A Note on the Gram Series

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Abstract

We obtain a new identity for the Gram series involving the incomplete gamma function by utilizing a classical integral. It is also shown that the Gram series satisfies a differential equation involving some strange new series.

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1. Introduction and Statement of Results

Let as usual $\mu(n)$ denote the Möbius function, which has the Dirichlet series generating function

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

for $s \in \mathbb{C}, \Re(s) > 1$, and $\zeta(s)$ is the Riemann zeta function. Gram (see [4] or [5]) offered the series

$$G(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln(x))^k}{kk!\zeta(k+1)},$$

as an approximation to the prime counting function $\pi(x)$, the number of primes less than or equal to $x$. As noted in [5], (2) is actually an equivalent form of the Riemann prime counting function

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)Li(x^{1/n})}{n}.$$
As an approximation, the series (3) has been shown to be better than Gauss’s approximation \( \pi(x) \sim Li(x) \) for \( x < 10^9 \). However, Littlewood (see [5]) showed that this formula is worse as \( x \to \infty \).

The Gram series has not received very much attention since its discovery [4], and so we offer some results that might be of some interest in its application to \( \pi(x) \). And, because \( G(x) \) and \( R(x) \) are essentially the same function, we will simply devote our attention to \( G(x) \).

**Theorem 1.1.** Let \( \gamma(s, x) \) be the lower incomplete gamma function, and let \( \gamma'(s, x) \) be its derivative with respect to \( s \). For \( 0 < x < 1 \), we have

\[
G(x) = 1 + \sum_{n=1}^{\infty} \frac{\mu(n)x^{1/n} \ln(\frac{\ln x}{n})}{n} + \sum_{n=1}^{\infty} \frac{\mu(n)\gamma'(1, \frac{\ln x}{n})}{n}.
\] (4)

In the following result, we see that (4) offers some further information about the function \( G(x) \) that might be of some interest.

**Corollary 1.2.** Let \( G'(x) \) be the derivative of \( G(x) \). Then for \( 0 < x < 1 \) we have

\[
G(x) = 1 + \sum_{n=1}^{\infty} \frac{\mu(n)x^{1/n} \ln(\frac{\ln x}{n})}{n} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} (\gamma'(2, \frac{\ln x}{n}) - \gamma(2, \frac{\ln x}{n}))
\]

\[
+ |\ln x| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \ln(\frac{\ln x}{n})x^{1/n} - x|\ln x| \frac{d}{dx}(x \ln x G'(x)).
\]

**2. Proofs of Results**

To establish our results we require some information about the logarithmic integral \( Li(x) \). This function has a clear relationship to the exponential integral \( Ei(x) \), by the formula \( Ei(\ln x) = Li(x) \). From [3], we have the well-known integral representation for \( Ei(x) \), given by

\[
Ei(-x) = -\gamma + e^{-x} \ln(x) + \int_0^x e^{-t} \ln(t)dt,
\] (5)

where \( x > 0 \).

**Proof of Theorem 1.1.** First recall that the lower incomplete gamma function is given by

\[
\gamma(s, x) = \int_0^x e^{-t}t^{s-1}dt.
\]
Now differentiating with respect to \( s \), and setting \( s = 1 \) gives

\[ \gamma'(1, x) = \int_0^x e^{-t} \ln(t) dt. \]  

(6)

Setting \( x = -\ln \frac{\ln x}{n} \) in (5) with \( 0 < x < 1 \), gives us

\[ \text{Li}\left(\frac{x_1}{n}\right) = -\gamma + x^{1/n} \ln\left(-\frac{\ln x}{n}\right) + \int_0^{-\ln \frac{\ln x}{n}} e^{-t} \ln(t) dt. \]  

(7)

Collecting (3), (6), and (7), we have

\[ G(x) = 1 + \sum_{n=1}^{\infty} \frac{(\ln(x))^k}{k! \zeta(k+1)} \]

\[ = 1 - \gamma \sum_{n=1}^{\infty} \frac{\mu(n)}{n} + \sum_{n=1}^{\infty} \frac{\mu(n) x^{1/n} \ln(-\frac{\ln x}{n})}{n} + \sum_{n=1}^{\infty} \frac{\mu(n) \gamma'(1, -\frac{\ln x}{n})}{n} \]

\[ = 1 + \sum_{n=1}^{\infty} \frac{\mu(n) x^{1/n} \ln(-\frac{\ln x}{n})}{n} + \sum_{n=1}^{\infty} \frac{\mu(n) \gamma'(1, -\frac{\ln x}{n})}{n}. \]

In the last line we have employed the formula [1, p.127]

\[ \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0. \]

**Proof of Corollary 1.2.** Using integration by parts it can be seen that

\[ s\gamma(s, x) = \gamma(s + 1, x) + x^s e^{-x}. \]

Dividing both sides by \( s \), differentiating with respect to \( s \), and setting \( s = 1 \), we obtain

\[ \gamma'(1, x) = \gamma'(2, x) - \gamma(2, x) + x \ln x e^{-x} - xe^{-x}. \]

Invoking this formula for \( \gamma'(1, x) \) on the right hand side of (4) gives

\[ G(x) = 1 + \sum_{n=1}^{\infty} \frac{\mu(n) x^{1/n} \ln\left(\frac{|\ln x|}{n}\right)}{n} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\gamma'(2, \frac{|\ln x|}{n}) - \gamma(2, \frac{|\ln x|}{n})\right) \]

\[ + |\ln x| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \ln\left(\frac{|\ln x|}{n}\right)x^{1/n} - |\ln x| \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} x^{1/n}. \]

Now note that \( G(x) \) obeys the differential formula

\[ G'(x) = \frac{1}{x \ln x} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} x^{1/n}. \]
Thus we have
\[
x \frac{d}{dx}(x \ln x G'(x)) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} x^{1/n}.
\]
Finally, using this formula in the last series implies the statement of the corollary.

3. Conclusion

It is interesting to note that repeating the integration by parts procedure in the proof of Corollary 1.2 leads to relations between \(G(x)\), and series of the form
\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^r} x^{1/n},
\]
where \(r\) is a positive integer. It is our hope that the results contained herein will stimulate further research on the function \(G(x)\).

References


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