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Blocks in Graphs of a Class of Claw-Free and 3-Colorable Graphs

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Abstract

We consider the class $\mathcal{G}_m$ of 3-colorable graphs containing neither claw nor hole of length more than m, where m is an integer ≥ 5. We give a complete description by a few basic graphs of the blocks containing a 5-hole in graphs of $\mathcal{G}_5$.

Mathematics Subject Classification: 05C75

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1 Introduction

The graphs considered in this paper are undirected, finite and simple. Let $G = (V, E)$ be a graph, where V is the vertex-set and E is the edge-set. For $X \subseteq V$, the subgraph of G *induced by* X is the subgraph with vertex-set X and edge-set all edges of G with both ends in X. The graph obtained from G by deleting X is denoted by $G \backslash X$. In the graph G, for every subset X of vertices, the *neighbourhood* $N(X)$ of X is the subset of vertices of $G \backslash X$ that have at least one neighbour in X. An edge x_1x_2 is called *independent* from an other edge $x'_1x'_2$ if $x_i \neq x'_j$ and x_i is no adjacent to x'_j for every $i = 1, 2$ and every $j = 1, 2$. A *path* is a subgraph of G described by a sequence $x_1x_2...x_k$ of

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distinct vertices of G and $x_i x_{i+1} \in E$ for every i, $1 \leq i \leq k-1$, vertices x_1 and x_k will be called the *endpoints* of the path. A *cycle* of G is a path $x_1 x_2 ... x_k$ with $x_k = x_1$; the *length* of a path or a cycle is the number of its vertices. A cycle with length three is a *triangle*. A *chord* of a path or a cycle $x_1 x_2 ... x_k$ is an edge $x_i x_j$ with $j \neq i \pm 1$. A *hole* in G is a chordless cycle with at least five vertices. We will frequently say $k-$hole instead of "hole of length k". Two vertices x and y in a chordless path (resp. hole) are *consecutive* if xy is an edge in the path (resp. the hole). A *clique* X in G is a subgraph of G such that every two vertices of X are adjacent; a clique with n vertices is denoted by K_n. A *claw* is a graph with four vertices a, b, c and d and three edges ab, ac and ad. A vertex is called *simplicial* if its neighbourhood is a clique.

A graph is said to be *F-free* if it does not contain an induced subgraph isomorphic to a given graph F.

The *line-graph* of G is the graph whose vertices are the edges of G and whose edges are the pairs of incident edges of G. A *block* in G is a subgraph of G that is 2-connected and is maximal with that property. It is well-known that the incidence graph of blocks and cut-vertices of a graph is a tree.

A $k-$*coloring* of the vertices of G is a mapping $c : V \rightarrow \{1, 2, \ldots, k\}$ for which every edge xy of G has $c(x) \neq c(y)$. The graph G is called $k-$*colorable* if it admits a $k-$coloring.

The treatment of the coloration problem by list which is a generalization of the classic coloration (see Häggkvist and Chetwynd [4]), of vertices or edges of a graph, would be less difficult if its structure is known. The class of claw-free graphs is natural to study in particular because it contains all line-graphs, studied by several authors (see Maffray and Reed [5]). Chvátal and Sbihi [3] discovered a decomposition of claw-free and 3-colorable graphs which are perfect; Gravier and Maffray [2] show that they are 3-list-colorable. We are interested in claw-free and 3-colorable graphs which are not necessarily perfect. It is always interesting to characterize this class of graphs; a complete description of graphs contributes in particular to the treatment of one of problems known to be difficult as the vertices list coloring. The structure of blocks of a graph can be useful to determine the structure of the associated incidence graph which will alow its coloration by list.

Let $m \geq 5$ be an integer and consider the class of graphs $\mathcal{G}_m$: $G \in \mathcal{G}_m$ if and only if G is claw-free, 3-colorable and contains no hole of length more than m.

In section 2, we give some general properties of claw-free graphs which are 3-colorable; section 3 describes the blocks of a given graph of $\mathcal{G}_5$ by a few basic graphs.

2 General properties

Let H_1 be a subgraph of a graph G; the *neighbourhood* of H_1 is a nonempty subgraph H_2 of G and disjoint from H_1 such that every vertex of H_2 has at least one neighbour in H_1; the graph $G' = H_2 \blacktriangleright H_1$ means that G' is the subgraph of G generated by the union of vertex-set of H_1 and of its neighbourhood H_2.

For a graph $G = (V, E)$ of $\mathcal{G}_m$ we will give some properties; the proofs being immediate, will be omitted. For every vertex u of G, $d(u)$ is the degree of u in G.

(p1) *For every vertex v of G, we have $d(v) \leq 4$. Consequently, $\Delta(G) \leq 4$ (the maximum degree in G).*

(p2) *Let C be a hole in G.*

- **(i)** *Every vertex v of $G \backslash C$ which is adjacent to a vertex of C is adjacent to at least two consecutive vertices of C; in particular, when $|C| = 5$, the set of vertices $N(v) \cap C$ induces a path.*
- **(ii)** *There is no distinct vertices of $G \backslash C$ with common neighbours on C.*
- **(iii)** *For every distinct vertices v and v' of $G \backslash C$, such that $N(v) \cap C \subset N(v') \cap C$, the two respective paths induced by $N(v)$ and $N(v')$, on C, have no common endpoints.*

(p3) *If $G = H \blacktriangleright C$ and C is a hole, then $|H| \leq |C|$, and consequently $|G| \leq 2|C|$.*

For any integer $n \geq 4$, a *stripe* T is a graph in which the vertex-set is a disjoint-union of two sets $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_{n-1}\}$ and whose edges are $x_i x_{i+1}$, $y_i y_{i+1}$, $x_i y_i$, $y_i x_{i+1}$, for $i = 1, 2, \ldots, n-2$ and $x_{n-1} x_n$, $x_{n-1} y_{n-1}$, $y_{n-1} x_n$; such a stripe will be denoted by $T = x_1 y_1 x_2 y_2 \ldots x_{n-1} y_{n-1} x_n$. Because a stripe T contains neither a claw nor a hole and its chromatic number is three, $T \in \mathcal{G}_m$.

The two following results due to Abbas and Saoula [1] summarize the neighbourhood structure of holes in graphs of $\mathcal{G}_m$.

Lemma 2.1 [*1*] *Let G be a graph of $\mathcal{G}_m$. Assume that P is a chordless path of length l and C is a $k-$hole. If $G = P \blacktriangleright C$ with $3 \leq l \leq k-2$, then C contains a chordless path P' with $l+1$ vertices such that the subgraph of G induced by $P \cup P'$ is a stripe.*

Theorem 2.2 [*1*] *Let G be a graph of $\mathcal{G}_m$ and C be a $k-$hole with $G = H \blacktriangleright C$. Let B be a connected component of H. We have:*

(1) If $k \equiv 1$ or $2 \mod 3$, then B is a triangle or a chordless path with length at most $k-1$.

(2) If $k \equiv 0 \mod 3$, then B is either a triangle, or a $k-$hole, or a chordless path with length at most k.

3 The class $\mathcal{G}_5$

Throughout this section G is a graph of $\mathcal{G}_5$ with $G = H \blacktriangleright C$, where $C = v_1v_2 \dots v_5v_1$ is a $5-$hole and the vertex-set of H is $\{w_1, \dots, w_l\}$. From property (p3) and Theorem 2.2, $l \leq 5$ and $\lambda(H) \leq 4$, where $\lambda(H)$ means the length of a largest chordless path in H.

3.1 Preliminary properties

The particular graphs F_1 and F_2 shown in Figure 1 will be denoted by $F_1 = w_1w_2 - v_1v_2v_3v_4$ and $F_2 = w_1'w_2' - v_1'v_2'v_3'$. The edge w_1w_2 in F_1 (resp. $w_1'w_2'$ in F_2) will be called *superedge* of F_1 (resp. F_2).

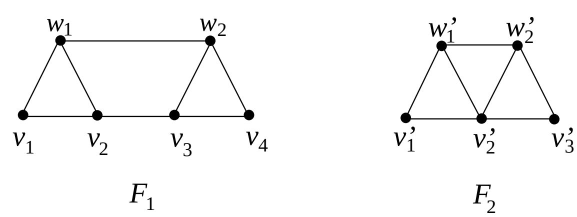

Figure 1: Particular graphs

Remark 3.1 *Let ww' be an edge of H.*

1. ww' is a superedge of F_1 or F_2 (because $|C| = 5$).

2. If ww' is a superedge of F_1, each of the two vertices w and w' has exactly two consecutive neighbours on C (because otherwise, G would not be $3-$colorable).

3. If ww' is a superedge of F_2, only one vertex w or w' has three consecutive neighbours on C (because otherwise, G would not be $3-$colorable or would contain a $6-$hole).

Lemma 3.2 *For $G = H \blacktriangleright C$ $(G \in \mathcal{G}_5)$, we have:*

(1) $\lambda(H) \leq 2$.

(2) H contains at most one edge.

Proof. (1) We only need to show that H contains no chordless path with length three. Suppose that the conclusion is false and let $P = w_1w_2w_3$ be such a path. By Lemma 2.1, there is a path P', contained in C, with four vertices such that $P \cup P'$ induces a stripe T in G; without loss of generality, we can consider $T = v_1w_1v_2w_2v_3w_3v_4$. Since G is 3-colorable, w_1v_5 or w_3v_5 is not an edge of G; thus $v_1w_1w_2v_3v_4v_5v_1$ or $v_1v_2w_2w_3v_4v_5v_1$ is a $6-$hole in G, a contradiction. Hence $\lambda(H) \leq 2$.

(2) Suppose that H contains two edges $e_1 = w_1w_2$ and $e_2 = w_3w_4$.
By (1), e_1 and e_2 are either independent or belong to a triangle. By 1. of Remark 3.1, each of the edges e_1 and e_2 is a superedge of F_i, $i = 1$ or 2, ($e_j \in F_i$, $j = 1$ or 2).

Case (a): e_1 and e_2 are independent:

Subcase a.1: $e_1 \in F_1$ and $e_2 \in F_1'$ (where F_1' is an other copy of F_1): assume that $F_1 = w_1w_2 - v_1v_2v_3v_4$ and $F_1' = w_3w_4 - v_1'v_2'v_3'v_4'$ ($\{v_1', v_2', v_3', v_4'\} \subset C$). Since w_3 and w_4 play the same role in F_1', we only consider w_3. The vertex w_3 may not have the same neighbours, on C, as w_1 or w_2 (2. of Remark 3.1 and (ii) of (p2)); so $N(w_3) \cap C$ is either $\{v_2, v_3\}$, or $\{v_4, v_5\}$, or $\{v_1, v_5\}$; since the length of C is five, v_2 and v_3 are neighbours of either w_3 or w_4, say $N(w_3) \cap C = \{v_2, v_3\}$; consequently $N(w_4) \cap C$ is $\{v_4, v_5\}$ or $\{v_1, v_5\}$, hence $v_1w_1w_2v_3w_3w_4v_5v_1$ or $v_2w_1w_2v_4v_5w_4w_3v_2$ is a $7-$hole in G, a contradiction.

Subcase a.2: $e_1 \in F_1$ and $e_2 \in F_2$: Let F_1 as in a.1 and $F_2 = w_3w_4 - v_1'v_2'v_3'$ ($\{v_1', v_2', v_3'\} \subset C$). Since the path $v_1'v_2'v_3'$ is without chord and $\Delta(G) \leq 4$, we have $v_2' = v_5$; by symmetry we can suppose that w_3 (resp. w_4) is adjacent to v_4 (resp. v_1); as w_1v_3 and w_2v_2 are not edges of G and $\Delta(G) \leq 4$, neither w_3 nor w_4 is adjacent to v_2 or v_3, so the cycle $v_1v_2v_3v_4w_3w_4v_1$ is a $6-$hole in G, a contradiction.

Subcase a.3: $e_1 \in F_2$ and $e_2 \in F_2'$: We consider $F_2 = w_1w_2 - v_1v_2v_3$ and $F_2' = w_3w_4 - v_1'v_2'v_3'$ ($\{v_1', v_2', v_3'\} \subset C$), where F_2' is an other copy of F_2. Since v_2 (resp.v_2') is of degree four in F_2 (resp. F_2') and $|C| = 5$, we have $v_2' \in \{v_4, v_5\}$; by 3. of Remark 3.1, we can suppose that w_2v_4 is an edge of G and w_1v_5 is not an edge of G; in this case, $v_2' = v_5$; consequently, v_1' and v_3' are in $\{v_1, v_4\}$, say $v_1' = v_4$ and $v_3' = v_1$; so $v_1w_1w_2v_4w_3w_4v_1$ is a $6-$hole in G, a contradiction.

Case (b): e_1 and e_2 are edges of a triangle $T' = w_1w_2w_3w_1$.

As every vertex w of T' is of degree two in T', w has precisely two consecutive neighbours on C. By 1. of Remark 3.1, there are two subcases:

Subcase b.1. $e_1 \in F_1$ with $F_1 = w_1w_2 - v_1v_2v_3v_4$. Since G is K_4-free, $N(w_3) \cap C$ is either $\{v_2, v_3\}$, $\{v_1, v_5\}$ or $\{v_4, v_5\}$ (the last two possibilities produce the same situation); each of the first two possibilities gives a 6−hole in G: induced by the vertices of $C \cup T' \setminus \{v_2, w_2\}$ or $C \cup T' \setminus \{v_1, w_2\}$, a contradiction.

Subcase b.2. $e_1 \in F_2$ with $F_2 = w_1w_2 - v_1v_2v_3$. As G is K_4-free, $N(w_3) \cap C$ is either $\{v_3, v_4\}$, $\{v_4, v_5\}$ or $\{v_1, v_5\}$ (the first and the last cases are identical), the first two possibilities produce a 6−hole in G: induced by $C \cup T' \setminus \{v_3, w_1\}$ or $C \cup T' \setminus \{v_2, w_3\}$, a contradiction. ■

3.2 Basic graphs of $\mathcal{G}_5$

Let B_i, for $i = 1, \ldots, 5$, be the basic graphs depicted in Figure 2; it is easy to verify that these graphs are in $\mathcal{G}_5$.

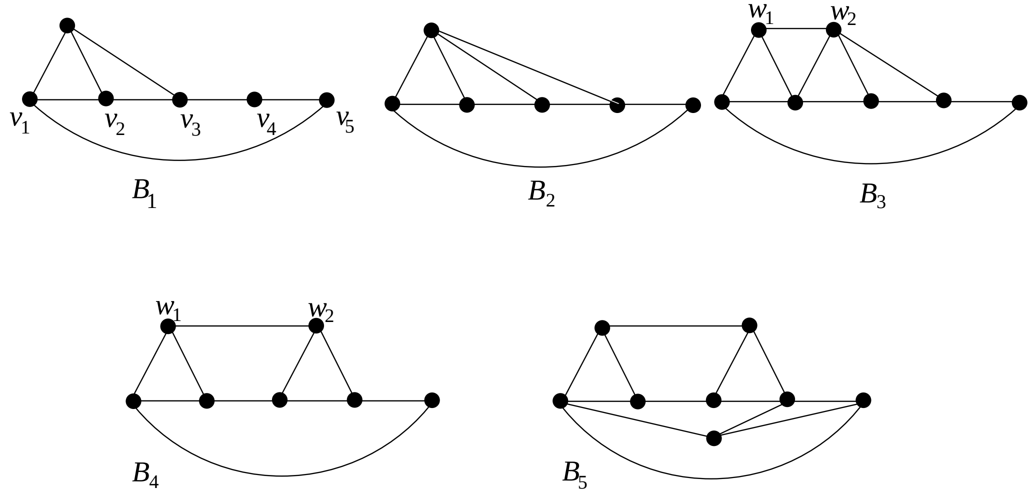

Figure 2: The basic graphs

Lemma 3.3 *Let $G = H \blacktriangleright C$ $(G \in \mathcal{G}_5)$. Assume that H contains no edge. If the degree (in G) of every vertex of H is at least three, then G is isomorphic to one of the basic graphs B_i, $i = 0, 1, 2$ or 3. (B_0 is a graph isomorphic to a 5−hole).*

Proof. If H contains no vertices of degree three or four, then we have $H = \phi$ and $G = C = B_0$. Assume now that $H \neq \phi$. Let w be a vertex of H with $d = d(w)$ (degree in G); since the neighbours of w on C are consecutive, we put $N(w) = \{v_i : 1 \leq i \leq d\}$ $(3 \leq d \leq 4)$. When $H \setminus \{w\} = \phi$ and $d = 3$ or 4, the graph G is isomorphic to B_1 or to B_2.

Case 1, $d = 4$: Suppose that there is a vertex $u \in H \setminus \{w\}$; as G is claw-free, $N(u) \neq N(w)$. If $d(u) = 4$, we have $N(u) = \{v_j, v_{j+1}, v_{j+2}, v_{j+3}\}$ for some

j, $2 \leq j \leq 5$ (the subscripts are counted modulo 5); for every 3-coloration of vertices of $C \cup \{u, w\}$, starting by coloring the vertices of the triangle $v_1 v_2 w v_1$, vertices u and v_5 receive the same color, a contradiction. If $d(u) = 3$, using the same arguments, a contradiction also arises. Consequently, $|H| = 1$; thus G is isomorphic to B_2.

Case 2, $d = 3$: Let $u \in H \setminus \{w\}$. It is clear that $d(u) \neq 4$, since otherwise, we return to the previous case and we would have $u = w$, which is not possible. So $d(u) = 3$ and thus $N(u) = \{v_j, v_{j+1}, v_{j+2}\}$ for some j, $2 \leq j \leq 5$. The case where $j = 2$ (resp. $j = 3$) is the same as the case when $j = 5$ (resp. $j = 4$). When $j = 3$, for any 3-coloration of vertices of $C \cup \{u, w\}$, the color-set of $\{w, v_2\}$ and $\{u, v_4\}$ is the same; hence v_1 or v_5 can not be colored; So this case is excluded; for $j = 2$, the set $C \cup \{u, w\}$ induces a subgraph isomorphic to B_3 (observe that H has no more than two vertices of degree three because otherwise, G would not be 3-colorable). ■

Lemma 3.4 *Let $G = H \blacktriangleright C$ ($G \in \mathcal{G}_5$). Suppose that H contains an edge. If the degree, in G, of every vertex of H is at least three, then G is isomorphic to B_3, B_4 or to B_5.*

Proof. Let $e = w_1 w_2$ be the unique edge of H (2. of Lemma 3.2); e is a superedge of F_1 or of F_2.

Case 1. $e \in F_1$ with $F_1 = w_1 w_2 - v_1 v_2 v_3 v_4$. When $H \setminus \{w_1, w_2\}$ contains no vertex of degree three or four, G is isomorphic to B_4. Let v be a vertex of $H \setminus \{w_1, w_2\}$. Since $3 \leq d = d(v) \leq 4$ (degree in G), we distinguish between two subcases.

Subcase 1.1, $d = 4$: As $|C| = 5$, the set $N(v)$ is either $\{v_2, v_3, v_4, v_5\}$ or $\{v_1, v_2, v_3, v_5\}$ because otherwise, at least one of the set $\{v_1, w_1, v, v_5\}$, $\{v_2, v_3, v, w_2\}$ and $\{v_2, v_3, v, w_1\}$ induces a claw in G; the two possibilities produce the same situation, the first means that $v_1 w_1 w_2 v_3 v v_5 v_1$ is a $6-$hole in G. So this case is impossible.

Subcase 1.2, $d = 3$: The set $N(v) \cap C$ is $\{v_j, v_{j+1}, v_{j+2}\}$ for some j, $1 \leq j \leq 5$ (the subscripts are counted modulo 5). $j = 4$ because otherwise either $\{v_1, w_1, v, v_5\}$, or $\{v_4, w_2, v, v_5\}$, or $\{v_3, w_2, v_2, v\}$, or $\{v_2, w_1, v, v_3\}$ induces a claw in G which is not possible. So $N(v) \cap C = \{v_4, v_5, v_1\}$. The subgraph $H \setminus \{w_1, w_2, v\}$ of G does not contain other vertices of degree three because otherwise the neighbours of such vertices on C will be non-consecutive. Hence G is isomorphic to B_5.

Case 2, $e \in F_2$, $F_2 = w_1 w_2 - v_1 v_2 v_3$: Only one vertex, among w_1 and w_2, must have three neighbours on C because otherwise either G would be not 3-colorable or it would contain a $6-$hole; we can assume that the edge $w_2 v_4$ exists. When $H \setminus \{w_1, w_2\}$ contains no vertex of degree three or four, G is isomorphic to B_3. Suppose that there is a vertex v in $H\setminus\{w_1, w_2\}$. v must be adjacent to at least three consecutive vertices among $\{v_1, v_3, v_4, v_5\}$. v is not adjacent to v_1 because otherwise $C \cup \{v, w_1, w_2\}$ induces a subgraph of G which is not 3-colorable; thus v is adjacent to v_3, v_4 and v_5; consequently $v_1w_1w_2v_3vv_5v_1$ is a $6-$hole in G, a contradiction. ■

3.3 Extension of a graph

Let G' be a graph. Let v and v' be any two adjacent vertices such that v (resp. v') is simplicial in $G'\setminus\{v'\}$ (resp. $G'\setminus\{v\}$). The graph, obtained from G' by adding a new vertex u adjacent exactly to v and v', is an extension of G' (by u). A graph G'' is an extension of G' by a set $U = \{u_1, \ldots, u_l\}$ if G'' is the last graph of the sequence $G'_0, G'_1, \ldots, G'_l$ where $G'_0 = G'$ and for every i, $1 \leq i \leq l$, the graph G'_i is an extension de G'_{i-1} by u_i.
Let G'' be an extension of the graph G'. It is clear that G' is a claw-free and 3-colorable graph if and only if G'' is claw-free and 3-colorable graph; since adding a vertex to G' to obtain G'' does not create a hole, $G' \in \mathcal{G}_m$ if and only if $G'' \in \mathcal{G}_m$ for every $m \geq 5$.

Let $G' \in \mathcal{G}_m$ such that $G' = H' \blacktriangleright C'$ where C' is a $k-$hole. A neighbourhood H' of C' is maximal if for every vertex $u \notin G'$ we have $(H' \cup \{u\}) \blacktriangleright C' \notin \mathcal{G}_m$, $(5 \leq k \leq m)$.

Notice that if B is a block of a graph G' and B contains a hole C', then the neighbourhood of C' is maximal (because B is 2-connected maximal).

As a consequence of the two previous lemmas, we have following result.

Theorem 3.5 *Let $G = H \blacktriangleright C \in \mathcal{G}_5$. If the neighbourhood H of C is maximal, then G is an extension of one of the graphs B_i, $i = 0, 1, \ldots, 5$.*

Proof. Let D be the set of vertices of H of degree two (in G). When $H\setminus D = \phi$; since neighbours of every vertex of D on C are consecutive, G is an extension of B_0. When $H\setminus D \neq \phi$; from Lemma 3.3 and Lemma 3.4, $G\setminus D$ is isomorphic to one of the graphs B_i, $i = 1, \ldots, 5$. Let us consider an index i such that G contains B_i and a vertex $w \in D$ which has neighbours v_j and v_{j+1} on C; the vertex v_j (resp. v_{j+1}) is simplicial in $G\setminus\{v_{j+1}\}$ (resp. $G\setminus\{v_j\}$) because otherwise, w would be a vertex of a claw contained in G which is impossible; hence G is an extension of B_i. ■

Remark 3.6

Let G' be a graph of $\mathcal{G}_5$. Suppose that G is a subgraph of G' with $G = H \blacktriangleright C$, and H contains two distinct non adjacent vertices, w and w', of degree two in G. The vertices w and w' are not connected by a chordless path P such that $P \backslash \{w, w'\}$ is contained in $G' \backslash G$ because otherwise, the subgraph of G' induced by $P \cup C$ will contain a hole of length at least six.

3.4 The main result

Now we can formulate our main result.

Theorem 3.7 *Let G' be a graph of $\mathcal{G}_5$ and B be a block of G'. If B contains a 5−hole, then B is an extension of one of the graphs B_i, $i = 0, 1, \ldots, 5$.*

Proof. Let B be a block of G' containing the 5−hole C. Let G be the subgraph of B induced by the vertices of C and all their neighbours (in G'), so $G = H \blacktriangleright C$ (G is 2-connected). Since B is 2-connected and maximal, H is a maximal neighbourhood of C. Using Theorem 3.5, G is one of the graphs $G_i = H_i \blacktriangleright C$, $i = 0, 1, \ldots, 5$ where G_i is an extension of a graph B_i. For $i = 1, 2$ or 3, since every vertex of H_i of degree at least three is not adjacent to any vertex of $B \backslash G_i$, we have $B \backslash G_i = \phi$; thus by Remark 3.6, B is isomorphic to one of G_i, which is an extension of B_i for $i = 0, 1, 2, 3$.
For $i = 4$ or 5: Let w_1 and w_2 be the two endpoints of the unique edge of H_i, for which the neighbours on C are $\{v_1, v_2\}$ and $\{v_3, v_4\}$ respectively (neighbours in G_i). Since B is 2-connected maximal and claw-free, there is a vertex u of $B \backslash G_i$ adjacent to the simplicial vertices w_1 and w_2; so $G'_i = G_i \cup \{u\}$ is an extension of G_i by the vertex u. We claim that, in B, for every vertex w of $H_i \backslash \{w_1, w_2\}$ we have:

(1) *u and w are not adjacent and*

(2) *u and w are not connected by a chordless path P such that $P \backslash \{u, w\}$ is nonempty and is contained in $B \backslash G'_i$.*

Let v be the vertex of B_4 which has three neighbours on C. Suppose (1) does not hold. Since the set of neighbours of w on C is either $\{v_1, v_5\}$, or $\{v_4, v_5\}$, or $\{v_2, v_3\}$, we have either $G'_i \backslash \{v_1, w_2\}$, or $G'_i \backslash \{v_1, v_5, w_2\}$, or $G'_i \backslash \{v_2, w_2, v\}$ is a hole of length at least six, a contradiction.
Instead of the edge uw let us consider a chordless path P of endpoints u and w such that $P \backslash \{u, w\} \subset B \backslash G'_i$; using Remark 3.6 and with the same arguments as in (1), the fact (2) can be established. We have $B \backslash G'_i = \phi$ because otherwise, there is a chordless P' containing a vertex u of $B \backslash G'_i$ and

connecting two vertices of $H_i \cup \{u\}$ which contradicts (2). Hence $B = G'_i$, therefore, B is an extension of B_i, $i = 4$ or 5. ■

As a consequence:

Corollary 3.8 *Let G' be a graph of $\mathcal{G}_5$ and B be a block of G'. If B contains one of the graphs B_i, $i = 0, 1, \ldots, 4$ or 5, then B is an extension of B_i by at most five vertices and every vertex of degree two in B is either a simplicial vertex of degree two in G' or a cut-vertex of G'.*

4 Conclusion

In terms of blocks, we have described the complete structure of graphs in a small class of claw-free graphs; such a description can contribute to the treatment of the list-colouring problem known to be difficult. In the case studied here, it becomes clear that the number of basic graphs allowing description of blocks depend on the length of the hole which they contain, it would be interesting to extend the class and determine properties which allow the description of basic graphs in an acceptable number.

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