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Kepler's Quartic Curve as a Model of Planetary Orbits

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Abstract

In this note the authors show that a curve which Kepler supposed as the Martian orbit lies on a quartic elliptic curve.

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1. Introduction

Plane algebraic curves have provided various model of the obits of planets. Hypotrochoids provide good representations of Ptolemaic model of the Martian orbit around the Earth. Those are rational curves. Many authors mentioned that Johannes Kepler supposed an "ovoid" as Mars's orbit around the Sun before he found that an ellipse was the true orbit of Mars (cf. [1], [2], [3], [4], [6],[9], [10]). Following the geometric construction of [3] or [11], we present three parameter representations of a model of planetary orbits which Kepler supposed.

On the Euclidean plane, we shall use Cartesian coordinstes (X, Y). We use the notations in [3] except for K (which was denoted in K" in [3] and K standed for another point in [3]). The points $C = (a+ae, 0)$, $D = (-a+ae, 0)$

are respective the aphelion and the perihelion points of a planet, where $0 < a$ is the radius of the deferent and e is the eccentricity with $0 < e < 1$. The Sun is fixed at the origin $A = (0, 0)$. The planet lies on a circle

$$(X - a\cos\beta)^2 + (Y - a\sin\beta)^2 = a^2e^2, \tag{1.1}$$

where the parameter β is said to be the eccentric anomaly. However the position of the planet is given by $(a\cos\beta + ae, a\sin\beta)$ in the Ptolemaic model, Kepler supposed a new model. Denote by Z the center $(a\cos\beta, a\sin\beta)$ of the above circle. Denote by B the center $(ae, 0)$ of the eccentric circle $(X - ae)^2 + Y^2 = a^2e^2$. Suppose that K is the intersection of the eccentric circle $(X - ae)^2 + Y^2 = a^2e^2$ and the straight line AZ, where we assume that $\overrightarrow{AK} = \lambda\overrightarrow{AZ}$ with $\lambda > 0$. The point $V = (ax, ay)$ of the planet corresponding to β lies on the circle (1.1) and the lengths of AK and AV are equal. We assume $0 < |\beta| < \pi$. There are two points satisfying the conditions for V. We choose (x, y) for which the modulus of the angle $\angle CAV$ is less than that of another. By some computations, we find that the point $V = (ax, ay)$ is given by

$$x = \cos\beta + e\sqrt{1 - e^2\sin^2\beta}, \quad y = (1 - e^2)\sin\beta. \tag{1.2}$$

Another intersection $V' = (a\tilde{x}, a\tilde{y})$ of the circle (1.1) and the circle $X^2 + Y^2 = AK^2$ is given by

$$\tilde{x} = (1 - 2e^2)\cos\beta + e\cos(2\beta)\sqrt{1 - e^2\sin^2\beta} + 2e^2\cos^3\beta,$$

$$\tilde{y} = \sin\beta\{1 + 2e\cos\beta\sqrt{1 - e^2\sin^2\beta} + e^2\cos(2\beta)\}.$$

We assume that $0 < \beta < \pi$. Under this condition the three angles $\angle CAV$, $\angle CAK$, $\angle CBK$ satisfy the equation

$$\angle CBK - \angle CAK = \angle CAK - \angle CAV \ (> 0). \tag{1.3}$$

The parameter β represents $\angle CAK$. By using $\phi = \angle CAK$, we obtain another representation of the point (x, y):

$$x = \frac{(1 + e^2)\cos\phi + 2e}{\sqrt{1 + e^2 + 2e\cos\phi}}, \quad y = \frac{(1 - e^2)\sin\phi}{\sqrt{1 + e^2 + 2e\cos\phi}}. \tag{1.4}$$

It is natural to represent the position of the point (x, y) by using the equated anomaly $\theta = \angle CAV$. It is given by

$$x = r\cos\theta, y = r\sin\theta, r = (1 - e^2)\sqrt{\frac{1 + e^2 + 2e\cos\theta}{1 + e^4 - 2e^2\cos(2\theta)}}. \tag{1.5}$$

The representation (1.2) provides a concrete parametrization of V. Its geometrical construction is given in [3], Section 6. The representation (1.4) provides another parametrization of V. Its geometrical construction and the property (1.3) is mentioned in [11] Ch. 30. This model (1.2), (1.4) or (1.5) was firstly mathematically formulated by Fladt in [4] (cf. [1],[9]). By correcting this model (1.2),(1.4) or (1.5), Kepler reached an elliptical orbit as the final model (cf.[5]). We present a graphic of the upper part ($0 \leq \beta \leq \pi$) of the orbit for $e = 0.8$ in Figure 1. A moving visual realization of the orbit is given for a less eccentricity in [11].

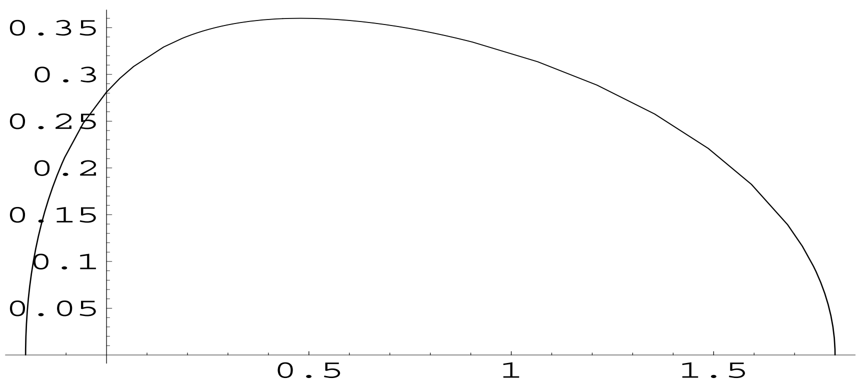

Figure 1: the upper part of the orbit model for $e = 0.8$

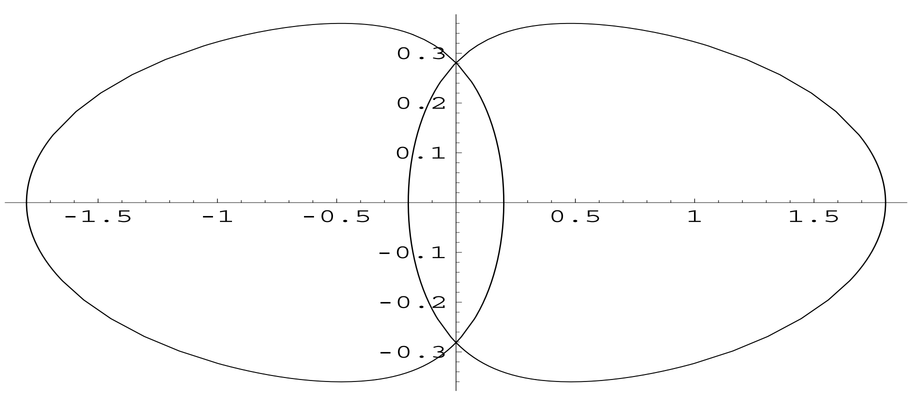

Figure 2: the real part of the quaric algebraic curve for $e = 0.8$

2. Quartic elliptic curves and their parameter representations

By straightforward computations, we find that the point (x, y) with parameter representations (1.2) or (1.4) satisfies an algebraic equation $F(1, x, y) = 0$ for the form

$$F(t, x, y; e) = (1 - e^2)^2 x^4 + 2(1 + e^4)x^2y^2 + (1 + e^2)^2 y^4$$
$$-2(1 - e^2)^2(1 + e^2)t^2(x^2 + y^2) + (1 - e^2)^4 t^4 = 0. \qquad (2.1)$$

The point (x, y) for $V' = (ax, ay)$ satisfies the following sextic equation:

$$S(x, y) = (x^2 + y^2)^3 - 2(1 + e^2)(x^2 + y^2)^2 + (1 - e^2)^2x^2 + (1 + e^2)^2y^2 = 0.$$

We remark that the parametrized curve (x, y) given by (1.2) also satisfies the equation $F(1, -x, y; e) = 0$. Kepler's second law of the planetary motion is stated that the integral

$$\int_0^{t_0} (x(t)y'(t) - y(t)x'(t))dt$$

is proportional to the time t_0, if the position of a planet at time t is represented by $(x(t), y(t))$. The half of the above integral is the area of the region swept by the line segment connecting the Sun and the planet between the time $0 \le t \le t_0$. Kepler provided some primitive numerical approximatios of the area. Motivated by these historical facts, we are interested in the integral of some function $G(x(\psi), y(\psi))$ with respect to some parameter ψ. If the curve $F(t, x, y; e) = 0$ were a rational curve, then such an integral would be rather easy for rational functions G by using some variable change.

We study the properties of the curve $F(t, x, y; e) = 0$ which are invariant under birational transformations. It will shows that the curve is an irreducible quartic elliptic curve and hence it is not a rational curve. So we will recognize that it is necessary to use elliptic functions to express the area.

We define a complex projective plane curve C_e on the complex projective plane $\mathbf{CP}^2$ by

$$C_e = \{[(t, x, y)] \in \mathbf{CP}^2 : F(t, x, y; e) = 0\}, \qquad (2.2)$$

for $0 < e < 1$. In the above the equivalence relation $(t_1, x_1, y_1) \equiv (t_2, x_2, y_2)$ in $\mathbf{C}^3 \backslash \{(0, 0, 0)\}$ by $(t_2, x_2, y_2) = k(t_1, x_1, y_1)$ for some $k \in \mathbf{C}$, $k \neq 0$ defines the quotient space $\mathbf{CP}^2$. We present a graphic of the real affine part of C_e for $e = 0.8$ in Figure 2.

An irreducible complex projective plane curve $F(t,x,y)=0$ is called an *elliptic curve* if its genus is 1. There are various equivalent conditions for the genus of the curve is 1. We use the following one: The curve $F(t,x,y)=0$ is transformed into a non-singular cubic curve

$$G(T,X,Y) = -TY^2 + a_0(X-\alpha T)(X-\beta T)(X-\gamma T) = 0$$

with $a_0(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)\neq 0$ by a birational transformation (cf. [6], [7]). An irreducible complex projective plane curve $F(t,x,y)=0$ is called *rational* if its genus is 0. It is known that $F(t,x,y)=0$ is rational if and only if it is transformed into a straight line by a birational transformation. The curve C_e has two ordinary double points at

$$(t,x,y) = (1,0,\frac{1-e^2}{\sqrt{1+e^2}}), \quad (t,x,y) = (1,0,-\frac{1-e^2}{\sqrt{1+e^2}}).$$

The curve C_e has no other singular point. By using this fact and the classification theory of quartic curves we can show that $F(\cdots;e)$ is irreducible in the polynomial ring $\mathbf{C}[t,x,y]$ and the curve C_e is an elliptic curve (cf.[7]). We shall analyze the curve C_e without using the classification theory.

Following [8] pp. 489-493, we express the curve C_e as the image of a cubic elliptic curve under a birational transformation. We consider the 2-parameter family of conic curves passing through the two singular points $(1,0,\pm(1-e^2)/\sqrt{1+e^2})$ and a non-singular point $(0,1,i)$. Denote by such a family as

$$\{\lambda G_1(t,x,y) + \mu G_2(t,x,y) + \nu G_3(t,x,y) : (\lambda,\mu,\nu)\neq(0,0,0)\}$$

where

$$G_1(t,x,y) = tx, \quad G_2(t,x,y) = (1-e^2)^2t^2 - (1+e^2)(x^2+y^2),$$

$$G_3(t,x,y) = (1-e^2)^2t^2 + i(1+e^2)xy - (1+e^2)y^2.$$

Using these conics, we define a correspondence $(t,x,y)\mapsto(t_1,x_1,y_1)$ of $\mathbf{CP}^2$ onto itself by the relations

$$t_1 = G_1 = tx, x_1 = G_2 - G_3 = -(1+e^2)(x+iy)x,$$

$$y_1 = G_3 = (1-e^2)^2t^2 + i(1+e^2)xy - (1+e^2)y^2.$$

For finite many points (t,x,y) of $\mathbf{CP}^2$, the above rule associate the point $(0,0,0)$. For instance, the rule associates $(0,0,0)$ with $(t,x,y)=(0,1,i)$. So the above correspondence is not exactly a map of $\mathbf{CP}^2$ onto itself. We adopt some exceptional points. Its inverse correspondence is given by the folowing

$$t = (1+e^2)t_1(x_1-y_1), x = -(1-e^2)^2(1+e^2)t_1^2 + x_1^2,$$

$$y = -i\{(1-e^2)^2(1+e^2)t_1^2 + x_1 y_1\}.$$

Substituting these equations into $F(t, x, y; e) = 0$, we have the equation

$$L(t_1, x_1, y_1; e) = 8e^2(1-e^2)^2(1+e^2)t_1^2 x_1$$

$$+(1+e^2)^2 y_1^3 + (1+e^2)^2 x_1 y_1^2 - (1-e^2)^2 x_1^2 y_1 - (1-e^2)^2 x_1^3 = 0,$$

and hence

$$L(t_1, 1, y_1; e) = 8e^2(1-e^2)^2(1+e^2)t_1^2 + (1+e^2)^2 y_1^3 + (1+e^2)^2 y_1^2 - (1-e^2)^2 y_1 - (1-e^2)^2$$

$$= 8e^2(1-e^2)^2(1+e^2)t_1^2 + (y_1+1)\{(e^2+1)y_1 - (1-e^2)\}\{(e^2+1)y_1 + (1-e^2)\}.$$

The curve $L(t_1, 1, y_1; e) = 0$ satisfies a canonical form of non-singular cubics. Thus we obtained the following result.

Theorem 2.1 *The complex projective curve C_e defined by (2.2) for a quartic form (2.1) is irreducible and it is transformed into a cibic elliptic curve by a birational transformation.*

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