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On t-Level R-Subgroups of Near-Rings

Kyung Ho Kim

Department of Mathematics, Chungju National University
Chungju 380-702, Korea
ghkim@cjnu.ac.kr

Abstract

Using t-norm T, we introduce the notion of t-level R-subgroup, and some related properties are investigated.

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1 Introduction

W. Liu [5] has studied fuzzy ideals of a ring, and many researchers are engaged in extending the concepts. S. Abou-Zaid [1] introduced the notion of a fuzzy subnear-ring, and studied fuzzy left (resp. right) ideals of a near-ring, and the present author [4] discussed further properties of fuzzy R-subgroups in near-rings. S. Abou-Zaid [1] also introduced the concept of R-subgroups of a near-ring. In this paper, using t-norm T, we introduce the notion of t-level R-subgroup, and some related properties are investigated.

2 Preliminaries

In this section we include some elementary aspects that are necessary for this paper.

By a *near-ring* we mean a non-empty set R with two binary operations "+" and "·" satisfying the following axioms:

(i) $(R, +)$ is a group,

(ii) $(R, \cdot)$ is a semigroup,

(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a left near-ring because it satisfies the left distributive law. We will use the word "near-ring" in stead of "left near-ring". We denote

xy instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in R$. A *two sided R-subgroup* of a near-ring R is a subset H of R such that

(i) $(H, +)$ is a subgroup of $(R, +)$,
(ii) $RH \subset H$,
(iii) $HR \subset H$.

If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R.

We now review some fuzzy logic concepts. A *fuzzy set* μ in a set R is a function $\mu : R \to [0, 1]$.

Let $(R, +, \cdot)$ be a near-ring. A fuzzy set μ in R is called a *fuzzy right* (resp. *left*) *R-subgroup* of R if

(1) μ is a fuzzy subgroup of $(R, +)$,
(2) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$), for all $r, x \in R$.

Definition 2.1. ([7]) By a *t-norm* T, we mean a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:

(T1) $T(x, 1) = x$,
(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$,
(T3) $T(x, y) = T(y, x)$,
(T4) $T(x, T(y, z)) = T(T(x, y), z)$,

for all $x, y, z \in [0, 1]$.

For a t-norm T on $[0, 1]$, denote by Δ_T the set of element $\alpha \in [0, 1]$ such that $T(\alpha, \alpha) = \alpha$, i.e., $\Delta_T := \{\alpha \in [0, 1] \mid T(\alpha, \alpha) = \alpha\}$.

Proposition 2.2. *Every t-norm T has a useful property:*

$$T(\alpha, \beta) \leq \min(\alpha, \beta)$$

for all $\alpha, \beta \in [0, 1]$.

Throughout this paper, all proofs are going to proceed the only left cases, because the right cases are obtained from similar method. In what follows, the term " fuzzy R-subgroup " (" T-fuzzy R-subgroup ") means "fuzzy left R-subgroup " (" T-fuzzy R-subgroup "), respectively.

3 t-level R-subgroups of near-rings

Definition 3.1. [4] A function $\mu : R \to [0, 1]$ is called a *T-fuzzy right* (resp. *left*) *R-subgroup* of R with respect to a t-norm T (briefly, a *T-fuzzy right* (resp. *left*) *R-subgroup of R*) if

(C1) $\mu(x-y) \geq T(\mu(x), \mu(y))$,
(C2) $\mu(xr) \geq \mu(x)$ (resp. $\mu(rx) \geq \mu(x)$)
for all $r, x \in R$.

It is easy to show that every fuzzy right (resp. left) R-subgrup is a T-fuzzy R-subgroup of R with $T(\alpha, \beta) = \alpha \wedge \beta$ for each $\alpha, \beta \in [0, 1]$

Definition 3.2. Let T be a t-norm. A fuzzy set A in R is said to satisfy *idempotent property* if $\mathrm{Im}(A) \subseteq \Delta_T$.

Proposition 3.3. *Let T be a t-norm on $[0,1]$. If A is an idempotent T-fuzzy R-subgroup of R, then we have $A(0) \geq A(x)$ for all $x \in R$.*

Proof. For every $x \in R$, we have

$$A(0) = A(x-x) \geq T(A(x), A(x)) = A(x).$$

This completes the proof. □

Proposition 3.4. *Let T be a t-norm on $[0,1]$. If A is an idempotent T-fuzzy R-subgroup of R, then the set*

$$A^{\omega} = \{x \in R \mid A(x) \geq A(\omega)\}$$

is an R-subgroup of a near-ring R.

Proof. Let $x, y \in A^{\omega}$. Then $A(x) \geq A(\omega)$ and $A(y) \geq A(\omega)$. Since A is an idempotent T-fuzzy R-subgroup of R, it follows that

$$A(x-y) \geq T(A(x), A(y)) \geq T(A(x), A(\omega)) \geq T(A(\omega), A(\omega)) = A(\omega).$$

Now let $r \in R, x \in A^{\omega}$. Then $A(rx) \geq A(x) \geq A(\omega)$. Thus, we have $A(x-y) \geq A(\omega)$ and $A(rx) \geq A(\omega)$, that is., $x - y \in A^{\omega}$ and $rx \in A^{\omega}$. This completes the proof. □

Corollary 3.5. *Let T be a t-norm. If A is an idempotent T-fuzzy R-subgroup of R, then the set*

$$A_R = \{x \in R \mid A(x) = A(0)\}$$

is an R-subgroup of a near-ring R.

Proof. From the Proposition 3.3, $A_R = \{x \in R \mid A(x) = A(0)\} = \{x \in R \mid A(x) \geq A(0)\}$, hence A_R is an R-subgroup of a near-ring R from Proposition 3.4. □

Let χ_I denote the characteristic function of a non-empty subset I of a near-ring R.

Theorem 3.6. *Let $I \subseteq R$. Then I is an R-subgroup of a near-ring R if and only if χ_I is a T-fuzzy R-subgroup of a near-ring R.*

Proof. Let I be an R-subgroup of R. Then it is easy to show that χ_I is an T-fuzzy R-subgroup of R. In fact, let $x, y \in I$ and $r \in R$. Then $x - y \in I$ and $rx \in I$. Hence

$$\chi_I(x-y) = 1 = T(\chi_I(x), \chi_I(y)) \text{ and } \chi_I(rx) \geq \chi_I(y) = 1.$$

If $x \in I$, $y \notin I$ (or $x \notin I$ and $y \in I$), then we have $\chi_I(x) = 1$ or $\chi_I(y) = 0$. This means that

$$\chi_I(x-y) \geq T(\chi_I(x), \chi_I(y)) = 0 \text{ and } \chi_I(rx) \geq \chi_I(x) = 0.$$

Conversely, suppose that χ_I is a T-fuzzy R-subgroup of R. Now let $x, y \in I$. Then $\chi_I(x-y) \geq T(\chi_I(x), \chi_I(y)) = 1$, and so $\chi_I(x-y) = 1$, that is, $x - y \in I$. Let $r \in R, x \in I$. Then $\chi_I(rx) \geq \chi_I(x) = 1$, and so $rx \in I$. This proves the theorem. □

Lemma 3.7. ([2]) *Let T be a t-norm. Then*

$$T(T(\alpha, \beta), T(\gamma, \delta)) = T(T(\alpha, \gamma), T(\beta, \delta))$$

for all $\alpha, \beta, \gamma, \delta \in [0, 1]$.

Proposition 3.8. *If A and B are T-fuzzy R-subgroups of a near-ring R, then $A \wedge B : R \to [0, 1]$ defined by*

$$(A \wedge B)(x) = T(A(x), B(x))$$

for all $x \in R$ is a T-fuzzy R-subgroup of R.

Proof. Let x, y and $r \in R$. Then we have

$$\begin{aligned}(A \wedge B)(x-y) &= T(A(x-y), B(x-y)) \geq T(T(A(x), A(y)), T(B(x), B(y))) \\ &= T(T(A(x), B(x)), T(A(y), B(y))) = T((A \wedge B)(x), (A \wedge B)(y))\end{aligned}$$

and

$$\begin{aligned}(A \wedge B)(rx) &= T(A(rx), B(rx)) \geq T(A(x), B(x)) \\ &= (A \wedge B)(x).\end{aligned}$$

This completes the proof. □

Definition 3.9. A fuzzy R-subgroup A of a near-ring R is said to be *normal* if $A(0) = 1$.

Theorem 3.10. *Let A be a T-fuzzy R-subgroup of near-ring R and let A^* be a fuzzy set in R defined by $A^*(x) = A(x) + 1 - A(0)$ for all $x \in R$. Then A^* is a normal T-fuzzy R-subgroup of a near-ring R containing A.*

Proof. For $x, y \in R$ and $r \in R$, we have

$$\begin{aligned} A^*(x-y) &= A(x-y) + 1 - A(0) \geq T(A(x), A(y)) + 1 - A(0) \\ &= T(A(x) + 1 - A(0), A(y) + 1 - A(0)) \\ &= T(A^*(x), A^*(y)) \end{aligned}$$

and

$$\begin{aligned} A^*(rx) &= A(rx) + 1 - A(0) \\ &\geq A(x) + 1 - A(0) \\ &= A^*(x). \end{aligned}$$

Hence A^* is a T-fuzzy R-subgroup of a near-ring R. Clearly, $A^*(0) = 1$ and $A \subset A^*$. □

Definition 3.11. Let A be a fuzzy subset of a set R, T a t-norm and $r \in [0, 1]$. Then we define a *t-level subset* of a fuzzy subset A as

$$A_r^T = \{x \in R \mid T(A(x), r) \geq r\}.$$

Theorem 3.12. *Let R be a near-ring and A a T-fuzzy R-subgroup of R. Then t-level subset A_α^T is an R-subgroup of R where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$.*

Proof. $A_\alpha^T = \{x \in M \mid T(A(x), \alpha) \geq \alpha\}$ is clearly nonempty. Let $x, y \in A_\alpha^T$. Then we have $T(A(x), \alpha) \geq \alpha$ and $T(A(y), \alpha) \geq \alpha$, Since A is a T-fuzzy R-subgroup of R, $A(x-y) \geq T(A(x), A(y))$ is satisfied. This means that

$$T(A(x-y), \alpha) \geq T(T(A(x), A(y)), \alpha) = T(A(x), T(A(y), \alpha)) \geq T(A(x), \alpha) \geq \alpha.$$

Hence $x - y \in A_\alpha^T$. Now let $r \in R$ and $x \in A_\alpha^T$. Then we have $T(A(x), \alpha) \geq \alpha$. Since A is a T-fuzzy R-subgroup of R, we have $A(rx) \geq A(x)$, and so $T(A(rx), \alpha) \geq T(A(x), \alpha) \geq \alpha$. This means that $rx \in A_\alpha^T$. Therefore A_α^T is an R-subgroup of R. □

Theorem 3.13. *Let R be a near-ring and A a fuzzy R-subgroup of R. Then t-level subset A_α^T is an R-subgroup of R where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$.*

Proof. $A_\alpha^T = \{x \in R \mid T(A(x), \alpha) \geq \alpha\}$ is clearly nonempty. Let $x, y \in A_\alpha^T$. Then we have $T(A(x), \alpha) \geq \alpha$ and $T(A(y), \alpha) \geq \alpha$. Since A is a fuzzy R-subgroup of R, $A(x-y) \geq \min\{A(x), A(y)\}$ is satisfied. This means that $T(A(x-y), \alpha) \geq T(\min(A(x), A(y)), \alpha)$. If $\min\{A(x), A(y)\} = A(x)$ or $\min\{A(x), A(y)\} = A(y)$, in two cases, we have $T(\min\{A(x), A(y)\}, \alpha) \geq \alpha$ since $x, y \in A_\alpha^T$. Therefore, $T(A(x-y), \alpha) \geq \alpha$. Thus we get $x - y \in A_\alpha^T$. It is easily seen that, as above, $rx \in A_\alpha^T$. Hence A_α^T is an R-subgroup of R. □

Theorem 3.14. *Let R be a near-ring and A be a fuzzy set of R such that A^T_α is an R-subgroup of R where $T(A(x), \alpha) \geq \alpha$ for all $\alpha \in [0, 1]$. Then A is a T-fuzzy R-subgroup of R.*

Proof. Let $x, y \in R$ and $T(A(x), \alpha_1) = \alpha_1$ and $T(A(y), \alpha_2) = \alpha_2$. Then $x \in A^T_{\alpha_1}$ and $y \in A^T_{\alpha_2}$. Let us assume $\alpha_1 < \alpha_2$. Then there follows that $T(A(x), \alpha_1) < T(A(y), \alpha_2)$ and $A^T_{\alpha_2} \subseteq A^T_{\alpha_1}$. So, $y \in A^T_{\alpha_1}$. Thus $x, y \in A^T_{\alpha_1}$ and since $A^T_{\alpha_1}$ is an R-subgroup of R, by hypothesis, $x - y \in A^T_{\alpha_1}$. Therefore we have

$$\begin{aligned} T(A(x-y), \alpha_1) &\geq \alpha_1 = T(A(x), \alpha_1) \\ &\geq T(A(x), T(A(y), \alpha_1)) \\ &= T(T(A(x), A(y)), \alpha_1). \end{aligned}$$

Thus we get $T(A(x-y), \alpha_1) \geq T(T(A(x), A(y)), \alpha_1)$. As a t-norm is monotone with respect to each variable and symmetric, we have $A(x-y) \geq T(A(x), A(y))$. Now let, $r \in R$ and $T(A(x), \alpha) = \alpha$. Then $x \in A^T_\alpha$. Since A^T_α is an R-subgroup of R, we have $rx \in A^T_\alpha$. Therefore $T(A(rx), \alpha) \geq \alpha$, and hence $T(A(rx), \alpha) \geq T(A(x), \alpha)$. So, we have $A(rx) \geq A(x)$. Thus A is a T-fuzzy R-subgroup of R. □

Definition 3.15. For each $i = 1, 2, 3, ..., n$, let A_i be a T-fuzzy R-subgroup in a near-ring R_i. Let T be a t-norm. Then the *T-product* of A_i $(i = 1, 2, ..., n)$ is the function $A_1 \times A_2 \times A_3 \times \cdots \times A_n : R_1 \times R_2 \times R_3 \times \cdots \times R_n \to [0, 1]$ defined

$$\begin{aligned} &(A_1 \times A_2 \times A_3 \times \cdots \times A_n)(x_1, x_2, x_3, \cdots, x_n) \\ &= T(A_1(x_1), A_2(x_2), A_3(x_3), ..., A_n(x_n)) \end{aligned}$$

for $x_i \in R_i$ $(i = 1, 2, ..., n)$.

Theorem 3.16. ([3]) *Let A and B be t-level subsets of the sets G and H, respectively, and let $\alpha \in [0, 1]$. Then $A \times B$ is also t-level subset of $G \times H$.*

Definition 3.17. Let R be a near-ring and A a T-fuzzy R-subgroup of R. The R-subgroup A^T_α is called *t-level R-subgroup* of R where $T(A(0), \alpha) \geq \alpha$ for $\alpha \in [0, 1]$.

Theorem 3.18. *Let R_1 and R_2 be two near-rings, and A and B T-fuzzy R-subgroups of R_1 and R_2, respectively. Then the t-level subset $(A \times B)^T_\alpha$, for $\alpha \in [0, 1]$, is an R-subgroups of $R_1 \times R_2$.*

Proof. $(A \times B)^T_\alpha = \{(x, y) \mid T((A \times B)(x, y), \alpha) \geq \alpha\}$. Since

$$\begin{aligned} T((A \times B)(0_{R_1}, 0_{R_2}), \alpha) &= T(T(A(0_{R_1}), B(0_{R_2})), \alpha)) \\ &= T(A(0_{R_1}), T(B(0_{R_2}), \alpha)) \\ &\geq T(A(0_{R_1}), \alpha) \geq \alpha, \end{aligned}$$

$(A \times B)^T_\alpha$ is nonempty. Let$(x_1, y_1), (x_2, y_2) \in (A \times B)^T_\alpha$. Then we have $T((A \times B)(x_1, y_1), \alpha) \geq \alpha$ and $T((A \times B)(x_2, y_2), \alpha) \geq \alpha$. Since $A \times B$ is an T-fuzzy R-subgroup of $R_1 \times R_2$, we get

$$(A\times B)((x_1, y_1)-(x_2, y_2)) = (A\times B)(x_1-x_2, y_1-y_2) = T(A(x_1-x_2), B(y_1-y_2)).$$

Since A and B are T-fuzzy R-subgroups, we get

$$\begin{aligned} T((A \times B)(x_1 - x_2, y_1 - y_2), \alpha) &\geq T(T(A(x_1 - x_2), B(y_1 - y_2)), \alpha) \\ &= T((A(x_1 - x_2), T(B(y_1 - y_2), \alpha) \\ &\geq T(A(x_1 - x_2), \alpha) \geq \alpha. \end{aligned}$$

Hence $(x_1, y_1) - (x_2, y_2) \in (A \times B)^T_\alpha$. Now let $(r_1, r_2) \in R_1 \times R_2$ and $(x_1, x_2) \in (A \times B)^T_\alpha$. Then we have

$$\begin{aligned} T((A \times B)(r_1, r_2)(x_1, x_2), \alpha) &= T((A \times B)(r_1x_1, r_2x_2), \alpha) \\ &\geq T(T(A(r_1x_1), B(r_2x_2)), \alpha) \\ &= T((A(r_1x_1), T(B(r_2x_2), \alpha)) \\ &\geq T(A(x_1), T(B(x_2), \alpha)) \\ &\geq T(A(x_1), \alpha) \geq \alpha. \end{aligned}$$

This means that $(x_1, x_2) \in (A \times B)^T_\alpha$. Therefore $(A \times B)^T_\alpha$ is an R-subgroup of $R_1 \times R_2$. □

Theorem 3.19. ([3]) *Let A and B be fuzzy sets of the sets G and H, respectively, T a t-norm and $\alpha \in [0, 1]$. Then $A^T_\alpha \times B^T_\alpha = (A \times B)^T_\alpha$.*

Theorem 3.20. *Let $A_1, A_2, A_3, ..., A_n$ be fuzzy R-subgroups under a minimum operation in near-rings $R_1, R_2, R_3, ..., R_n$, respectively, and let $\alpha \in [0, 1]$. Then*

$$(A_1 \times A_2 \times \cdots \times A_n)^T_\alpha = A^T_{1\alpha} \times A^T_{2\alpha} \times \cdots \times A^T_{n\alpha}.$$

Proof. Let $(a_1, a_2, a_3, ..., a_n) \in (A_1 \times A_2 \times \cdots \times A_n)^T_\alpha$. Then we have

$$\begin{aligned} &T(\min((A_1 \times A_2 \times \cdots \times A_n)(a_1, a_2, a_3, ..., a_n), \alpha) \\ &= T(\min(A_1(a_1), A(_2(a_2), ..., A_n(a_n)), \alpha). \end{aligned}$$

For all $i = 1, 2, ..., n$, $\min(A_1(a_1), A(_2(a_2), ..., A_n(a_n)) = A_i(a_i)$. This gives us

$$\begin{aligned} &T(\min(A_1(a_1), A(_2(a_2), ..., A_n(a_n)), \alpha) \\ &= T(A_i(a_i), \alpha) \geq \alpha. \end{aligned}$$

Thus we have $a_i \in A^T_{i\alpha}$. That is, $(a_1, a_2, a_3, ..., a_n) \in A^T_{1\alpha} \times A^T_{2\alpha} \times \cdots \times A^T_{n\alpha}$. Similarly, $(a_1, a_2, a_3, ..., a_n) \in A^T_{1\alpha} \times A^T_{2\alpha} \times \cdots \times A^T_{n\alpha}$. Then, for all $i = 1, 2, ..., n$, we have $a_i \in A^T_{i\alpha}$. That is, $T(A_i(a_i), \alpha) \geq \alpha$. Since $\min(A_1(a_1), A(_2(a_2), ..., A_n(a_n)) = A_i(a_i)$ and $T(A_i(a_i), \alpha) \geq \alpha$, we have

$$\begin{aligned} &T((A_1 \times A_2 \times \cdots \times A_n)(a_1, a_2, a_3, ..., a_n), \alpha) \\ &= T(\min(A_1(a_1), A(_2(a_2), ..., A_n(a_n)), \alpha) \\ &= T(A_i(a_i), \alpha) \geq \alpha. \end{aligned}$$

Thus we have $(a_1, a_2, a_3, ..., a_n) \in (A_1 \times A_2 \times \cdots \times A_n)^T_\alpha$. □

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