A Note on Summability of Wavelet Packet Series

Abdullah

Zakir Husain College, University of Delhi
Jawahar Lal Nehru Marg, New Delhi-110 025, India
abd_khan11002@yahoo.com

Firdous Ahmad Shah

Department of Applied Mathematics
BGSB University, Rajouri, Jammu-185131, India
fashah_jmi@yahoo.co.in, fashah79@gmail.com

Abstract

Wavelet expansions have been the focus of many research papers in the last few years. The present paper deals with the study the strong convergence of wavelet packet expansions of periodic functions.

Mathematics Subject Classification: 40A30, 42C15

Keywords: Multiresolution analysis, Periodic wavelet packets, Convergence, Strong summability

1. Introduction

Wavelet packet functions provide a new class of orthogonal expansions in $L^2(\mathbb{R})$ and comprise a rich family of building block functions, localized in time and offer more flexibility than wavelets in representing different types of signals. It is a simple, but very powerful extension of wavelets and multiresolution analysis (MRA). The power of wavelet packet lies in the fact that we have much more freedom in deciding which basis function is to be used to represent the given function.

The problem of convergence of the wavelet series has been studied by Meyer [7], Walter [9, 10], and Kelly et al. [5, 6]. Meyer was amongst the first to study convergence results for wavelet expansions. He has shown that the regular wavelet expansions converge in $L^p$, $1 \leq p < \infty$ and also in $L^\infty$. 
for expansions of uniformly continuous functions, the expansion of continuous functions converge everywhere. The results in [7] were based on the assumption of so called regularity for the basic wavelets and their derivatives. In addition, Walter [9, 10] established pointwise convergence results for regular wavelet expansions of continuous functions. Kelly et al. [5, 6] have extended and obtained results analogous to those obtained by Carleson [2] and Hunt [4] for the Fourier series. In contrast, the results in [5, 6] assumed only that the wavelets being used be bounded by radial decreasing $L^1$-functions. In [6], it is shown that the wavelet expansions of a function belonging to $L^p$ converges pointwise everywhere on the Lebesgue set of a given function, for $1 \leq p < \infty$.

Recently, Ahmad and Kumar [1] have generalized the results of Kelly et al. [5, 6], Walter [9, 10] and Hernández and Weiss [3] for wavelet packet setting and have shown that such expansions of $L^p(\mathbb{R})$ functions ($1 \leq p \leq \infty$) converges pointwise almost everywhere on the Lebesgue set of the functions being expanded where as Shah and Ahmad [8] have obtained analogous results outside the Lebesgue set of a function being expanded without assuming the radial decreasing conditions.

Motivated and inspired by the importance of wavelet packets, in the present paper, we study the strong convergence of wavelet packet expansions of periodic functions.

2. Preliminaries

For basic ideas, results on wavelets, wavelet packets and multiresolution analysis, we refer to [3, 7, 11].

We construct wavelet packets from multiresolution analysis. In general, consider two sequences $\{\alpha_n\}_{n \in \mathbb{Z}}$ and $\{\beta_n\}_{n \in \mathbb{Z}}$ in $\ell^2(\mathbb{Z})$. Let $\mathbb{H}$ be a Hilbert space with orthonormal basis $\{e_k\}_{k \in \mathbb{Z}}$. Then, the sequences

$$f_{2n} = \sqrt{2} \sum_{k \in \mathbb{Z}} \alpha_{2n-k} e_k, \quad f_{2n+1} = \sqrt{2} \sum_{k \in \mathbb{Z}} \beta_{2n-k} e_k$$

are orthonormal bases of two orthogonal closed subspaces $\mathbb{H}_1$ and $\mathbb{H}_0$, respectively, such that

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_0.$$

Using this “splitting trick” we now define the basic wavelet packets associated with the scaling function $\varphi$ as defined in MRA.

Let $\omega_0 = \varphi$. The basic wavelet packets $\omega_n$, $n = 0, 1, 2, \ldots$, associated with
the scaling function $\varphi$ are defined recursively by

$$
\begin{align*}
\omega_{2n}(x) &= \sqrt{2} \sum_{k \in \mathbb{Z}} h_k \omega_n(2x - k) \\
\omega_{2n+1}(x) &= \sqrt{2} \sum_{k \in \mathbb{Z}} g_k \omega_n(2x - k).
\end{align*}
$$

(2.1)

It follows from the above definition that $\omega_1 = \psi$ is a mother wavelet and the set
\[
\{\omega_n(x - k) : n = 0, 1, ..., k \in \mathbb{Z}\}
\]
is an orthonormal basis for the Hilbert space $L^2(\mathbb{R})$.

Corresponding to some orthogonal scaling function $\varphi = \omega_0$, the family of
wavelet packets $\{\omega_n\}$ defines a family of subspaces of $L^2(\mathbb{R})$ as follows:

$$
U^n_j = \text{span}\{2^j \omega_n(2^j x - k) : k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \ldots
$$

(2.2)

Observe that

$$
U^0_j = V_j \quad \text{and} \quad U^1_j = W_j
$$

so that the orthogonal decomposition can be written as

$$
U^0_{j+1} = U^0_j \oplus U^1_j.
$$

(2.3)

A generalization of this result for other values of $n = 1, 2, 3, \ldots$ can be
written as

$$
U^n_{j+1} = U^{2n}_j \oplus U^{2n+1}_j, \quad j \in \mathbb{Z}.
$$

(2.4)

Now, we state a lemma which will be used in the proof of the preceding results.

**Lemma 2.1.** For each $j = 1, 2, \ldots$, decomposition trick (2.4) gives
\begin{align}
W_j &= U_j^1 = U_{j-1}^2 \oplus U_{j-1}^3 \\
&= U_{j-2}^4 \oplus U_{j-2}^5 \oplus U_{j-2}^6 \oplus U_{j-2}^7 \\
&\vdots \\
&= U_{j-k}^{2k} \oplus U_{j-k}^{2k+1} \oplus \ldots \oplus U_{j-k}^{2k+1-1} \\
&\vdots \\
&= U_0^{2^j} \oplus U_0^{2^j+1} \oplus \ldots \oplus U_0^{2^j+1-1},
\end{align}

(2.5)

where \( U_j^n \) is defined in (2.2). Moreover, for each \( j = 1, 2, \ldots ; k = 1, 2, \ldots , j \) and \( m = 0, 1, 2, \ldots \) \( , 2^k - 1 \), and the set \( \{ 2^{j-k}\omega_p(2^{j-k}x - \ell) : \ell \in \mathbb{Z} \} \) is an orthonormal basis of \( U_{j-k}^p \) where \( p = 2^k + m \). However, all the elements of this basis have the general form

\( \omega_{j,n,k}(x) = 2^{j/2}\omega_n(2^jx - k). \)

(2.6)

If a function \( f \in L^2(\mathbb{R}) \), then

\( f(x) \sim \sum_{j \in \mathbb{Z}} \sum_{n = 2^p}^{2^{p+1}-1} \sum_{k \in \mathbb{Z}} C_{\ell,n,k} \omega_{\ell,n,k}(x) \)

(2.7)

where \( \ell = j - p, p = 0 \) if \( j < 0 \) and \( p = 0, 1, 2, \ldots , j \) if \( j \geq 0 \); will be a wavelet packet expansion of \( f \) and \( C_{\ell,n,k} \) the wavelet packet coefficients defined as

\( C_{\ell,n,k} = \langle f, \omega_{\ell,n,k} \rangle. \)

(2.8)

Wavelet packet have been introduced as a flexible method for time-frequency analysis of signals combining the advantages of windowed Fourier and wavelet analysis. Similarly periodic wavelet packets provide an interesting alternative to Fourier series.

By periodizing the basis functions to period 1 an MRA for \( L^2(\mathbb{R}) \) is transformed into an MRA for \( L^2(0, 1) \) (see [11]). Let \( \{ \omega_n : n \in \mathbb{Z} \} \) be the family of nonstationary wavelet packets discussed above. For \( n, j \in \mathbb{N} \) and \( 0 \leq k < 2^j \) define general periodic wavelet packets \( \omega_{n,j,k}^{\text{per}} \) by

\( \omega_{n,j,k}^{\text{per}}(x) = \sum_{\ell \in \mathbb{Z}} 2^{j/2}\omega_n(2^j(x + \ell) - k). \)
We now define an operator \( S_m f \) associated with periodic wavelet packets \( \omega_{n}^{\text{per}} \) as

\[
S_m f = \sum_{n=2^p}^{2^{p+1}-1} \sum_{k=0}^{m} \langle f, \omega_{\ell,n,k}^{\text{per}} \rangle \omega_{\ell,n,k}^{\text{per}},
\]

(2.9)

3. Main Results

Let \( D_m \) \((m \in \mathbb{N})\) denotes the set of dyadic step functions that are constants on the intervals \([k2^{-m},(k+1)2^{-m})\) \((0 < k \leq 2^m)\) and \( D = \bigcup_{m=1}^{\infty} D_m \). Any function \( g \in D \) generates a bounded linear functional \( \sigma_g \) on a Banach space \( B \) by

\[
\sigma_g f = \int_0^1 f g \quad \text{for } f \in B.
\]

We have

\[
|\sigma_g f| \leq \|g\|_{\infty} \|f\|_1 \leq \|g\|_{\infty} \|f\|_{B}.
\]

Now if we take \( B = L^q \) and define

(3.1) \[
\|g\|_p = \|\sigma_g\| = \sup_{\|f\|_q \leq 1} \int_0^1 f g \quad \text{for any } g \in D,
\]

then clearly

(3.2) \[
\left| \int_0^1 f g \right| \leq \|f\|_p \|g\|_p, \quad f \in L^q, \; g \in D.
\]

Let us write

\[
P_{j} f(x) = \sum_{m=0}^{2^{j-1}} S_m f(x) \chi_{[m2^{-j},(m+1)2^{-j})},
\]

and

\[
A_{j} = \sum_{m=0}^{2^{j-1}} C^{\text{per}}_{\ell,n,k} \chi_{[m2^{-j},(m+1)2^{-j})},
\]

where \( S_m f \) is defined by (2.9) and \( \chi_A \) is the characteristic function of \( A \subset \mathbb{R} \).

Now, we define an operator

\[
T_{j}(x,y) = 2^{-j} \sum_{k=0}^{2^{j-1}} C^{\text{per}}_{\ell,0,k} \varphi_{j,k}^{\text{per}}(x) \varphi_{j,k}^{\text{per}}(y)
\]
\[= 2^{-j} \sum_{n=2^p}^{2^p+1-1} \sum_{m<j} \sum_{k=0}^{2^j-1} \omega_{\ell,n,k}^{\text{per}}(x) \omega_{\ell,n,k}^{\text{per}}(y),\]

where \(\ell = m - p, p = 0\) if \(m < 0\) and \(p = 0, 1, 2, \ldots, m\) if \(0 \leq m < j\).

**Theorem 3.1.** Let \(f\) be a continuous periodic function with period 1. Then

\[
\left\| \left( 2^{-j} \sum_{m=0}^{2^j-1} |S_m f|^p \right) \right\|_p \leq C_q \|f\|_\infty,
\]

if and only if

\[
\|T\|_1 \leq C_q \|A\|_q.
\]

Furthermore,

\[
\lim_{j \to \infty} \|P_j f(x) - f(x)\|_p = 0 \quad \text{uniformly in } x \in [0, 1].
\]

**Proof.** By (3.1) we have

\[
\left( 2^{-j} \sum_{m=0}^{2^j-1} |S_m f|^p \right)^{\frac{1}{p}} = \|P_j f(x)\|_p = \sup_{\|A\|_p \leq 1} \int_0^1 P_j f(x) A_j
\]

\[
= \sup_{\|A\|_p \leq 1} 2^{-j} \sum_{m=0}^{2^j-1} C_{\ell,n,k}^{\text{per}} S_m f(x)
\]

\[
= \sup_{\|A\|_p \leq 1} \int_0^1 2^{-j} \sum_{m=0}^{2^j-1} C_{\ell,n,k}^{\text{per}} K(x, y) f(y) dy
\]

\[
\leq \|f\| \sup_{\|A\|_p \leq 1} \|T(x, y)\|_1
\]

\[
\leq C_q \|f\|_\infty \quad \text{(by (3.4))}
\]

where

\[
K(x, y) = \sum_{k=0}^{2^j-1} \varphi_{j,k}^{\text{per}}(x) \overline{\varphi_{j,k}^{\text{per}}(y)}
\]
\[
2^{p+1} - 1 = \sum_{n=2^p}^{2^j-1} \sum_{m<j} \sum_{k=0}^{2^j-1} \omega_{\ell,n,k}(x) \omega_{\ell,n,k}(y),
\]

where \(\ell = m - p\), \(p = 0\) if \(m < 0\) and \(p = 0, 1, 2, ..., m\) if \(0 \leq m < j\).

Conversely suppose that Equation (3.3) holds, then we have

\[
\|T_j(x,y)\|_1 = \sup_{\|f\|_\infty \leq 1} \int_0^1 2^{-j} \sum_{m=0}^{2^j-1} C_{\ell,n,k}^\text{per} T_m(0,y)f(y) \, dy
\]

\[
= \sup_{\|f\|_\infty \leq 1} 2^{-j} \sum_{m=0}^{2^j-1} C_{\ell,n,k}^\text{per} S_m f(0)
\]

\[
= \sup_{\|f\|_\infty \leq 1} \int_0^1 P_j f(0) A_j 
\]

\[
\leq \sup_{\|f\|_\infty \leq 1} \|A_j\|_q \|P_j f(0)\|_p
\]

\[
\leq \|A_j\|_q \sup_{\|f\|_\infty \leq 1} \left\| \left( 2^{-j} \sum_{m=0}^{2^j-1} |S_m f(0)|^p \right)^{\frac{1}{p}} \right\|_\infty
\]

\[
\leq C_q \|A_j\|_q \quad \text{(by (3.3))}
\]

Now

\[
P_\ell(x) - f(x) = \sum_{m=0}^N (S_m f(x) - f(x)) \chi_{[m2^{-\ell},(m+1)2^{-\ell})}
\]

for any \(\ell \geq N \geq 2^j\). Therefore,

\[
\|P_\ell(x) - f(x)\| \leq \sum_{m=0}^N \|S_m f - f\|_\infty \|\chi_{[m2^{-\ell},(m+1)2^{-\ell})}\|_p
\]

and hence the final result is followed, since the limit in all \(L^p\)-spaces \((p < \infty)\) of the characteristic function of \([0, 2^{-j})\) is 0.

References


Received: February 8, 2008