K}$ of K with the following properties:

1. $\mathcal{K}$ is a compact subset of $L^2(\Omega)$ and is a bounded subset of $\mathcal{D}(A)$;
2. $\mathcal{K}$ is an invariant set of $S(t)$, i.e., $S(t)\mathcal{K} \subset \mathcal{K}$ for every $t > 0$;
3. $\mathcal{K}$ is an absorbing set of $S(t)$, i.e., for any bounded set B of K, there exists a time $t_B > 0$ such that $S(t)B \subset \mathcal{K}$ for every $t \ge t_B$.

Since $\mathcal{K}$ is an invariant set of $S(t)$ and since $S(t)$ is continuous on $\mathcal{K}$ with respect to the L^2-topology (due to (2.6)), $(S(t), \mathcal{K}, L^2(\Omega))$ defines a dynamical system.

Problem (2.1) thus determines a dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$ in the universal space $L^2(\Omega)$ with the phase space $\mathcal{K}$. In particular, every trajectory $S(\cdot)u_0$ starting from $u_0 \in K$ enters in the phase space $\mathcal{K}$ in a finite time.

3. Exponential attractors

In this section, we shall construct exponential attractors for the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.

Let us first recall the definition of exponential attractor presented by Eden et al. [3]. Consider a dynamical system $(S(t), \mathcal{K}, X)$ in a universal space X (cf. [1, 17]), X being a Banach space. We assume that the phase space $\mathcal{K}$ is a compact subset of X and that the nonlinear semigroup $S(t)$ is continuous in the sense that a mapping $G(t, u) = S(t)u$ is continuous from $[0, \infty) \times \mathcal{K}$ into $\mathcal{K}$.

Let $\mathcal{A} = \bigcap_{0 \le t < \infty} S(t)\mathcal{K}$. Then, $\mathcal{A}$ is a nonempty compact set of X and is the global attractor of $(S(t), \mathcal{K}, X)$, namely, it holds that

$$\lim_{t \to \infty} h(S(t)\mathcal{K}, \mathcal{A}) = 0.$$

In what follows, $h(B_1, B_2)$ denotes the Hausdorff pseudo-distance

$$h(B_1, B_2) = \sup_{u \in B_1} \inf_{v \in B_2} \|u - v\|_X$$

for any two subsets B_1 and B_2 of $\mathcal{K}$.

A subset $\mathcal{M}$ of $\mathcal{K}$ is called an exponential attractor of $(S(t), \mathcal{K}, X)$ if $\mathcal{M}$ satisfies the following conditions:

1. $\mathcal{M}$ is a compact subset of X containing the global attractor $\mathcal{A}$ (i.e., $\mathcal{A} \subset \mathcal{M} \subset \mathcal{K}$) and has a finite fractal dimension $d_F(\mathcal{M}) < \infty$;
2. $\mathcal{M}$ is an invariant set of $S(t)$, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t > 0$;
3. $\mathcal{M}$ attracts $\mathcal{K}$ at an exponential rate

$$h(S(t)\mathcal{K}, \mathcal{M}) \le Ce^{-\delta t}, \qquad 0 \le t < \infty$$

with some exponent $\delta > 0$ and a constant $C > 0$.

Let X be a Hilbert space. In the paper [3], the authors presented also a sufficient condition for the semigroup $S(t)$ in order that $(S(t), \mathcal{K}, X)$ enjoys the exponential attractor. Assume that there exists time $0 < t^* < \infty$ which satisfies the following conditions:

1. There exist some exponent $0 \le \delta < \frac{1}{4}$ and an orthogonal projection P of finite rank N such that, for each pair u, v of vectors of $\mathcal{K}$, either

$$\|S^*u - S^*v\| \le \delta \|u - v\| \tag{3.1}$$

or

$$\|(I - P)(S^*u - S^*v)\| \le \|P(S^*u - S^*v)\| \tag{3.2}$$

holds, where $S^* = S(t^*)$;

2. The mapping $G(t, u) = S(t)u$ is Lipschitz continuous on $[0, t^*] \times \mathcal{K}$, i.e.,

$$\|G(t, u) - G(s, v)\| \le L\big(|t - s| + \|u - v\|\big), \qquad t,\ s \in [0, t^*];\ u,\ v \in \mathcal{K}. \tag{3.3}$$

Condition (3.1)-(3.2) is called the squeezing property of $S(t^*)$. According to [3, Theorem 3.1], the squeezing property (3.1)-(3.2) together with (3.3) in fact enables us to construct an exponential attractor $\mathcal{M}$ of $(S(t), \mathcal{K}, X)$ with fractal dimension

$$d_F(\mathcal{M}) \le N \max\left\{1, \frac{\log(\frac{2L}{\delta} + 1)}{\log(\frac{1}{4\delta})}\right\} + 1. \tag{3.4}$$

When a dynamical system $(S(t), \mathcal{K}, X)$ is determined from the Cauchy problem of an abstract evolution equation like (2.1), the authors of [3] showed also some convenient method for verifying the squeezing properties of $S(t)$. Consider (2.1) in a Hilbert space X in which the linear operator A is a positive definite self-adjoint operator of X. Let the problem determine a dynamical system $(S(t), \mathcal{K}, X)$ with some compact phase space $\mathcal{K}$. We assume that the nonlinear operator $F(u)$ satisfies a Lipschitz condition of the form

$$\|F(u) - F(v)\| \leq C\|A^{\frac{1}{2}}(u - v)\|, \qquad u, v \in \mathcal{K}. \tag{3.5}$$

Then, it is possible to conclude that, for any $0 < t^* < \infty$, the nonlinear operator $S(t^*)$ fulfils (3.1)-(3.2) with a suitable exponent $0 \leq \delta < \frac{1}{4}$ and a projection P of finite rank N. Indeed, see [3, Proposition 3.1].

In the second half of this section, let us apply the general method to our dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$ which was reviewed in the preceding section. To this end, it now suffices to verify that (3.3) and (3.5) are fulfilled.

Write

$$G(t, u) - G(s, v) = [S(t)u - S(s)u] + [S(s)u - S(s)v], \qquad u, v \in \mathcal{K}.$$

Since $\mathcal{K}$ is a bounded subset of $\mathcal{D}(A)$, it follows that

$$\begin{aligned}\|S(t)u - S(s)u\| &= \left\|\int_s^t \frac{dS(\tau)}{d\tau} u \, d\tau\right\| = \left\|\int_s^t [-AS(\tau)u + F(S(\tau)u)]d\tau\right\| \\ &\leq \sup_{w\in\mathcal{K}} \|-Aw + F(w)\| \, |t - s| \leq L_1|t - s|.\end{aligned}$$

In the meantime, let $0 < t^* < \infty$ be arbitrarily fixed. We already established that

$$\|S(s)u - S(s)v\| \leq L_2\|u - v\|, \qquad 0 \leq s \leq t^*; \ u, v \in \mathcal{K}$$

due to (2.6). Therefore, (3.3) is fulfilled. (3.5) has already been verified by (2.2).

We hence establish the following theorem.

Theorem 3.1. *The dynamical system* $(S(t), \mathcal{K}, L^2(\Omega))$ *enjoys an exponential attractor* $\mathcal{M}$ *with dimension given by* (3.4).

It is possible to substitute any Sobolev space $H^\theta(\Omega)$, where $0 < \theta < 4$, for the present universal space $L^2(\Omega)$. As an analogy of [5, Corollary 5.1], we can show the following result.

Corollary 3.1. *For each* $0 < \theta < 4$*, the exponential attractor* $\mathcal{M}$ *constructed above for* $(S(t), \mathcal{K}, L^2(\Omega))$ *is an exponential attractor of* $(S(t), \mathcal{K}, H^\theta(\Omega))$*, too.*

Proof. By the interpolation estimate, we have

$$\|u - v\|_{H^\theta} \leq C\|u - v\|_{H^4}^{\frac{\theta}{4}}\|u - v\|_{L^2}^{\frac{4-\theta}{4}} \leq C\delta^{H^4}(\mathcal{K})^{\frac{\theta}{4}}\|u - v\|_{L^2}^{\frac{4-\theta}{4}} \qquad u, v \in \mathcal{K}, \tag{3.6}$$

where $\delta^{H^4}(\mathcal{K})$ is the diameter of $\mathcal{K}$ in $H^4(\Omega)$. This means that, in $\mathcal{K}$, the L^2-topology is equivalent to the H^θ-topology.

Noting this fact, we can see in a direct way that $\mathcal{M}$ is an exponential attractor even in the topology of $H^\theta(\Omega)$. In particular, we observe that

$$d_F^{H^\theta}(\mathcal{M}) \leq 4(4-\theta)^{-1} d_F(\mathcal{M}),$$

here $d_F^{H^\theta}(\mathcal{M})$ denotes the fractal dimension of $\mathcal{M}$ measured in the space $H^\theta(\Omega)$. $\square$

4. Lyapunov function

In this section, we shall construct a Lyapunov function $\Psi(u)$ for the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.

Let $u_0 \in \mathcal{K}$ and let $S(t)u_0 = u(t; u_0) = u(t)$ be the global solution to (1.1) with initial function u_0. Multiply the equation of (1.1) by $\frac{\partial \overline{u}}{\partial t}$ and integrate the product in Ω. Then,

$$\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx = -a \int_\Omega \Delta^2 u \cdot \frac{\partial \overline{u}}{\partial t}\, dx - \mu \int_\Omega \left[\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right] \frac{\partial \overline{u}}{\partial t}\, dx.$$

Since $\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\Delta u = 0$ on $\partial\Omega$, we have

$$\int_\Omega \Delta^2 u \cdot \frac{\partial \overline{u}}{\partial t}\, dx = \int_\Omega \Delta u \cdot \frac{\partial}{\partial t}\Delta \overline{u}\, dx.$$

Furthermore, taking the real parts of both hand sides, we have

$$\mathrm{Re} \int_\Omega \Delta^2 u \cdot \frac{\partial \overline{u}}{\partial t}\, dx = \int_\Omega \frac{1}{2}\left(\frac{\partial}{\partial t}\Delta u \cdot \Delta \overline{u} + \Delta u \cdot \frac{\partial}{\partial t}\Delta \overline{u}\right) dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2 dx.$$

In the meantime, it is seen that

$$\int_\Omega \left[\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right] \frac{\partial \overline{u}}{\partial t}\, dx = -\int_\Omega \frac{\nabla u}{1+|\nabla u|^2} \cdot \nabla \frac{\partial \overline{u}}{\partial t}\, dx.$$

Therefore,

$$\begin{aligned}
\mathrm{Re} &\int_\Omega \left[\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right)\right] \frac{\partial \overline{u}}{\partial t}\, dx \\
&= -\int_\Omega \frac{1}{1+|\nabla u|^2}\, \frac{1}{2}\left(\nabla \frac{\partial u}{\partial t} \cdot \nabla \overline{u} + \nabla u \cdot \nabla \frac{\partial \overline{u}}{\partial t}\right) dx \\
&= -\frac{1}{2}\int_\Omega \frac{1}{1+|\nabla u|^2}\, \frac{\partial}{\partial t}|\nabla u|^2 dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega \log\left(1+|\nabla u|^2\right) dx.
\end{aligned}$$

Hence, we obtain that

(4.1)
$$\frac{d}{dt}\int_\Omega [a|\Delta u|^2 - \mu \log(1+|\nabla u|^2)]\,dx = -2\int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx \le 0, \qquad 0<t<\infty.$$

This indeed shows that the functional

$$\Psi(u) = \int_\Omega [a|\Delta u|^2 - \mu \log(1+|\nabla u|^2)]\,dx, \qquad u \in H^2(\Omega) \tag{4.2}$$

is a Lyapunov function for the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.

Theorem 4.1. *Along any trajectory $S(\cdot)u_0$, where $u_0 \in \mathcal{K}$, the function $\Psi(S(t)u_0)$ is monotonically decreasing and has a limit as $t \to \infty$. For $u_0 \in \mathcal{K}$ and $0 < t_0 < \infty$, $\overline{u} = S(t_0)u_0$ is an equilibrium if and only if $[\frac{d}{dt}\Psi(S(t)u_0)]_{|t=t_0} = 0$.*

Proof. By (4.1), we have $\frac{d}{dt}\Psi(S(t)u_0) \le 0$ for $0<t<\infty$. Therefore, $\Psi(S(t)u_0)$ is monotonically decreasing. Since $\log(1+|\nabla u|^2) \le |\nabla u|^2$, we have

$$\Psi(S(t)u_0) \ge -\mu \int_\Omega \log(1+|\nabla S(t)u_0|^2)\,dx \ge -\mu\|S(t)u_0\|_{H^1}^2 \ge -\mu \sup_{w\in\mathcal{K}} \|w\|_{H^1}^2,$$

that is, $\Psi(S(t)u_0)$ is bounded from below. Therefore, $\Psi(S(t)u_0)$ has a limit as $t \to \infty$.

Let $\overline{u} = S(t_0)u_0$. By (4.1), we have

$$\left[\tfrac{d}{dt}\Psi(S(t)u_0)\right]_{|t=t_0} = -2\left\|\left[\tfrac{d}{dt}S(t)u_0\right]_{|t=t_0}\right\|^2 = -2\|-A\overline{u} + F(\overline{u})\|^2.$$

Hence, the second assertion is verified. □

5. ω-limit sets

We shall investigate asymptotic behavior of the trajectory $S(\cdot)u_0$ for each $u_0 \in \mathcal{K}$. For $u_0 \in \mathcal{K}$, the ω-limit set $\omega(u_0)$ of $S(\cdot)u_0$ is defined by

$$\omega(u_0) = \bigcap_{t\ge 0} \overline{\{S(\tau)u_0;\ t \le \tau < \infty\}} \qquad \text{(closure in the topology of } L^2(\Omega)),$$

namely, $\overline{u} \in \omega(u_0)$ if and only if there exists a time sequence $\{t_n\}$ tending to ∞ such that $S(t_n)u_0 \to \overline{u}$ in $L^2(\Omega)$. Since

$$\overline{\{S(t)u_0;\ 0 \le t < \infty\}} \subset \mathcal{K}$$

and $\mathcal{K}$ is a compact set, $\omega(u_0)$ is nonempty set. Moreover, we easily verify that $\omega(u_0)$ is a strictly invariant set of $S(t)$, i.e.,

$$S(t)\big(\omega(u_0)\big) = \omega(u_0) \qquad \text{for every} \quad 0 < t < \infty. \tag{5.1}$$

We prove that the ω-limit set consists of equilibria.

Theorem 5.1. *For any $u_0 \in \mathcal{K}$, $\omega(u_0)$ consists of equilibria of the dynamical system $(S(t), \mathcal{K}, L^2(\Omega))$.*

Proof. Let $\overline{u} \in \omega(u_0)$ be any vector. There exists a sequence $t_n \to \infty$ such that $S(t_n)u_0 \to \overline{u}$ in $L^2(\Omega)$. Thanks to (3.6), we observe that $S(t_n)u_0$ is convergent to $\overline{u}$ in the topology of $H^2(\Omega)$, too. Therefore, by Theorem 4.1, it is concluded that

$$\Psi(\overline{u}) = \lim_{n\to\infty} \Psi(S(t_n)u_0)) = \inf_{0\le t<\infty} \Psi(S(t)u_0) \equiv \Psi_\infty(u_0).$$

This means that the value of the Lyapunov function is constant for vectors in $\omega(u_0)$.

Let $\overline{u} \in \omega(u_0)$. Since (5.1) implies $S(t)\overline{u} \in \omega(u_0)$ for every $t > 0$, it follows that $\Psi(S(t)\overline{u}) \equiv \Psi_\infty(u_0)$ for $t > 0$. As a consequence, $\frac{d}{dt}\Psi(S(t)\overline{u}) \equiv 0$. By Theorem 4.1 again, we conclude that $S(t)\overline{u}$ is an equilibrium for every $t > 0$. Hence, for every $s > 0$, $S(s)S(t)\overline{u} = S(t)\overline{u}$. Then, $S(t)S(s)\overline{u} = S(t+s)\overline{u} = S(s)S(t)\overline{u} = S(t)\overline{u}$. Letting $t \to 0$, we conclude that $S(s)\overline{u} = \overline{u}$. Thus, $\overline{u}$ is shown to be an equilibrium. □

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